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BOUNDARY VALUE PROBLEMS FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Sufficient conditions are given to ensure the existence of solutions for the boundary value problem

$$(1) \qquad y(t) = T(t)\phi(0) + \int_0^t T(t-s)F(y_s)ds \quad 0 \leqq t \leqq b$$

$$(*) \qquad My_0 + Ny_b = \psi\ , \quad \psi \in C(=C([-r, 0]; B) \text{ by def.})\ .$$

It is assumed that $T(t)$, $t \geqq 0$, is a strongly continuous semigroup of bounded linear operators on the Banach space B and $T(t)$, $t \geqq 0$, has infinitesimal generator A. The function F is continuous from C to B and M and N are bounded linear operators defined on C.

Denote by C the Banach space of continuous functions from $[-r, 0]$ into the Banach space B, where for each $\varphi \in C$, $||\varphi||_C = \sup_{-r \leqq \theta \leqq 0} \sup ||\varphi(\theta)||$. Let A be the infinitesimal generator of a strongly continuous semigroup of linear operators $T(t)$, $t \geqq 0$ mapping B into B and satisfying $|T(t)| \leqq e^{\omega t}$ for some real ω. We let F be a nonlinear continuous function from C into B. If $y(t)$ is a continuous function from $[0, T]$ to B for some $T > 0$, define the element $y_t \in C$ by $y_t(\theta) = y(t + \theta)$. Throughout this paper the reference $y(t)$ is a solution of Equation (1) (*) will mean $y(t)$ satisfies Equation (1) and the boundary condition (*). The statement $y(\varphi)(t)$ is a solution of Equation (1) will mean $y(t)$ satisfies Equation (1) and the initial condition $y_0 = \varphi$. The notation Equation (1) without (*) will always denote the initial value problem.

In a recent paper [8] C. Travis and G. Webb have considered initial value problems for Equation (1). With F satisfying

$$(2) \qquad ||F(\varphi) - F(\bar{\varphi})|| \leqq L||\varphi - \bar{\varphi}||_C$$

for some $L > 0$ and φ, $\bar{\varphi} \in C$, Travis and Webb obtain the existence of unique solutions of Equation (1) for each $\varphi \in C$. In another paper W. E. Fitzgibbon [2] has shown that global solutions of Equation (1) exist if F satisfies for each $\varphi \in C$

$$(3) \qquad ||F(\varphi)|| \leqq K_1||\varphi||_C + K_2 \quad \text{for some} \quad K_1, K_2 \in R\ ,$$

and if $T(t)$, $t > 0$ is compact.

When Equation (1) has unique solutions for each $\varphi \in C$, the mapping $U(t)\varphi = y_t(\varphi)$ is well defined for each $t \geqq 0$ and $\varphi \in C$. Here $y_t(\varphi)$ represents the element of C such that $y(\varphi)(t)$ is a solution of

Equation (1). If F satisfies (2) the following estimate from [8] is true:

$$\|U(t)\varphi - U(t)\bar{\varphi}\|_C \leqq e^{(\omega+L)t}\|\varphi - \bar{\varphi}\|_C \quad \text{if} \quad \omega \geqq 0 \tag{4}$$

for all $t \geqq 0$. Throughout this paper it will be assumed that $\omega \geqq 0$.

If F satisfies (3), then we have for each $\varphi \in C$ and $0 \leqq t \leqq b$

$$\|U(t)\varphi\|_C = \|y_t(\varphi)\|_C = \sup_{-r\leqq\theta\leqq 0}\left\|T(t+\theta)\varphi(0) + \int_0^{t+\theta} T(t+\theta-s)F(y_s)ds\right\|$$
$$\leqq e^{\omega t}\|\varphi\|_C + e^{\omega t}\int_0^t e^{-\omega s}K_1\|y_s(\varphi)\|_C + K_2 ds .$$

This implies that

$$\|y_t(\varphi)\|_C \leqq \bar{K}_1\|\varphi\|_C + \bar{K}_2 \tag{5}$$

where $\bar{K}_1 = e^{(\omega+K_1)b}$ and $\bar{K}_2 = e^{(\omega+K_1)b}K_2 b$.

It is shown in [8] that if the semigroup $T(t)$, $t \geqq 0$ is compact for $t > 0$, then the solution mapping $U(t)\varphi = y_t(\varphi)$ is compact in φ for each fixed $t > r$.

Equation (1) is the integrated form of the functional differential equation

$$\begin{aligned} &y'(t) = Ay(t) + F(y_t) \quad 0 \leqq t \leqq b \\ &y_0 = \varphi . \end{aligned} \tag{6}$$

Our results then can be applied to partial functional differential equations of the form

$$\begin{aligned} &v(_t x, t) = v_{xx}(x, t) + f(v(x, t-r)) && 0 \leqq t \leqq b,\ 0 \leqq x \leqq l \\ &v(0, t) = v(l, t) = 0 && t \geqq 0 \\ &\alpha(x, t)v(x, t) + \beta(x, t)v(x, b+t) = \psi(x, t) && -r \leqq t \leqq 0,\ 0 \leqq x \leqq l . \end{aligned}$$

Boundary value problems of the type Equation (6) (*) have been studied recently by R. Fennell and P. Waltman [1], G. Reddien and G. Webb [7] and P. Waltman and J. S. W. Wong [9] when $B = R^n$. The work here extends results found in [7] and [9] to Equation (1) (*) when B is infinite dimensional. Certain technical difficulties arise when B is infinite dimensional. For example, the solution mapping $U(t)\varphi$ for Equation (1) is not compact as is the case when $B = R^n$, see J. Hale [4]; this is a problem when trying to apply standard fixed point theorems. This difficulty is overcome by assuming the semigroup $T(t)$, $t \geqq 0$ is compact for $t > 0$. It will become clear that our results depend on the operators M and N, the Lipschitz constant L, and the length of the interval b.

Define $S(b)\varphi = x_b(\varphi)$; $x_b(\varphi)$ is the element of C such that $x(\varphi)(t)$ is the unique solution of the system

$$(7)\qquad \begin{aligned} x(t) &= T(t)\varphi(0) & t &\geqq 0 \\ x_0 &= \varphi & \varphi &\in C\,. \end{aligned}$$

Notice that $S(b)$ is a special case of $U(b)\varphi \equiv y_b(\varphi)$ where $y(\varphi)(t)$ is the solution of Equation (1) for the initial function $\phi \in C$. That is, the mapping $S(b)$ is $U(b)$ when $F \equiv 0$. Also, if the semigroup $T(t)$, $t \geqq 0$ is compact for $t > 0$, we have that $U(b)$ is compact and therefore $S(b)$ is compact.

We also have need to consider the system

$$(8)\qquad \begin{aligned} z(t) &= \int_0^t T(t-s)F(y_s(\varphi))ds & 0 &\leqq t \leqq b \\ z &\equiv 0 & &\text{on } [-r, 0] \end{aligned}$$

where $y(\varphi)(t)$ is the solution of Equation (1) for the initial function $\varphi \in C$.

PROPOSITION 1. *Let F satisfy condition* (2).

(a) *Suppose $(M+N)^{-1}$ exists with the range $R((U(b)-I))$ of $U(b)-I$ contained in $D((M+N)^{-1})$, that $\|(M+N)^{-1}N(U(b)-I)\|_{\rm Lip} < 1$ $(b > r)$ and $\psi \in D((M+N)^{-1})$, then solutions of Equation* (1) (*) *exist and are unique.*

(b) *Suppose $(M+NS(b))^{-1}$ exists with $R(N(U(b)-S(b))) \subset D((M+NS(b))^{-1})$ and $\|(M+NS(b))^{-1}N(U(b)-S(b))\|_{\rm Lip} < 1$ $(b > r)$, then solutions of Equation* (1) (*) *exist and are unique.*

Proof. For an initial function $\varphi \in C$ and its corresponding unique solution of Equation (1) we have

$$My_0 + My_b = M\varphi + NU(b)\varphi = (M + NU(b))\varphi\,.$$

Therefore, in order to solve the boundary value problem Equation (1) (*) we must solve the operator equation

$$(M + NU(b))\varphi = \psi\,.$$

In case (a) we can write Equation (6) in the form

$$(M + N + N(U(b) - I))\varphi = \psi$$

and in case (b) in the form

$$(M + NS(b) + N(U(b) - S(b)))\varphi = \psi\,.$$

Since $(M+N)^{-1}$ exists in (a) and $(M+NS(b))^{-1}$ exists in (b) the above equations become

$$(9)\qquad (I + (M + N)^{-1}N(U(b) - I))\varphi = (M + N)^{-1}\psi\ ,$$

and

$$(9')\qquad (I + (M + NS(b))^{-1}N(U(b) - S(b)))\varphi = (M + NS(b))^{-}\psi$$

when $\psi \in D((M + N)^{-1})$ or $\psi \in D((M + NS(b))^{-1})$. The equations (9) and (9′) are in the form $x + Sx = y$ with $||S||_{\mathrm{Lip}} < 1$ and so are uniquely solvable.

Given an initial function $\varphi \in C$ and the solution $y(\varphi)(t)$ of Equation (1) we can write

$$(10)\qquad \begin{array}{l} y(\varphi)(t) = x(\varphi)(t) + z(0)(t) \\ y_t(\varphi) = x_t(\varphi) + z_t(0) \end{array} \quad 0 \leqq t \leqq b$$

where $x(\varphi)(t)$ and $z(0)(t)$ are solutions of Equations (7) and (8), respectively. Using the identity (10) we have the following corollary to Proposition 1(b).

COROLLARY TO PROPOSITION 1(b). *If operator* $(M + NS(b))^{-1}$ *exists on* C *and* $||(M + NS(b))^{-1}N||\,e^{(L+\omega)b} < 1$ $(b > r)$, *then the boundary value problem Equation* (1) (*) *has a unique solution.*

Proof. We show that the mapping $(M + NS(b))^{-1}N(U(b) - S(b))$ is a strict contraction:

$$\begin{aligned} &||(M + NS(b))^{-1}N(U(b) - S(b))\varphi - (M + NS(b))^{-1}(U(b) - S(b))\bar{\varphi}||_C \\ &\quad \leqq ||(M + NS(b))^{-1}N|| \sup_{-r \leqq \theta \leqq 0} \left|\left| \int_0^{b+\theta} T(b + \theta - s)(F(y_s(\varphi)) - F(y_s(\bar{\varphi})))ds \right|\right| \\ &\quad \leqq ||(M + NS(b))^{-1}N||\,Le^{\omega b}||\varphi - \bar{\varphi}||_C \int_0^b e^{LS}ds \\ &\quad < ||(M + NS(b))^{-1}N||\,e^{(\omega+L)b}||\varphi - \bar{\varphi}||_C < ||\varphi - \bar{\varphi}||_C\ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for all}\quad \varphi, \bar{\varphi} \in C\ . \end{aligned}$$

The result now follows by Proposition 1(b).

PROPOSITION 2. *Let* F *satisfy condition* (2). *If the mapping* M^{-1} *exists on* C *with* $||M^{-1}N||\,e^{(L+\omega)b} < 1$ $(b > r)$, *then Equation* (1) (*) *has a unique solution.*

Proof. For an initial function $\varphi \in C$ and its corresponding solution $y(\varphi)(t)$ of Equation (1), we have $My_0 + Ny_b = (M + NU(b))\varphi$. Thus, for the equation $(M + NU(b))\varphi = \psi$, $\psi \in C$, we can write $(I + M^{-1}NU(b))\varphi = M^{-1}\psi$. From (4) we have that

$$\begin{aligned} ||M^{-1}NU(b)\varphi - M^{-1}NU(b)\bar{\varphi}||_C &\leqq ||M^{-1}N||\,||U(b)\varphi - U(b)\bar{\varphi}||_C \\ &\leqq ||M^{-1}N||\,e^{(L+\omega)b}||\varphi - \bar{\varphi}||_C < ||\varphi - \bar{\varphi}||_C \end{aligned}$$

for all $\varphi, \bar{\varphi} \in C$. The mapping $M^{-1}NU(b)$ is a strict contraction and so the equation $(I + M^{-1}NU(b))\varphi = M^{-1}\psi$ has a unique solution for each $\psi \in C$. The result easily follows.

Using the identity (10) we are able to extend a result found in [9].

PROPOSITION 3. *The two point boundary value problem Equation* (1) (*) *has a solution if and only if* $Nz_b(0) \in \psi + R(M + NS(b))$, $\psi \in C$, $b > r$.

Proof. Given an initial function $\varphi \in C$, and its corresponding solution $y(\varphi)(t)$ of Equation (1) we have by (10) that

$$My_0(\varphi) + Ny_b(\varphi) = M\varphi + N(x_b(\varphi) + z_b(0)) = (M + NS(b))\varphi + Nz_b(0) .$$

If $\psi \in C$ and $My_0(\varphi) + Ny_b(\varphi) = \psi$, we obtain $\psi = (M + NS(b))\varphi + Nz_b(0)$; this gives $Nz_b(0) = \psi - (M + NS(b))\varphi$ and so $Nz_b(0) \in \psi + R(M + NS(b))$.

If there exists a solution ϕ of $Nz_b(0) = \varphi + R(M + NS(b))\varphi$, define $v = -\varphi$. Then for the solution $y(v)(t)$ of Equation (1) we have

$$\begin{aligned} My_0(v) + Ny_b(v) &= Mv + Nx_b(v) + Nz_b(0) \\ &= (M + NS(b))v + Nz_b(0) \\ &= -(M + NS(b))\varphi + Nz_b(0) = \psi . \end{aligned}$$

Therefore the boundary value problem is solved.

The following result is due to A. Granas [3].

PROPOSITION 4. *If T is a compact operator mapping the Banach space X into X and satisfying* $\overline{\lim}_{\|x\|\to\infty} \|Tx\|/\|x\| < 1$, *then* $R(I - T) = X$.

PROPOSITION 4. (i) *Suppose the semigroup* $T(t)$, $t \geqq 0$ *is compact,* (ii) *F takes closed bounded sets of C into bounded sets in B, and* $\lim_{\|\varphi\|_C\to\infty} \|F(\varphi)\|/\|\varphi\|_C = 0$, (iii) *there exist unique solutions to the initial value problem Equation* (1), $(M + NS(b))^{-1}$ $(b > r)$ *exists on C as a bounded operator. Then the boundary value problem Equation* (1) (*) *has a solution.*

Proof. Condition (ii) implies that there exists K_1 and K_2 such that $\|F(\varphi)\| \leqq K_1\|\varphi\|_C + K_2$ for all $\varphi \in C$, so that global solutions for Equation (1) exist [2]. Furthermore, we can find constants $\bar{K}_1$ and $\bar{K}_2$ such that condition (5) is true. Let φ_n be a sequence of functions in C such that $\|\varphi_n\|_C \to \infty$ as $n \to \infty$ and define $\beta_n = \sup_{0\leqq t\leqq b} \|y_t(\varphi_n)\|_C$. Note that $\beta_n \leqq \bar{K}_1\|\varphi_n\|_C + \bar{K}_2$ for each n. Let ε

be such that $0 < \varepsilon < 1/b\bar{K}_1 e^{\omega b}\|(M + NS(b))^{-1}N\|$, then by (ii) there exists $h > 0$ such that if $\|\varphi\|_C > h$, $\|F\varphi\| \leqq \varepsilon\|\varphi\|_C$. We define $R = \max\{\|F(\varphi)\|: \|\varphi\|_C \leqq h\}$ then

$$\begin{aligned}\|(M &+ NS(b))^{-1}N(U(b) - S(b))\varphi_n\| \\ &\leqq \sup_{-r \leqq \theta \leqq 0} \|(M + NS(b))^{-1}N\| \int_0^b \|T(b + \theta - s)\| \|F(y_s(\varphi_n))\| ds \\ &\leqq \|(M + NS(b))^{-1}N\| e^{\omega b} \int_0^b \|F(y_s(\varphi_n))\| ds \\ &\leqq \|(M + NS(b))^{-1}N\| e^{\omega b} b \max\{R, \varepsilon(\bar{K}_1\|\varphi_n\|_C + \bar{K}_2)\} .\end{aligned}$$

If $\beta_n \to \infty$ as $n \to \infty$, we have $\overline{\lim}_{n\to\infty} \|(M + NS(b))^{-1}N(U(b) - S(b))\varphi_n\|_C/\|\varphi_n\|_C < 1$ and if β_n bounded as $n \to \infty$ then $\overline{\lim}_{n\to\infty} \|(M + NS(b))^{-1}N(U(b) - S(b))\varphi_n\|/\|\varphi_n\|_C = 0$. Notice that $U(b)$ exists by (iii) and that by (i) $(M + NS(b))N(U(b) - S(b))$ is compact. Thus by Proposition A there is a solution to $(I + (M + NS(b))^{-1}N(U(b) - S(b)))\varphi = (M + NS(b))^{-1}\psi$ and the proposition is proved.

To prove Proposition 5 we need the following result of Z. Nashed and J. S. W. Wong [5].

PROPOSITION B. *If A_1 is a strict contraction on a Banach space X, i.e., $\|A_1x - A_1y\| \leqq \gamma\|x - y\|$ $(0 < \gamma < 1)$, $x, y \in X$, and A_2 is a compact mapping on X such that $\lim_{\|x\|\to\infty} \|A_2x\|/\|x\| = \beta < 1 - \gamma$, then $R(I - (A_1 + A_2)) = X$.*

PROPOSITION 5. (i) *If the semigroup $T(t)$, $t \geqq 0$ is compact for $t > 0$,* (ii) *F takes closed bounded sets of C into bounded sets in B, and $\lim_{\|\theta\|_C\to\infty} \|F(\varphi)\|/\|\varphi\| = 0$,* (iii) *there exist unique solutions to the initial value problem Equation* (1), (iv) *M^{-1} exists on C as a bounded operator and $\|M^{-1}N\|e^{\omega b} < 1$ $(b > r)$. Then the boundary value problem Equation* (1) (*) *has a solution.*

Proof. Given an initial function $\varphi \in C$, we can write

$$y_b(\varphi)(\theta) = T(b + \theta)\varphi(0) + \int_0^{b+\theta} T(b + \theta - s)F(y_s(\varphi))ds$$

where $y(\varphi)(t)$ is the solution of Equation (1) corresponding to φ. Define the operators A_1 and A_2 on C as follows:

$$(A_1\varphi)(\theta) = T(b + \theta)\varphi(0) \quad \text{and} \quad (A_2\varphi)(\theta) = \int_0^{b+\theta} T(b + \theta - s)F(y_s(\varphi))ds .$$

The operator A_2 is compact by (i) and for $\varphi, \bar{\varphi} \in C$ we have

$$\|M^{-1}NA_1\varphi - M^{-1}NA_1\bar{\varphi}\|_C \leqq \|M^{-1}N\|e^{\omega b}\|\varphi - \bar{\varphi}\|_C .$$

By (iv) the operator $M^{-1}NA_1$ is Lipschitz with Lipschitz constant $\gamma \leqq \|M^{-1}N\| e^{\omega b} < 1$.

Let $\varphi_n \in C$ such that $\|\varphi_n\|_C \to \infty$ as $n \to \infty$ and define $\beta_n = \sup_{0 \leqq t \leqq b} \|y_t(\varphi_n)\|_C$. As in the proof of Proposition 4 we have constants K_1, K_2, $\bar{K}_1$, $\bar{K}_2$ such that $\|F(\varphi)\| \leqq K_1\|\varphi\|_C + K_2$ and $\|y_t(\varphi)\|_C \leqq \bar{K}_1\|\varphi\|_C + \bar{K}_2$; therefore, we have $\beta_n \leqq \bar{K}_1\|\varphi_n\|_C + \bar{K}_2$. If the sequence β_n has limit infinity as n approaches infinity, then by (ii)

$$\overline{\lim_{n\to\infty}} \|M^{-1}NA_2\varphi_n\|_C/\|\varphi_n\|_C \leqq \overline{\lim_{n\to\infty}} \|M^{-1}N\| e^{\omega b}\varepsilon \int_0^b (\bar{K}_1\|\varphi_n\|_C + \bar{K}_2)ds/\|\varphi_n\|_C$$
$$\leqq \|M^{-1}N\| e^{\omega b}\varepsilon b \in \bar{K}_1 ,$$

where $\varepsilon > 0$ is arbitrary. Thus if we choose $\varepsilon < 1 - \gamma/\|M^{-1}N\| e^{\omega b} b\bar{K}_1$, then $\overline{\lim}_{n\to\infty} \|M^{-1}NA_2\varphi_n\|_C/\|\varphi_n\|_C < 1 - \gamma$. If the sequence β_n is bounded, then $\overline{\lim}_{n\to\infty} \|M^{-1}NA_2\varphi_n\|_C/\|\varphi_n\|_C = 0 < 1 - \gamma$. Applying Proposition B, we see that for each $\psi \in C$ there exists a solution φ of

$$(I + M^{-1}N(A_1 + A_2))\varphi = M^{-1}\psi .$$

From the above equation we can solve the boundary value problem Equation (1) (*).

To illustrate our results we consider the partial functional differential equation

$$\begin{aligned} &w_t(x, t) = w_{xx}(x, t) + f(w(x, t - r)) && 0 \leqq t \leqq b \quad 0 \leqq x \leqq l \\ &w(0, t) = w(l, t) = 0 && t \geqq 0 . \end{aligned}$$

Here f is a real-valued, Lipschitz continuous and continuously differentiable function. We let $B = L_2[0, l]$, and define A and F respectively as:

A: $D(A) \to B$ by $Au = \ddot{u}$, $D(A) = \{u \in B \mid u$ and $\dot{u}$ are absolutely continuous, $\ddot{u} \in B$ and $u(0) = u(l) = 0\}$ and F: $C \to B$ by $F(\varphi)(x) = f(\varphi(-r)(x))\varphi \in C$ and $x \in [0, l]$. It is known that A generates a strongly continuous semigroup $T(t)$, $t \geqq 0$ such that $T(t)$ is compact for $t > 0$ and $w = 0$, see A. Pazy [6, pages 9 and 47]. The function F is Lipschitz continuous and continuously differentiable.

If we let $M = I$, $N = 1/4\ I$, then $(M + N)^{-1} = 4/5\ I$ and

$$\begin{aligned} &\|(M + N)^{-1}N(U(b) - I)\varphi - (M + N)^{-1}N(U(b) - I)\bar{\varphi}\|_C \\ &\quad \leqq \|(M + N)^{-1}N\|(\|U(b)\varphi - U(b)\bar{\varphi}\|_C + \|\varphi - \bar{\varphi}\|_C) \\ &\quad \leqq 1/5(\|y_b(\varphi) - y_b(\bar{\varphi})\|_C + \|\varphi - \varphi\|_C) \leqq 1/5(e^{Lb}\|\varphi - \bar{\varphi}\|_C + \|\varphi - \varphi\|_C) \\ &\quad \leqq 1/5(e^{Lb} + 1)\|\phi - \bar{\varphi}\|_C . \end{aligned}$$

Part (a) of Proposition 1 is applicable if $1/5(e^{Lb} + 1) < 1$. This is true if $Lb < ln4$.

If the operators $M = I$ and $N = -1/4\ I$ then

$$\begin{aligned}||M + NS(b)\varphi||_C &= \sup_{-r\leqq\theta\leqq 0} ||(M + NS(b)\varphi)(\theta)|| \\ &= \sup_{-r\leqq\theta\leqq 0} ||\varphi(\theta) - 1/4T(b+\theta)\varphi(0)|| \\ &\geqq \sup_{-r\leqq\theta\leqq 0} ||\varphi(\theta)|| - 1/4||\varphi(0)|| \geqq ||\varphi||_C - 1/4||\varphi||_C = 3/4||\varphi||_C .\end{aligned}$$

The above estimate implies that $(M + NS(b))^{-1}$ exists on C and $||(M + NS(b))^{-1}|| \leqq 4/3$, furthermore

$$\begin{aligned}&||(M + NS(b))^{-1}N(U(b) - S(b))\varphi - (M + NS(b))^{-1}N(U(b) - S(b))\bar{\varphi}||_C \\ &\quad \leqq ||(M + NS(b))^{-1}N|| \, ||(U(b) - S(b))\varphi - (U(b) - S(b))\bar{\varphi}||_C \\ &\quad \leqq ||(M + NS(b))^{-1}N|| e^{Lb}||\varphi - \bar{\varphi}||_C \leqq 4/3 \cdot 1/4e^{Lb}||\varphi - \bar{\varphi}||_C \\ &\quad = 1/3e^{Lb}||\varphi - \bar{\varphi}||_C .\end{aligned}$$

Here if $Lb < ln\ 3$ then $1/3e^{Lb} < 1$, and the corollary to Proposition 1(b) applies.

If $M = I$ and $N = -1/2I$ Proposition 1 is not readily applicable since we can obtain only the following estimate:

$$||(M + N)^{-1}N(U(b) - I)||_{\text{Lip}} \leqq ||(M + N)^{-1}N||(e^{Lb} + 1) \leqq e^{Lb} + 1 .$$

The term $e^{Lb} + 1$ cannot be less than 1 for any positive numbers L and b. Similarly we have

$$||(M + NS(b))^{-1}N(U(b) - S(b))||_{\text{Lip}} \leqq ||(M + NS(b))^{-1}N|| e^{Lb} \leqq e^{Lb}$$

and e^{Lb} cannot be less than 1 and positive for any L and b. Proposition 2, however, is easily applied since $||M^{-1}N||e^{Lb} = 1/2e^{Lb} < 1$ if $0 < Lb < ln\ 2$.

If we define $F(\varphi)(x) = f(\varphi(-r)(x)) = \varphi^{1/4}(-r)(x)$, then

$$\begin{aligned}||F(\varphi)||/||\varphi||_C &= \left(\int_0^l |\varphi^{1/2}(-r)(x)|dx\right)^{1/2} \Big/ \sup_{-r\leqq\theta\leqq 0} \int_0^l \varphi^2(\theta)(x)|dx \\ &\leqq l^{3/8}\left(\int_0^l |\varphi^2(-r)(x)|dx\right)^{1/8} \Big/ \sup_{-r\leqq\theta\leqq 0} \int_0^l |\varphi^2(\theta)(x)|dx \\ &\leqq l^{3/8}\left(\sup_{-r\leqq\theta\leqq 0} \int_0^l |\varphi^2(\theta)(x)|dx\right)^{1/8} \Big/ \sup_{-r\leqq\theta\leqq 0} \int_0^l |\varphi^2(\theta)(x)|dx\end{aligned}$$

and $\lim_{||\varphi||_C\to\infty} ||F(\varphi)||/||\varphi|| = 0$. Furthermore, F takes closed bounded sets of C into bounded sets of $B = L_2[0, l]$. Letting $M = I$ and $N = -1/4\ I$, both $(M + NS(b))^{-1}$ and M^{-1} exist, and Propositions 4 and 5 can be applied to obtain solutions of

$$y(t) = T(t)\varphi(0) + \int_0^t T(t + \theta - s)y_s^{1/4}(-r)(\cdot)ds \tag{11}$$

$$My_0 + Ny_b = \psi \quad b > r . \tag{*}$$

Notice that the length of the interval b does not enter into the discussion for the above example, other than b is required to be greater than r.

The next theorem handles periodic boundary conditions, i.e., the boundary condition $y_0 = y_b$.

PROPOSITION 6. *Suppose F satisfies condition* (2). *If the operator $M + NS(b)$ has a bounded inverse defined on C such that $\|(M + NS(b))^{-1}\| < d$ for some $d > 0$ and for all (r, γ) where γ satisfies $\gamma > r$ and $d\|N\|e^{(L+\omega)r} = 1$, then the boundary value problem Equation* (1) (*) *has a unique solution.*

Proof. For a function $\psi \in C$ define the mapping $H: C \to C$ by

$$H\varphi = (M + NS(b))^{-1}\psi - (M + NS(b))^{-1}N(U(b) - S(b))\varphi .$$

We have for $\varphi, \bar{\varphi} \in C$

$$\begin{aligned}\|H\varphi - H\bar{\varphi}\|_C &= \|(M + NS(b))^{-1}N(U(b) - S(b))\varphi \\ &\quad - (M + NS(b))^{-1}N(U(b) - S(b))\bar{\varphi}\|_C \\ &\leqq \|(M + NS(b))^{-1}N\|\,\|z_b(\varphi) - \bar{z}_b(\bar{\varphi})\|_C \\ &\leqq d\|N\| \sup_{-r\leqq\theta\leqq 0} \|z(\varphi)(b + \theta) - \bar{z}(\bar{\varphi})(b + \theta)\| \\ &\leqq d\|N\| \int_0^b e^{\omega(b-s)}\|F(y_s(\varphi)) - F(y_s(\bar{\varphi}))\|ds \\ &\leqq d\|N\|e^{\omega b}L \int_0^b e^{-\omega s}\|y_s(\varphi) - y_s(\bar{\varphi})\|_C ds \\ &\leqq d\|N\|e^{(L+\omega)b}Lb\|\varphi - \bar{\varphi}\|_C .\end{aligned}$$

The operator H is a contraction if b is sufficiently small and the boundary value problem is uniquely solvable.

REMARK. Proposition 4 also handles periodic boundary conditions since again the only requirement on M and N is the existence of $(M + NS(b))^{-1}$. The inverse of $M + NS(b)$ exists with domain C if and only if the boundary value problem Equation (7)(*) has a unique solution for each $\psi \in C$.

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