PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 1, 1980

WHEN IS A BIPARTITE GRAPH A RIGID FRAMEWORK?

E. D. BOLKER AND B. ROTH

We find the dimension of the space of stresses for all realizations of the complete bipartite graph $K_{m,n}$ in R^d. That allows us to determine the infinitesimal rigidity or infinitesimal flexibility of such frameworks. The results lead both to the generic classification of $K_{m,n}$ in R^d and to a description of the realizations which deviate from the generic behavior.

1. Introduction. A *framework* in $\boldsymbol{R}^d$ is a finite sequence $p_1, \cdots, p_v$ of points in $\boldsymbol{R}^d$ called *vertices* together with a nonempty set E of sets $\{i, j\}$ such that $1 \leqq i < j \leqq v$. Such a framework is a *realization* in $\boldsymbol{R}^d$ of the graph E on the abstract vertex set $\{1, \cdots, v\}$. The natural and correct tendency is to think of $p_1, \cdots, p_v$ as a set of vertices in $\boldsymbol{R}^d$; we formally define it as a sequence because we wish to allow $p_i = p_j$ even when $i \neq j$ and because we often sum over the index set $\{1, \cdots, v\}$. We refer to the segment $[p_i, p_j]$ for $\{i, j\} \in E$ as an *edge* of the framework. Note that edges may have length zero. A *stress* of a framework is a real valued function ω on E such that for each vertex p_i

$$(1) \qquad \sum_{\{j:\{i,j\}\in E\}} \omega_{\{i,j\}}(p_i - p_j) = 0\ , \quad 1 \leqq i \leqq v\ .$$

If one thinks of a framework as a physical object whose edges are stiff rods then the scalars $\omega_{\{i,j\}}$ can be thought of as an assignment to each edge of a compression or tension, depending on whether $\omega_{\{i,j\}}$ is positive or negative. Then Equation (1) says the forces at each vertex are in equilibrium.

We study stresses of frameworks because, roughly speaking, their existence indicates that some edges are redundant. The more edges, the more likely a framework is to be infinitesimally rigid, but the larger the space of stresses, the less likely. A short but precise account of these relationships, and of the connections with generic rigidity, appears in § 4.

Frameworks realizing bipartite graphs contain no triangles. Hence the rigidity of such frameworks is noteworthy. Mathematicians and engineers knew in the 19th century that a plane realization of $K_{3,3}$ is infinitesimally rigid except when its six vertices lie on a conic. The graphs $K_{4,6}$ and $K_{5,5}$ realized in space have a chance of being generically rigid; whether they actually are was asked at the special session on rigidity at the Syracuse meeting of the American Mathematical Society in October, 1978.

In this paper, we compute the dimension of the space of stresses for every realization of $K_{m,n}$ in $\boldsymbol{R}^d$. With that information one can determine the generic classification of $K_{m,n}$ in $\boldsymbol{R}^d$ and also describe the realizations in which the framework deviates from its generic behavior. We find that these deviations occur for two reasons: when the affine span of the m-set or the n-set of vertices is of lower dimension than expected, or when the vertices lie on more than the expected number of quadric surfaces.

2. The space of stresses. We shall study the framework in $\boldsymbol{R}^d$ which realizes the complete bipartite graph $K_{m,n}$ by locating the vertices at $A = (a_1, \cdots, a_m) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d = \boldsymbol{R}^{md}$ and $B = (b_1, \cdots, b_n) \in \boldsymbol{R}^{nd}$. This framework, which we denote $K(A, B)$, has for edges the segments $[a_i, b_j]$ for $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$. Then we can regard a stress of $K(A, B)$ as an $m \times n$ matrix $\omega = (\omega_{ij})$ such that

$$\sum_{j=1}^{n} \omega_{ij}(a_i - b_j) = 0\,, \quad 1 \leqq i \leqq m \tag{2}$$

and

$$\sum_{i=1}^{m} \omega_{ij}(b_j - a_i) = 0\,, \quad 1 \leqq j \leqq n\,. \tag{2'}$$

This is just Equation (1) for the framework $K(A, B)$. To avoid sinking in a morass of indices we will often think of A and B as sets, of ω as (ω_{ab}), $a \in A$, $b \in B$, and write the stress conditions (2,2′) as

$$\sum_{b} \omega_{ab}(a - b) = 0 \quad \text{for all} \quad a \in A \tag{3}$$

and

$$\sum_{a} \omega_{ab}(b - a) = 0 \quad \text{for all} \quad b \in B\,. \tag{3'}$$

Let $\Omega = \Omega(A, B)$ be the vecter space of stresses of the framework $K(A, B)$. Equations (3, 3′) say that $\omega \in \Omega$ if and only if, setting $\rho_a = \sum_b \omega_{ab}$ and $\gamma_b = \sum_a \omega_{ab}$, we have

$$\sum_{b} \omega_{ab} b = \rho_a a \quad \text{for all} \quad a \in A \tag{4}$$

and

$$\sum_{a} \omega_{ab} a = \gamma_b b \quad \text{for all} \quad b \in B\,. \tag{4'}$$

This is the first hint of the importance of the row and column sums ρ_a, γ_b of ω. We begin with an analysis of the stresses for which the row and column sums are all zero.

For $X = (x_1, \cdots, x_k) \in \boldsymbol{R}^d \times \cdots \times \boldsymbol{R}^d = \boldsymbol{R}^{kd}$, let

$$\bar{X} = \left\{\sum_{i=1}^{k} \mu_i x_i \colon \sum_{i=1}^{k} \mu_i = 1\right\}$$

be the *affine span* of X and

$$D(X) = \left\{(\lambda_1, \cdots, \lambda_k) \in \boldsymbol{R}^k \colon \sum_{i=1}^{k} \lambda_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 0\right\} \tag{5}$$

be the space of *affine dependencies* of X. Then

$$\dim D(X) + \dim \bar{X} = k - 1 . \tag{6}$$

For vectors $\alpha = (\alpha_1, \cdots, \alpha_m) \in \boldsymbol{R}^m$ and $\beta = (\beta_1, \cdots, \beta_n) \in \boldsymbol{R}^n$, let $\alpha \otimes \beta$ be the $m \times n$ matrix $(\alpha_i \beta_j)$, and let $D(A) \otimes D(B)$ be the space spanned by $\{\alpha \otimes \beta \colon \alpha \in D(A), \beta \in D(B)\}$.

THEOREM 1. *$D(A) \otimes D(B)$ is the subspace W of $\Omega(A, B)$ consisting of those stresses all of whose row and column sums are zero.*

Proof. There is much to check, but most of it is straightforward multilinear algebra. If $\omega = \alpha \otimes \beta$ for $\alpha \in D(A)$ and $\beta \in D(B)$ then for all $a \in A$,

$$\rho_a = \sum_b \omega_{ab} = \sum_b \alpha_a \beta_b = \alpha_a \sum_b \beta_b = 0$$

because $\beta \in D(B)$. Similarly, $\gamma_b = 0$ for all $b \in B$. Thus ω has zero row and column sums. To see that it is a stress, we check Equations (4, 4′). For all $a \in A$,

$$\sum_b \omega_{ab} b = \sum_b \alpha_a \beta_b b = \alpha_a \sum_b \beta_b b = 0 = \rho_a a$$

because $\beta \in D(B)$ and $\rho_a = 0$. A similar argument establishes (4′). Thus $\omega = \alpha \otimes \beta \in W$, and so $D(A) \otimes D(B) \subset W$. To prove the reverse inclusion, it suffices to show $\dim W = \dim D(A) \dim D(B)$ since $\dim D(A) \otimes D(B) = \dim D(A) \dim D(B)$. Equations (4, 4′) tell us that a stress ω belongs to W if and only if every row of ω is in $D(B)$ and every column is in $D(A)$. Regarding stresses as linear maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$, we then have $\omega \in W$ if and only if $\operatorname{im} \omega \subset D(A)$ and $\operatorname{im} \omega^t \subset D(B)$, where ω^t is the transpose of ω. But $\operatorname{im} \omega^t = (\ker \omega)^\perp \subset D(B)$ if and only if $D(B)^\perp \subset \ker \omega$ if and only if ω can be factored through $\boldsymbol{R}^n / D(B)^\perp$. Therefore W is the space of linear maps from $\boldsymbol{R}^n / D(B)^\perp$ to $D(A)$ and thus has dimension

$$(n - (n - \dim D(B))) \dim D(A) = \dim D(A) \dim D(B)$$

as desired.

Next we observe that a natural geometrical condition may force some row and column sums to vanish.

LEMMA 2. *If $a \notin \bar{B}$ (respectively $b \notin \bar{A}$) then $\rho_a = 0$ (respectively $\gamma_b = 0$) for all stresses of $K(A, B)$.*

Proof. Suppose $\omega \in \Omega$. If $\rho_a \neq 0$ then Equation (4) implies $\sum_b(\omega_{ab}/\rho_a)b = a$. Since $\sum_b(\omega_{ab}/\rho_a) = 1$, that says $a \in \bar{B}$.

COROLLARY 3. *If $A \cap \bar{B} = B \cap \bar{A} = \varnothing$ then $\Omega(A, B) = D(A) \otimes D(B)$.*

Usually, however, some vertices in A will belong to the affine span of B and vice versa, so some row and column sums need not vanish. To analyze that situation, let $T: \Omega \to \boldsymbol{R}^{m+n}$ be given by

$$T(\omega) = (\cdots, \rho_a, \cdots, -\gamma_b, \cdots) , \quad a \in A, b \in B .$$

The minus sign preceding the column sums is merely a technical device. Theorem 1 says that $\ker T = D(A) \otimes D(B)$, so we will know all about Ω when we have characterized $\operatorname{im} T$. To that end, for $X = (x_1, \cdots, x_k) \in \boldsymbol{R}^{kd}$ we let

$$D^2(X) = \left\{(\lambda_1, \cdots, \lambda_k) \in \boldsymbol{R}^k \colon \sum_{i=1}^k \lambda_i(x_i \otimes x_i) = 0 , \right.$$
$$\left. \sum_{i=1}^k \lambda_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^k \lambda_i = 0 \right\} .$$

That is,

$$D^2(X) = D(X) \cap D(x_1 \otimes x_1, \cdots, x_k \otimes x_k) ,$$

the simultaneous affine dependencies of $x_1, \cdots, x_k$ and $x_1 \otimes x_1, \cdots, x_k \otimes x_k$. Finally, for $\xi = (\xi_1, \cdots, \xi_d) \in \boldsymbol{R}^d$, let $\underline{\xi} = (\xi, 1) = (\xi_1, \cdots, \xi_d, 1) \in \boldsymbol{R}^{d+1}$.

LEMMA 4. *$D(X)$ is the space of linear dependencies of $\underline{x}_1, \cdots, \underline{x}_k$, and $D^2(X)$ is the space of linear dependencies of $\underline{x}_1 \otimes \underline{x}_1, \cdots, \underline{x}_k \otimes \underline{x}_k$.*

Proof. The first assertion follows easily from the definition in (5). The second is true because the coordinates of $\underline{\xi} \otimes \underline{\xi}$ are the products $\xi_i\xi_j$, $1 \leqq i, j \leqq d$, the linear terms ξ_i, $1 \leqq i \leqq d$, and the constant 1.

LEMMA 5. *For $(A, B) = (a_1, \cdots, a_m, b_1, \cdots, b_n)$, $\operatorname{im} T \subset D^2(A, B)$.*

Proof. Suppose $\omega \in \Omega(A, B)$ and $T\omega = (\cdots, \rho_a, \cdots, -\gamma_b, \cdots)$. To prove the lemma note first that

$$\sum_a \rho_a - \sum_b \gamma_b = \sum_{a,b} \omega_{ab} - \sum_{a,b} \omega_{ab} = 0 .$$

Then, using Equation (4′),

$$\sum_a \rho_a a - \sum_b \gamma_b b = \sum_a \sum_b \omega_{ab} a - \sum_b \gamma_b b$$
$$= \sum_b \sum_a \omega_{ab} a - \sum_b \gamma_b b = \sum_b \gamma_b b - \sum_b \gamma_b b = 0 ,$$

so $T\omega \in D(A, B)$. Finally, using Equations (4, 4′) and the bilinearity of $\otimes$,

$$\sum_a \rho_a (a \otimes a) - \sum_b \gamma_b (b \otimes b) = \sum_a a \otimes (\sum_b \omega_{ab}) a - \sum_b (\sum_a {}_{ab}) b \otimes b$$
$$= \sum_a (a \otimes \sum_b \omega_{ab} b) - \sum_b (\sum_a \omega_{ab} a) \otimes b = \sum_{a,b} \omega_{ab} a \otimes b$$
$$- \sum_{a,b} \omega_{ab} a \otimes b = 0 .$$

Therefore $T\omega \in D^2(A, B)$.

However, the reverse inclusion fails. Not every $\lambda \in D^2(A, B)$ comes from a stress of $K(A, B)$. That is because Lemma 2 forces some coordinates of $T\omega$ to vanish. When we take that into account we can characterize the subspace $\operatorname{im} T$ of $D^2(A, B)$. Let $C = (A \cap \bar{B}, B \cap \bar{A})$. More precisely, $C = (c_1, \cdots, c_k)$ consists of those vertices of $A = (a_1, \cdots, a_m)$ which lie in $\bar{B}$ and those of $B = (b_1, \cdots, b_n)$ which lie in $\bar{A}$. We naturally regard $D^2(C)$ as the subspace of $D^2(A, B)$ consisting of those vectors which are zero at coordinate places corresponding to vertices $a \notin \bar{B}$ and $b \notin \bar{A}$.

THEOREM 6. *For* $C = (A \cap \bar{B}, B \cap \bar{A})$, $\operatorname{im} T = D^2(C)$.

Proof. Suppose $\omega \in \Omega(A, B)$. Then Lemma 2 says that $\rho_a = 0$ for all $a \notin \bar{B}$ and $\gamma_b = 0$ for all $b \notin \bar{A}$, so Lemma 5 gives $T\omega \in D^2(C)$. Thus $\operatorname{im} T \subset D^2(C)$. To establish the reverse inclusion, suppose $\lambda \in D^2(C)$. Regard λ as a vector $(\cdots, \mu_a, \cdots, \nu_b, \cdots) \in D^2(A, B)$ for which $\mu_a = 0$ if $a \notin \bar{B}$ and $\nu_b = 0$ if $b \notin \bar{A}$. We shall exhibit a stress ω of $K(A, B)$ for which $T\omega = \lambda$. Let $A_1 \subset A$ be an affine basis for $\bar{A}$ and A_2 the remaining vertices of A. Define B_1 and B_2 in a similar fashion. To simplify notation, we write x, s, y, t for generic vertices of A_1, A_2, B_1, B_2, respectively. Thus, for example, $\sum_a = \sum_x + \sum_s$. Each element u of $\bar{A}$ can be written uniquely as an affine combination of the elements x of A_1. We write the coefficients as u^x, so that

$$u = \sum_x u^x x \quad \text{where} \quad \sum_x u^x = 1 . \tag{7}$$

Similarly, each $v \in \bar{B}$ has a unique expression as

$$v = \sum_y v^y y \quad \text{where} \quad \sum_y v^y = 1 . \tag{7′}$$

We shall soon need the fact that Equations (7.7′) are equivalent to

$$(8) \qquad \underline{u} = \sum_x u^x \underline{x}$$

and

$$(8') \qquad \underline{v} = \sum_y v^y \underline{y} \ ,$$

the unique ways to express $\underline{u}$, $u \in \bar{A}$, and $\underline{v}$, $v \in \bar{B}$, as linear combinations of the vectors $\underline{x}$, $x \in A_1$, and $\underline{y}$, $y \in B_1$.

Define the $m \times n$ matrix $\omega = (\omega_{ab})$ as

$$\begin{array}{cc} & \begin{array}{cc} y \in B_1 & \qquad t \in B_2 \end{array} \\ \begin{array}{c} x \in A_1 \\ \\ s \in A_2 \end{array} & \left[\begin{array}{c:c} \omega_{xy} = \mu_x x^y + \sum_t \nu_t t^x t^y & \omega_{xt} = -\nu_t t^x \\ \hdashline \omega_{sy} = \mu_s s^y & \omega_{st} = 0 \end{array}\right] . \end{array}$$

Note that if $x \notin \bar{B}$ then x^y is undefined. But then $\mu_x = 0$, so we adopt the convention that $\mu_x x^y = 0$. Similarly, for $t \notin \bar{A}$ and $s \notin \bar{B}$, we set $\nu_t t^x = \mu_s s^y = 0$.

We now verify that $T\omega = \lambda$ and that $\omega \in \Omega(A, B)$. That requires eight kinds of calculations—two for the vertices in each of A_1, A_2, B_1, B_2.

(i) The vertices $x \in A_1$. First, the x row sum ρ_x of ω is

$$\begin{aligned} \rho_x = \sum_b \omega_{xb} &= \sum_y \omega_{xy} + \sum_t \omega_{xt} = \sum_y (\mu_x x^y + \sum_t \nu_t t^x t^y) - \sum_t \nu_t t^x \\ &= \mu_x + \sum_t \nu_t t^x \sum_y t^y - \sum_t \nu_t t^x = \mu_x \ . \end{aligned}$$

Second, the stress condition for vertex x is true because

$$\begin{aligned} \sum_b \omega_{xb} b &= \sum_y \omega_{xy} y + \sum_t \omega_{xt} t = \sum_y (\mu_x x^y + \sum_t \nu_t t^x t^y) y - \sum_t \nu_t t^x t \\ &= \mu_x x + \sum_t \nu_t t^x \sum_y t^y y - \sum_t \nu_t t^x t = \mu_x x = \rho_x x \ . \end{aligned}$$

(ii) The vertices $s \in A_2$. We have

$$\rho_s = \sum_b \omega_{sb} = \sum_y \omega_{sy} + \sum_t \omega_{st} = \sum_y \mu_s s^y + 0 = \mu_s \ ,$$

as desired. Moreover, the stress condition is true since

$$\sum_b \omega_{sb} b = \sum_y \omega_{sy} y + \sum_t \omega_{st} t = \sum_y \mu_s s^y y = \mu_s s = \rho_s s \ .$$

(iii) The argument for the vertices $t \in B_2$ is similar to the one above for $s \in A_2$.

(iv) The vertices $z \in B_1$. Now the fun starts, for here we use the fact that $\lambda \in D^2(A, B)$ to link the sets A_1, A_2, B_1, B_2. The vertex in B_1 is called z rather than y because in the course of the argument we will be summing over $y \in B_1$.

The z column sum of ω is

$$\begin{aligned}\gamma_z = \sum_a \omega_{az} = \sum_x \omega_{xz} + \sum_s \omega_{sz} &= \sum_x (\mu_x x^z + \sum_t \nu_t t^x t^z) + \sum_s \mu_s s^z \\ &= \sum_x \mu_x x^z + \sum_s \mu_s s^z + \sum_t \nu_t t^z \;. \end{aligned} \tag{9}$$

Now we use the fact that $\lambda = (\cdots, \mu_a, \cdots, \nu_b, \cdots)$ belongs to $D^2(A, B) \subset D(A, B)$. Then Lemma 4 implies

$$\sum_x \mu_x \underline{x} + \sum_s \mu_s \underline{s} + \sum_y \nu_y \underline{y} + \sum_t \nu_t \underline{t} = 0 \;. \tag{10}$$

Use Equation (8′) to express each of the vectors $\underline{x}$, $\underline{s}$, $\underline{t}$ in Equation (10) in terms of $\underline{y}$, $y \in B_1$. That yields

$$\sum_y (\sum_x \mu_x x^y + \sum_s \mu_s s^y + \nu_y + \sum_t \nu_t t^y)\underline{y} = 0 \;. \tag{11}$$

Since the vectors $\underline{y}$, $y \in B_1$, are linearly independent, every coefficient in Equation (11) vanishes. In particular,

$$\sum_x \mu_x x^z + \sum_s \mu_s s^z + \nu_z + \sum_t \nu_t t^z = 0 \;. \tag{12}$$

Combining Equations (9) and (12) leads to $\gamma_z = -\nu_z$, the desired conclusion.

We follow the same pattern to verify the stress condition for vertex z. We have

$$\begin{aligned}\sum_a \omega_{az} a = \sum_x \omega_{xz} x + \sum_s \omega_{sz} s &= \sum_x (\mu_x x^z + \sum_t \nu_t t^x t^z) x + \sum_s \mu_s s^z s \\ &= \sum_x \mu_x x^z (\sum_y x^y y) + \sum_s \mu_s s^z (\sum_y s^y y) + \sum_t \nu_t t^z t \\ &= \sum_y (\sum_x \mu_x x^z x^y + \sum_s \mu_s s^z s^y) y + \sum_t \nu_t t^z t \;. \end{aligned} \tag{13}$$

Now we use the fact that $\lambda \in D^2(A, B)$. Lemma 4 implies

$$\sum_x \mu_x \underline{x} \otimes \underline{x} + \sum_s \mu_s \underline{s} \otimes \underline{s} + \sum_y \nu_y \underline{y} \otimes \underline{y} + \sum_t \nu_t \underline{t} \otimes \underline{t} = 0 \;. \tag{14}$$

Use Equation (8′) to express each of the vectors $\underline{x}$, $\underline{s}$, $\underline{t}$ in Equation (14) in terms of $\underline{y}$, $y \in B_1$. Expanding and collecting terms, one obtains

$$\sum_{y, y' \in B_1} (\sum_x \mu_x x^y x^{y'} + \sum_s \mu_s s^y s^{y'} + \nu_y \delta(y, y') + \sum_t \nu_t t^y t^{y'}) \underline{y} \otimes \underline{y}' = 0 \;, \tag{15}$$

where $\delta(y, y')$ is the Kronecker delta. But the linear independence of the vectors $\underline{y}$, $y \in B_1$, implies the linear independence of the vectors $\underline{y} \otimes \underline{y}'$, y, $y' \in B_1$, so every coefficient in Equation (15) vanishes. In particular, for all $y \in B_1$ we have

$$\sum_x \mu_x x^y x^z + \sum_s \mu_s s^y s^z + \nu_y \delta(y, z) + \sum_t \nu_t t^y t^z = 0 . \tag{16}$$

Combining Equations (13) and (16) gives

$$\begin{aligned}\sum_a \omega_{az} a &= \sum_y (-\nu_y \delta(y, z) - \sum_t \nu_t t^y t^z) y + \sum_t \nu_t t^z t \\ &= -\nu_z z - \sum_t \nu_t t^z t + \sum_t \nu_t t^z t = -\nu_z z = \gamma_z z ,\end{aligned}$$

the desired conclusion.

We now know, from Theorems 1 and 6, that

$$\dim \Omega(A, B) = \dim D(A) \dim D(B) + \dim D^2(C) .$$

Our next task is to discover the geometrical meaning of $D^2(C)$.

3. Quadrics. We now show that the space $D^2(C)$ of simultaneous affine dependencies of $C = (c_1, \cdots, c_k) \in \boldsymbol{R}^{kd}$ and $(c_1 \otimes c_1, \cdots, c_k \otimes c_k)$ can be conveniently described in terms of quadric polynomials which vanish on C. *A quadric polynomial* is simply a polynomial

$$q(\xi_1, \cdots, \xi_d) = \sum_{1 \leqq i \leqq j \leqq d} \sigma_{ij} \xi_i \xi_j + \sum_{1 \leqq i \leqq d} \sigma_i \xi_i + \sigma$$

and a *quadric surface* is the set of zeros of a nontrivial quadric polynomial. A quadric polynomial is determined by its $d(d + 3)/2 + 1$ coefficients σ_{ij}, σ_i, σ and the set of quadric polynomials forms a vector space in the obvious way.

What should we mean by the space $Q(C)$ of quadric polynomials which vanish on C? If the affine span $\bar{C}$ of C is all of $\boldsymbol{R}^d$ the answer is clear, but we must take some care when $\bar{C} \neq \boldsymbol{R}^d$. Then we want our polynomials to be defined on $\boldsymbol{R}^h$, where $h = \dim \bar{C}$. If $\bar{C}$ is a singleton, define $\dim Q(C) = \dim \bar{C} = 0$. Now suppose $h \geqq 1$. Choose an affine transformation $T: \boldsymbol{R}^d \to \boldsymbol{R}^h$ which maps $\bar{C}$ onto $\boldsymbol{R}^h$. Let Q be the $h(h + 3)/2 + 1$ dimensional vector space of quadric polynomials on $\boldsymbol{R}^h$ and define

$$Q(C) = \{q \in Q : q(Tc) = 0 \quad \text{for all} \quad c \in C\} .$$

In this definition $Q(C)$ depends on the choice of T, but we will soon seen that its dimension is independent of T. Then we regard any one of these isomorphic subspaces $Q(C)$ of Q as the space of quadric polynomials which vanish on $C = (c_1, \cdots, c_k)$.

LEMMA 7. *Let $Tc = (\xi_1, \cdots, \xi_h) \in \boldsymbol{R}^h$ for $c \in C$. Then $Q(C)$ is the space of linear dependencies among the rows of the $(h(h + 3)/2 + 1) \times k$ matrix $M(T)$ whose* cth *column is the vector*

$$(\xi_1^2, \cdots, \xi_h^2, \xi_1\xi_2, \cdots, \xi_{h-1}\xi_h, \xi_1, \cdots, \xi_h, 1) .$$

Proof. Just examine the meaning of $q(Tc) = 0$.

It follows from Lemma 7 that

$$\dim Q(C) \geqq \max\left\{0, \ \frac{1}{2}h(h+3) + 1 - k\right\} . \tag{17}$$

Generically, that is, for a dense open subset of points $C = (c_1, \cdots, c_k) \in \boldsymbol{R}^{kd}$, we have both $h = \min\{k-1, d\}$ and equality in (17). For example, the space of quadric polynomials which vanish on ten points in $\boldsymbol{R}^3$ is generally trivial, while there is usually a unique quadric surface through nine points in $\boldsymbol{R}^3$.

LEMMA 8. *Let* $TC = (\cdots, Tc, \cdots)$ *for* $C = (\cdots, c, \cdots)$. *Then* $D^2(C) = D^2(TC)$.

Proof. Because T is affine it is the sum of a linear map L: $\boldsymbol{R}^d \to \boldsymbol{R}^h$ and a constant vector $\xi \in \boldsymbol{R}^h$. Suppose $\lambda = (\cdots, \lambda_c, \cdots) \in D^2(C)$. Then $\sum_c \lambda_c = 0$, $\sum_c \lambda_c c = 0$, and $\sum_c \lambda_c c \otimes c = 0$. Thus

$$\sum_c \lambda_c Tc = L(\sum_c \lambda_c c) + (\sum_c \lambda_c)\xi = 0$$

and, by the bilinearity of $\otimes$,

$$\begin{aligned}\sum_c \lambda_c Tc \otimes Tc &= \sum_c \lambda_c (Lc + \xi) \otimes (Lc + \xi) \\ &= L(\sum_c \lambda_c c \otimes c)L^t + L(\sum_c \lambda_c c) \otimes \xi + \xi \otimes L(\sum_c \lambda_c c) \\ &\quad + (\sum_c \lambda_c)\xi \otimes \xi = 0 .\end{aligned}$$

Therefore $\lambda \in D^2(TC)$. The reverse inclusion follows from a similar argument since there exists an affine map S: $\boldsymbol{R}^h \to \boldsymbol{R}^d$ with $STc = c$ for all $c \in C$.

THEOREM 9. *Let* $C = (c_1, \cdots, c_k) \in \boldsymbol{R}^{kd}$ *and* $h = \dim \bar{C}$. *Then*

$$\dim D^2(C) = \dim Q(C) + k - \frac{1}{2}h(h+3) - 1 .$$

Proof. If $\bar{C}$ is a singleton then the k vertices in C coincide so $D^2(C) = \{\lambda \in \boldsymbol{R}^k \cdot \sum_{i=1}^k \lambda_i = 0\}$. Therefore $\dim D^2(C) = k - 1$. Since $h = 0$ and $\dim Q(C) = 0$ by definition, the theorem is true in this case.

Suppose $h \geqq 1$ and let T: $\boldsymbol{R}^d \to \boldsymbol{R}^h$ be an affine map which takes

$\bar{C}$ onto $\boldsymbol{R}^h$. Then $\lambda \in D^2(C)$ if and only if $\lambda \in D^2(TC)$ (Lemma 8) if and only if λ is a linear dependency among the vectors $\underline{Tc} \otimes \underline{Tc}$, $c \in C$ (Lemma 4) if and only if λ is orthogonal to the rows of $M(T)$. Regarding $M(T)$ as a linear map on $\boldsymbol{R}^k$, we have $D^2(C) = \ker M(T)$. Hence

$$\operatorname{rank} M(T) = k - \dim D^2(C) \,, \tag{18}$$

which shows that $\operatorname{rank} M(T)$ is independent of T. Now Lemma 7 implies

$$\dim Q(C) = \frac{1}{2} h(h + 3) + 1 - \operatorname{rank} M(T) \,, \tag{19}$$

which shows that $\dim Q(C)$ is also independent of T. Combining Equations (18) and (19) completes the proof.

Similar results relating dependencies among tensor-squares of vectors to the vectors lying on common quadric surfaces appear in Baclawski and White [3].

Theorems 1, 6, and 9 together give our main result, the dimension of the space of stresses of $K(A, B)$.

THEOREM 10. *Let* $C = (A \cap \bar{B}, B \cap \bar{A}) = (c_1, \cdots, c_k) \in \boldsymbol{R}^{kd}$ *and* $h = \dim \bar{C}$. *Then the dimension of* $\Omega(A, B)$ *is*

$$\dim D(A) \dim D(B) + \dim Q(C) + k - h(h + 3)/2 - 1 \,.$$

4. **Rigidity and infinitesimal rigidity.** We now examine the implications of Theorem 10 for the rigidity and infinitesimal rigidity of realizations of $K_{m,n}$ in $\boldsymbol{R}^d$. The definitions are complete but the discussion is brief. More detail may be found in Asimow and Roth [1, 2], Crapo [4], and Roth [5]. Let

$$R = \{(A, B) = (a_1, \cdots, a_m, b_1, \cdots, b_n) \colon a_i \,, \quad b_j \in \boldsymbol{R}^d\} \,,$$

the manifold of all realizations of $K_{m,n}$ in $\boldsymbol{R}^d$. The *edge function* of $K_{m,n}$ is the function $f \colon R \to \boldsymbol{R}^{mn}$ defined by

$$f(A, B) = (\cdots, \|a_i - b_j\|^2, \cdots) \,, \quad 1 \leqq i \leqq m \,, \quad 1 \leqq j \leqq n \,.$$

The coordinates of $f(A, B)$ are the squares of the lengths of the mn edges of $K(A, B)$. Thus $f^{-1}(f(A, B))$ is the set of realizations $(A', B') \in R$ such that $K(A, B)$ and $K(A', B')$ have corresponding edge lengths equal. The derivative $df(A, B)$ of f at (A, B) has $(m + n)d$ columns, each d-tuple of which consists of the d partial derivaties with respect to the coordinates of a vertex of $K(A, B)$, and mn rows, each arising from an edge of $K(A, B)$. The row coming from the edge $[a_i, b_j]$ has only two d-tuples of nonzero entries. These are $2(a_i - b_j)$ in the d-tuple of columns corresponding to vertex a_i and $2(b_j - a_i)$ in the d-tuple of columns corresponding to b_j. Thus the $mn \times (m+n)d$ matrix $df(A, B)/2$ has the form

$$\begin{array}{c|c} & \begin{array}{ccccccccc} a_1 & \cdot & \cdot & \cdot & a_m & b_1 & \cdot & \cdot & b_n \end{array} \\ \begin{array}{c} [a_1, b_1] \\ \vdots \\ [a_1, b_n] \\ \vdots \\ [a_m, b_n] \end{array} & \begin{bmatrix} a_1 - b_1 & 0 & \cdots & 0 & b_1 - a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_1 - b_n & 0 & & 0 & 0 & & 0 & b_n - a_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_m - b_n & 0 & & 0 & b_n - a_m \end{bmatrix} \end{array}.$$

Equations (2, 2′) then say that a stress of the framework $K(A, B)$ is just a linear dependency among the rows of $df(A, B)$.

To see the connection with infinitesimal rigidity regard $df(A, B)$ as a linear map from $\boldsymbol{R}^{(m+n)d}$ to $\boldsymbol{R}^{mn}$. A vector $(u, v) = (u_1, \cdots, u_m, v_1, \cdots, v_n) \in \boldsymbol{R}^{(m+n)d}$ belongs to $\ker df(A, B)$ if and only if we have

$$(a_i - b_j)\cdot(u_i - v_j) = 0\ , \quad 1 \leqq i \leqq m\ , \quad 1 \leqq j \leqq n\ . \tag{20}$$

We call the vectors in $\ker df(A, B)$ the *infinitesimal motions* of $K(A, B)$. To see why the name is appropriate let $G = (G_1, \cdots, G_m)$ be a smooth map from $\boldsymbol{R}$ to $\boldsymbol{R}^{md}$ and $H = (H_1, \cdots, H_n)$ a smooth map from $\boldsymbol{R}$ to $\boldsymbol{R}^{nd}$. Suppose $G(0) = A$, $H(0) = B$ and set $u = G'(0)$, $v = H'(0)$. Then Equation (20) is satisfied provided the derivative of $||G_i(t) - H_j(t)||^2$ vanishes at $t = 0$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Moreover, every element of $\ker df(A, B)$ is the derivative at $t = 0$ of a smooth map $(G, H)\colon \boldsymbol{R} \to \boldsymbol{R}^{(m+n)d}$ beginning at (A, B) which instantaneously preserves the squares of edge lengths.

There are some infinitesimal motions which result from Euclidean motions. We say that $(A, B), (A', B') \in R$ are *congruent* if there exists a rigid motion S of $\boldsymbol{R}^d$ such that $Sa_i = a'_i$, $1 \leqq i \leqq m$, and $Sb_j = b'_j$, $1 \leqq j \leqq n$. Let $T(A, B)$ be the tangent space at (A, B) to the submanifold of R consisting of those points congruent to (A, B). Then it is not hard to show that

$$\dim T(A, B) \leqq \frac{1}{2} d(d+1) \tag{21}$$

with equality in (21) whenever the affine span of (A, B) is $\boldsymbol{R}^d$. Finally, it is easy to verify that $T(A, B) \subset \ker df(A, B)$.

We say that the framework $K(A, B)$ is *infinitesimally rigid* when $T(A, B) = \ker df(A, B)$, which says that all infinitesimal motions of $K(A, B)$ arise from rigid motions, and *infinitesimally flexible* otherwise. Since stresses of $K(A, B)$ are dependencies among the rows of $df(A, B)$ while infinitesimal motions of $K(A, B)$ are elements of the kernel of $df(A, B)$, we have

$$mn - \dim \Omega = \operatorname{rank} df = (m + n)d - \dim \ker df .$$

Therefore

$$\dim \ker df(A, B) = \dim \Omega(A, B) + (m + n)d - mn . \tag{22}$$

(This same device was used in Theorem 9 to relate $D^2(C)$ and $Q(C)$. In that case the matrix was $M(T)$.)

We now turn to the connection between infinitesimal rigidity and rigidity. A framework $K(A, B)$ is *flexible* if there exists a continuous map (G, H): $[0, 1] \to \boldsymbol{R}^{(m+n)d}$ such that $(G(0), H(0)) = (A, B)$, $(G(t), H(t)) \in f^{-1}(f(A, B))$ for all $t \in [0, 1]$, but $(G(t), H(t))$ is not congruent to (A, B) for some $t \in (0, 1]$. That says precisely that $K(A, B)$ is flexible if its vertices can be continuously moved in such a way that edge lengths are preserved but the distance between some pair of vertices changes. When $K(A, B)$ is not flexible we say it is *rigid*. Let

$$r = \max \{\operatorname{rank} df(A, B) \colon (A, B) \in R\} ,$$

the largest possible value of the rank of the derivative of the edge function. We say that $(A, B) \in R$ is a *regular point* of f if rank $df(A, B) = r$. At regular points infinitesimal flexibility and flexibility are equivalent, as are infinitesimal rigidity and rigidity ([1, Theorem] and [2, § 3]). The regular points of f form a dense open subset of R, for which we always have infinitesimal rigidity or always have infinitesimal flexibility ([1, Corollary 2] and [2, §3]). We now investigate which alternative—generic rigidity or generic flexibility—occurs for $K_{m,n}$ in $\boldsymbol{R}^d$.

Workers studying rigidity always begin with the naive hope that the generic rigidity of realizations of a graph with e edges and v vertices is equivalent to the inequality

$$e \geqq vd - \frac{1}{2} d(d + 1) \tag{23}$$

which for $K_{m,n}$ is

$$mn \geqq (m + n)d - \frac{1}{2} d(d + 1) .$$

We shall see that this hope is justified when $d = 2$ or 3 but vain for $d \geqq 4$.

THEOREM 11. *If $mn < (m + n)d - d(d + 1)/2$, then all realiza-*

tions of $K_{m,n}$ *in* $\boldsymbol{R}^d$ *are infinitesimally flexible.*

Proof. Inequality (21), Equation (22) and the hypothesis imply

$$\dim T(A, B) \leqq \frac{1}{2}d(d+1) < (m+n)d - mn \leqq \dim \ker df(A, B)$$

so $T(A, B) \neq \ker df(A, B)$ for all $(A, B) \in R$. Thus $K_{m,n}$ is always infinitesimally flexible and therefore generically flexible.

In $\boldsymbol{R}^3$ the hypothesis of Theorem 11 is satisfied except for $m = n = 1$, $m = 4$ and $n \geqq 6$, or $m, n \geqq 5$. Realizations of $K_{1,1}$ are always rigid, so the naive hope will be borne out when we prove the following theorem.

THEOREM 12. $K_{4,n}$, $n \geqq 6$, *and* $K_{m,n}$, m, $n \geqq 5$, *are generically rigid in* $\boldsymbol{R}^3$.

Proof. Consider first $K_{4,n}$ for $n \geqq 6$. Let U be the dense open set of realizations $(A, B) \in R$ satisfying $\bar{A} = \bar{B} = \boldsymbol{R}^3$ (so $C = (A, B)$) and $\dim Q(C) = 0$. For such realizations $D(A) = \{0\}$, $k = n + 4$ and $h = \dim \bar{C} = 3$ so Theorem 10 says $\dim \Omega(A, B) = n - 6$ and hence, from Equation (22),

$$\dim \ker df(A, B) = n - 6 + 3(n + 4) - 4n = 6 .$$

But for $(A, B) \in U$, $\dim T(A, B) = 6$ and thus $K(A, B)$ is infinitesimally rigid. Since U and the set of regular points have nonempty intersection (in fact, every point of U is regular), we conclude that $K(A, B)$ is infinitesimally rigid at all regular points. Hence $K_{4,n}$ is generically rigid in $\boldsymbol{R}^3$ for $n \geqq 6$.

A similar argument works for $K_{m,n}$ when m, $n \geqq 5$. We let U consist of those realizations (A, B) for which $\bar{A} = \bar{B} = \boldsymbol{R}^3$ (so $C = (A, B)$) and $\dim Q(C) = 0$. Then $\dim D(A) = m - 4$, $\dim D(B) = n - 4$, $k = m + n$ and $h = 3$. It follows as above that $\dim \ker df(A, B) = 6 = \dim T(A, B)$ for all $(A, B) \in U$. Thus $K_{m,n}$ is generically rigid in $\boldsymbol{R}^3$ provided m, $n \geqq 5$.

Similar results hold in the plane. That is, the generic classification of $K_{m,n}$ in $\boldsymbol{R}^2$ is given by a comparison of $e = mn$ and $2v - 3 = 2(m + n) - 3$. (The proof, which is easier than that of Theorem 12, is left to reader. Theorem 14 in the next section contains part of the proof). However, the analog for $d = 4$ is false.

EXAMPLE 13. Consider $K_{6,7}$ in $\boldsymbol{R}^4$. Then $e = 42 = 4v - 10$ so inequality (23) is satisfied and we might expect $K_{6,7}$ to be generically rigid in $\boldsymbol{R}^4$. However, let U be the dense open set of realizations $(A, B) \in R$ with $\bar{A} = \bar{B} = \boldsymbol{R}^4$ (so $C = (A, B)$) and $\dim Q(C) = 2$. (See the discussion following inequality (17)). For these realizations $\dim D(A) = 1$, $\dim D(B) = 2$, $k = 13$ and $h = 4$. Theorem 10 then implies $\dim \Omega(A, B) = 2$. Hence, from Equation (22), $\dim \ker df(A, B) = 12$. But $\dim T(A, B) = 10$ and thus $K(A, B)$ is infinitesimally flexible for all $(A, B) \in U$. Since elements of U are regular points, $K(A, B)$ is infinitesimally flexible at all regular points. Therefore $K_{6,7}$ is generically flexible in $\boldsymbol{R}^4$.

5. Special realizations. In this section we apply Theorem 10 to some nongeneric realizations and discuss several examples. We begin with the classical theorem on $K_{3,3}$ in the plane and its generalization.

THEOREM 14. *A realization of $K_{3,3}$ in $\boldsymbol{R}^2$ is infinitesimally rigid unless its six vertices lie on a conic.*

Proof. Consider first the case in which $\bar{A} = \bar{B} = \boldsymbol{R}^2$. Then $D(A) = D(B) = \{0\}$, $C = (A, B)$, $k = 6$ and $h = 2$. Theorem 10 implies $\dim \Omega(A, B) = \dim Q(C)$ and thus, from Equation (22),

$$\dim \ker df(A, B) = \dim Q(C) + 3 .$$

But $\dim T(A, B) = 3$ for such realizations so $K(A, B)$ is infinitesimally flexible if and only if the six vertices (A, B) lie on a conic. By analyzing special cases and using the fact that the union of two lines is a conic one can show that the same result holds without the restriction $\bar{A} = \bar{B} = \boldsymbol{R}^2$.

The natural generalization of Theorem 14 to $\boldsymbol{R}^d$, $d > 2$, is only partly true.

THEOREM 15. *A realization (A, B) of $K_{d+1,d(d+1)/2}$ in $\boldsymbol{R}^d$ for which $\bar{A} = \bar{B} = \boldsymbol{R}^d$ is infinitesimally rigid unless its $d(d+3)/2 + 1$ vertices lie on a quadric surface.*

Proof. Our hypotheses imply $D(A) = \{0\}$, $C = (A, B)$, $k = d(d+3)/2 + 1$ and $h = d$. Thus $\dim \Omega(A, B) = \dim Q(C)$ by Theorem 10, and hence, from Equation (22),

$$\dim \ker df = \dim\ Q(C) + \frac{1}{2}d(d+1) = \dim Q(C) + \dim\ T(A, B) .$$

Therefore $K(A, B)$ is infinitesimally rigid unless $\dim Q(C) > 0$, that is, unless its vertices lie on a quadric surface.

Theorem 15 when $d = 2$ is somewhat weaker than Theorem 14. Our next example shows that it is as strong as possible for arbitrary d. The requirement that $\bar{A} = \bar{B} = \boldsymbol{R}^d$ cannot be dropped for $d \geqq 3$. Moreover, the fact that the infinitesimal rigidity of $K_{3,3}$ in $\boldsymbol{R}^2$ is independent of the partition of the six vertices into the sets A and B does not generalize.

EXAMPLE 16. Let u_1, u_2, u_3 be the three coordinate vectors in $\boldsymbol{R}^3$. Consider the realization (A, B) of $K_{4,6}$ in $\boldsymbol{R}^3$ where $A = (\pm u_1, \pm u_2)$ and $B = (0, \pm u_3, u_1 + u_2, u_1 + u_3, u_2 + u_3)$. Then $\dim \bar{A} = 2$, $\bar{B} = \boldsymbol{R}^3$, $\dim D(A) = 1$ and $\dim D(B) = 2$. In this case $C = (A, 0, u_1 + u_2)$ and $Q(C) = \{0\}$. Moreover, $Q(A, B) = \{0\}$ too, so the ten vertices lie on no quadric surface. Nevertheless, $\dim \Omega(A, B) = 2$ so

$$\dim \ker df(A, B) = 8 > 6 = \dim T(A, B) .$$

Thus $K(A, B)$ is infinitesimally flexible even though its vertices do not lie on a quadric surface in $\boldsymbol{R}^3$. If instead we let $A = (0, u_1, u_2, u_3)$ and B be the remaining six of the ten vertices then Theorem 15 applies and $K(A, B)$ is infinitesimally rigid.

Since this paper grew out of our investigation of the generic rigidity of $K_{m,m}$ in $\boldsymbol{R}^3$, it seems fitting to conclude with a look at some realizations of this graph when $m = 5$ and 4.

EXAMPLE 17. Consider $K_{5,5}$ in $\boldsymbol{R}^3$. If $\bar{A} = \bar{B} = \boldsymbol{R}^3$ then $\dim D(A) = \dim D(B) = 1$, $C = (A, B)$, $k = 10$ and $h = 3$ so $\dim \Omega(A, B) = \dim Q(C) + 1$. Thus generically $K_{5,5}$ has a one dimensional stress space and is rigid. Locating the ten vertices on a quadric surface will increase the dimension of the stress space. We can also create additional stresses by reducing the affine span of A or B. If, for instance, $A = (0, \pm u_1, \pm u_2)$ and $B = (\pm u_3, u_1 + u_2, u_1 + u_3, u_2 + u_3)$ then $\dim D(A) = 2$, $\dim D(B) = 1$, $C = (A, u_1 + u_2)$, $Q(C) = \{0\}$, $k = 6$ and $h = 2$. Then $\dim \Omega(A, B) = 2$. We can even make both kinds of singularities occur simultaneously. With A as above let $B = (\pm u_3, u_1 + u_3, -u_1 + u_3, 2u_2)$. Then $\dim D(A) = 2$, $\dim D(B) = 1$, $C = (A, 2u_2)$, $\dim Q(C) = 1$, $k = 6$ and $h = 2$, so $\dim \Omega(A, B) = 3$.

Finally, the infinitesimal rigidity of various tensegrity (or cabled) frameworks follows from the results of this paper. For example, if (A, B) is a realization of $K_{5,5}$ in $\boldsymbol{R}^3$ with $\bar{A} = \bar{B} = \boldsymbol{R}^3$ and $Q(C) = \{0\}$ then its stress space $\Omega(A, B)$ is one dimensional and the framework $K(A, B)$ is infinitesimally rigid. In fact, $\Omega(A, B) =$

$D(A) \otimes D(B)$ is spanned by $\omega = \alpha \otimes \beta$ where $\alpha \in D(A)$, $\beta \in D(B)$ and $\alpha, \beta \neq 0$. Replacing all the edges $[a_i, b_j]$ of $K(A, B)$ with $\omega_{ij} = \alpha_i \beta_j < 0$ by cables gives an infinitesimally rigid tensegrity structure. (See [4, § 7] or [6, Theorem 5.2] for details.)

Our last example deals with two models which we have actually built. We can clearly see how they behave but understanding that behavior mathematically is quite another matter. The reader is invited to join us in our efforts to develop an applicable theory.

EXAMPLE 18. Consider $K_{4,4}$ in $\boldsymbol{R}^3$. Theorem 11 shows that in any realization it has too few edges to be infinitesimally rigid. Suppose $A = (\pm u_1, \pm u_2)$ and that $\bar{B}$ is a plane parallel to the plane $\bar{A}$. Then $A \cap \bar{B} = B \cap \bar{A} = \phi$ so $C = \phi$ and $\Omega = D(A) \otimes D(B)$ is one dimensional. Using the signs of the stress coefficients gives a way to build $K(A, B)$ as a tensegrity framework. Since

$$\dim \ker df(A, B) - \dim T(A, B) = 1 + 24 - 16 - 6 = 3 ,$$

when $K(A, B)$ is built entirely of rods it will have three independent infinitesimal degrees of freedom which do not correspond to Euclidean motions. Replacing some rods by cables may increase that number. Suppose now that B is A raised by one unit and rotated through $\pi/2$: $B = (u_3 + (\pm u_1 \pm u_2)/\sqrt{2})$. Then $K(A, B)$ has edges of two lengths. When we built the model we discovered that the long edges meet at interior points. We had to choose which rods would pass over, which under at those intersections. Figures 1 and 2 are top views of two possible sets of choices. We built Figure 1 as a tensegrity framework. Only one of its infinitesimal degrees of freedom is palpable. Where the long rods pass the long

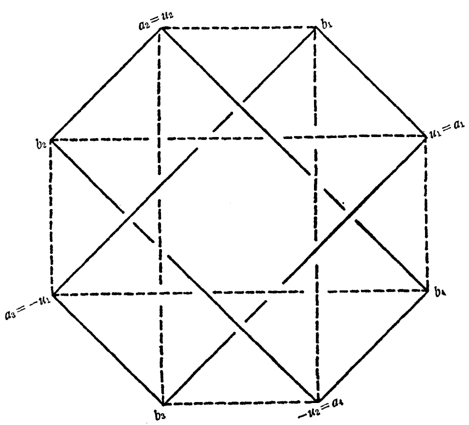

FIGURE 1. Top view of a realization of $K_{4,4}$ in $\boldsymbol{R}^3$ as a tensegrity mechanism. Vertices a_i are in the plane of the paper, vertices b_j in a parallel plane above it. Rods are solid, cables dashed.

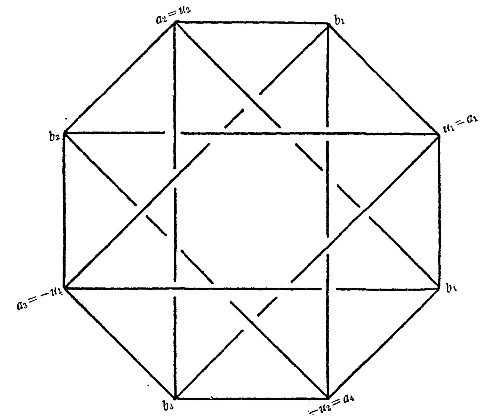

FIGURE 2. Top view of a realization of $K_{4,4}$ in $\boldsymbol{R}^3$ as a framework with two independent degrees of freedom. Vertices a_i are in the plane of the paper. Vertices b_j start out in a parallel plane above it, but do not remain coplanar as the mechanism flexes.

cables they rub. We think it is that rubbing which constrains $\bar{A}$ and $\bar{B}$ to be parallel planes, whose distance varies as the model flexes. That constraint keeps $\dim \Omega = 1$ throughout the motion and allows a tensegrity mechanism: the cables stay taut as the model flexes.

Building Figure 2 as a tensegrity framework would give an enormous number of actual degrees of freedom (perhaps ten?). When we built it entirely of rods we found just two actual degrees of freedom. Exercising them independently destroyed the coplanarity of B, but not of A. Then $B \cap \bar{A} = \phi$, $A \cap \bar{B} = A = C$, $h = \dim Q(C) = 2$, $k = 4$ and

$$\dim \Omega = 0 + 2 + 4 - 6 = 0 .$$

Such points (A, B) are regular points of f so $K(A, B)$ has precisely two infinitesimal (or equivalently actual) degrees of freedom once the coplanarity of B is lost.

REFERENCES

1. L. Asimow and B. Roth, *The rigidity of graphs*, Trans. Amer. Math. Soc., **245** (1978), 279-289.

2. ———, *The rigidity of graphs II*, J. Math. Anal. Appl., **68** (1979), 171-190.

3. K. Baclawski and N. White, *Higher order independence in matroids*, J. London Math. Soc., (2), **19** (1979), 193-202.

4. H. Crapo, *Structural rigidity*, Structural Topology, **1** (1979), 26-45.

5. B. Roth, *Rigid and flexible frameworks*, Amer. Math. Monthly, to appear.

6. B. Roth and W. Whiteley, *Tensegrity frameworks*, Trans. Amer. Math. to appear.

Received August 28, 1979. The authors are grateful to the University of Washington for its hospitality during our visits there.

University of Massachusetts
Boston, MA 02125
and
University of Wyoming
Laramie, WY 82071