PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 2, 1980

ON THE DECOMPOSITION OF STATES OF SOME *-ALGEBRAS

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We study the direct integral decomposition of Op^*-algebras defined on a metrizable, dense domain of a separable Hilbert space. Applications to the decomposition into irreducible representations and into extremal states of representations and states of *-algebras with a countable dominating subset are given.

0. Introduction. This paper is concerned with the extension of the reduction theory for bounded operators [2] to algebras of unbounded operators and, as an application, with the decomposition of representations and states of *-algebras.

In a previous paper [3] we gave a meaning to the direct integral decomposition of unbounded operators, by considering them as bounded operators between several Hilbert spaces. Considering then, families $\mathscr{A}$ of unbounded operators defined on a *common* dense domain $\mathscr{D}$ of a separable Hilbert space $\mathscr{H}$ (the so-called Op^*-algebras [8]) and considering the decomposition $\mathscr{H} = \int_{\Lambda} \mathscr{H}(\lambda)d\mu(\lambda)$ of the Hilbert space with respect to an Abelian von Neumann algebra $\mathscr{M}$ contained in the strong commutant of $\mathscr{A}$, we have been able to define for almost every $\lambda \in \Lambda$, a domain $\mathscr{D}(\lambda)$ in $\mathscr{H}(\lambda)$ such that any $A \in \mathscr{A}$ can be written as $A = \int_{\Lambda} A(\lambda)d\mu(\lambda)$ where $A(\lambda)$ is a continuous operator from $\mathscr{D}(\lambda)$ into itself. Then, to any countable subalgebra $\mathscr{A}_0 \subset \mathscr{A}$ corresponds for almost every $\lambda \in \Lambda$ a countable Op^*-algebra $\mathscr{A}_0(\lambda)$ on $\mathscr{D}(\lambda)$.

However, in order to be sure that the domain $\mathscr{D}(\lambda)$ is nonzero and even dense in $\mathscr{H}(\lambda)$, almost everywhere, we had to ask that the $\mathscr{A}$-graph topology on $\mathscr{D}$ is metrizable and this assumption characterizes the class of Op^*-algebras we want to consider. For examples of such Op^*-algebras we refer to [13], [14].

In this paper, we extend the decomposition to uncountable Op^*-algebras. If we look in [2] how it is possible to get the reduction of a von Neumann algebra once one is able to decompose a single operator or a countable set of operators, one sees that the important things are the following:

(a) a von Neumann algebra is separable in the strong operator topology.

(b) this strong operator topology is metrizable.

(c) If $A^i \to A$ is this topology any if $\{A^i\}$ and A are decomposed, then $A^i(\lambda) \to A(\lambda)$ almost everywhere.

In our case we shall assume that $\mathscr{A}$ is separable in some topology (described in § II) related to the quasi-uniform topologies of [9] and involving the strong topology of $\mathscr{H}$. Then $\mathscr{A}$ will be also separable in another topology, weaker but metrizable and we shall be able to use the result (c) above. For almost every $\lambda \in \Lambda$ we shall construct Op^*-algebras $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ such that any $A \in \mathscr{A}$ is decomposed in a direct integeral $A = \int A(\lambda)d\mu(\lambda)$ with $A(\lambda) \in \mathscr{A}(\lambda)$ a.e. As we shall see, our choice of topology is also such that the subset $\mathscr{A}_0$ which is dense in $\mathscr{A}$ by assumption, will have the same weak commutant and strong commutant as $\mathscr{A}$. This will be useful when we prove that the decomposition of $\mathscr{A}$ is irreducible.

In § III, we apply our result to the decomposition of representations of separable locally convex *-algebras into irreducible representations. We consider *-algebras with a countable dominating subset [12] in order that the domains of any of their representations fulfill our assumption of metrizability. The fact that the algebras we consider are separable will imply that strongly continuous representations of them in a Hilbert space are automatically separable in the particular topology of § II so that we can apply the previous result (Theorem 2.5).

We end this section with a few words about symmetric and self-adjoint algebras for which the situation is simpler [12, 6, 4]. Finally in § IV we decompose into extremal states, states of barrelled, locally convex, separable *-algebras, dominated by a countable subset.

Let us notice that we do not know in general if the extremal states obtained in that decomposition are continuous or not, contrary to what occurs in [1], [5] where the case of nuclear *-algebras is treated. In our case, as in the bounded case, there is no continuous map between the initial structure (algebra, representation, state) and the λ-component in the decomposition.

1. Definitions and preliminaries. In this section we define the framework in which we are going to work and recall some of the results of [3] which will be useful in the sequel.

1.1. In this paper $\mathscr{H}$ will always denote a separable Hilbert space and $\mathscr{D}$ a dense domain in $\mathscr{H}$.

Let $\mathscr{L}_+(\mathscr{D})$ denote the set of all linear operators A such that:

(a) the domain $D(A) = \mathscr{D}$ and $A\mathscr{D} \subseteq \mathscr{D}$

(b) the adjoint operator A^* satisfies $D(A^*) \supseteq \mathscr{D}$ and $A^*\mathscr{D} \subseteq \mathscr{D}$

$\mathscr{L}_+(\mathscr{D})$ is a *-algebra with the involution $A \to A^+ \equiv A^* \restriction_{\mathscr{D}}$. Any

*-subalgebra $\mathscr{A}$ of $\mathscr{L}_+(\mathscr{D})$ containing $1 \upharpoonright_{\mathscr{D}}$ is called an Op^*-algebra [8] and will sometimes be denoted by $(\mathscr{A}, \mathscr{D})$. Because $\mathscr{A}$ consists of unbounded operators we must distinguish between the weak and the strong commutant [1] both contained in the set $B(\mathscr{H})$ of bounded operators on $\mathscr{H}$:

(a) the weak commutant

$$(1) \qquad \mathscr{A}'_w \equiv \{C \in B(\mathscr{H}) \mid (f, CAg) = (A^+f, Cg) \quad \forall A \in \mathscr{A} \quad \forall f, g \in \mathscr{D}\}$$

(b) the strong commutant

$$(2) \quad \mathscr{A}'_s \equiv \{C \in B(\mathscr{H}) \mid C\mathscr{D} \subseteq \mathscr{D},\ CAf = ACf,\ \forall A \in \mathscr{A},\ \forall f \in \mathscr{D}\}\ .$$

We have $\mathscr{A}_s' \subseteq \mathscr{A}_w'$ and an Op^*-algebra is called irreducible if its weak commutant consists of scalar multiples of the identity operator [12], [1]. (The commutants will sometimes be denoted by $(\mathscr{A}, \mathscr{D})'_s$ and $(\mathscr{A}, \mathscr{D})'_w$ when confusion is possible).

1.2. Let $(\mathscr{A}, \mathscr{D})$ be an Op^*-algebra and assume that there exists an Abelian von Neumann algebra $\mathscr{M} \subset \mathscr{A}_s'$, containing 1 and maximal in the sense that $\mathscr{M} = \mathscr{M}' \cap \mathscr{A}_w'$. ($\mathscr{M}'$ denotes here the usual commutant for bounded operators.) This assumption is justified by the extension theory developed in [1]. Any $(\mathscr{A}, \mathscr{D})$ admits an extension $(\hat{\mathscr{A}}, \hat{\mathscr{D}})$ in a bigger Hilbert space $\hat{\mathscr{H}}$, for which such a $\mathscr{M}$ exists.

By the reduction theory for von Neumann algebras [2] we know that there exists a compact metrizable space Λ, a positive regular Borel measure μ on Λ, and a μ-measurable field $\lambda \to \mathscr{H}(\lambda)$ of Hilbert spaces such that

$$(3) \qquad \mathscr{H} \cong \int_\Lambda \mathscr{H}(\lambda) d\mu(\lambda)$$

and such that $\mathscr{M}$ consists of diagonalized operators in that decomposition ($\mathscr{M} \cong L^\infty(\Lambda, \mu)$). $\mathscr{M}'$ is then the set of bounded operators in $\mathscr{H}$ which are decomposed in that direct integral.

1.3. If we equip the domain $\mathscr{D}$ with the topology defined by the set of *all* graph-norms $\{\|f\|_A^2 \equiv \|f\|^2 + \|Af\|^2 \mid A \in \mathscr{A}\}$ and called in short the $\mathscr{A}$-graph topology, we get that every $B \in \mathscr{A}$ is a continuous operator from $\mathscr{D}$ into itself. The completion of $\mathscr{D}$ in this topology coincide with $\bigcap_{A \in \mathscr{A}} D(\bar{A})$ (where $\bar{A}$ denotes the operator-closure of A; $D(\bar{A})$ provided with the graph-norm $\|\ \|_{\bar{A}}$ is a Hilbert space). In [3] § 3.3.1 we had to impose an additional assumption on $\mathscr{D}$: we asked $\mathscr{D}$ to be metrizable i.e., the $\mathscr{A}$-graph topology is actually given by a countable set of norms.

In other words, there exists a countable subset $\mathscr{B}_0 \subset \mathscr{A}$ such that the $\mathscr{A}$-graph topology is equivalent to the $\mathscr{B}_0$-graph topology (this happens for instance if we consider representations of abstract *-algebras with a countable dominating subset as defined in [12]). With this metrizability assumption on $\mathscr{D}$, we proved in [3] that *for almost every* $\lambda \in \Lambda$ there exists a domain $\mathscr{D}(\lambda)$ dense in $\mathscr{H}(\lambda)$, which is moreover a Frechet space and that each $A \in \mathscr{A}$ can be written as:

$$A = \int_\Lambda A(\lambda) d\mu(\lambda) \tag{4}$$

where $A(\lambda)$ is a.e. a continuous operator from $\mathscr{D}(\lambda)$ into itself. (Without the metrizability assumption on $\mathscr{D}$ we should have been able to defined $\mathscr{D}(\lambda)$ but not to assure that it is nonzero on a set of measure one in Λ. For more details see [3] § 3.)

For two elements $A, A' \in \mathscr{A}$, $A + A'$, AA' and A^+ are also decomposable in the above sense and the algebraic relations are preserved for almost every $\lambda \in \Lambda$ i.e.,

$$(A + A')(\lambda) = A(\lambda) + A'(\lambda) \quad \text{a.e.}$$
$$(AA')(\lambda) = A(\lambda)A'(\lambda) \quad \text{a.e.}$$

and

$$A^+(\lambda) = A(\lambda)^+ \equiv (A(\lambda))^* \restriction_{\mathscr{D}(\lambda)} \quad \text{a.e.}$$

Finally for any countable Op^*-algebra $\mathscr{A}_0 \subset \mathscr{A}$ there exists for almost every $\lambda \in \Lambda$, countable Op^*-algebras $\mathscr{A}_0(\lambda)$ on $\mathscr{D}(\lambda)$ such that

$$\mathscr{A}_0 = \int_\Lambda \mathscr{A}_0(\lambda) d\mu(\lambda) . \tag{5}$$

If $\mathscr{A}_0$ is "big enough" in the sense that the chosen $\mathscr{M}$ satisfies $\mathscr{M} \subset (\mathscr{A}_0)'_s$ and $\mathscr{M} = \mathscr{M}' \cap (\mathscr{A}_0)'_w$ then the $\mathscr{A}_0(\lambda)$ are irreducible Op^*-algebras.

In the next section we are going to extend this last result to the decomposition of $\mathscr{A}$ itself (assumed to be uncountable but separable in a suitable topology).

A last remark about the topology on $\mathscr{D}(\lambda)$ that we have not described so far: this topology is shown to be equivalent to the $\mathscr{B}_0(\lambda)$-graph topology [3] (as the topology on $\mathscr{D}$ was the $\mathscr{B}_0$-graph topology) and all $A(\lambda)$'s obtained by the decomposition of any $A \in \mathscr{A}$, are continuous with respect to this topology.

II. Decomposition of Op^*-algebras into irreducible components.

2.1. Topology on $\mathscr{A}$.

Let $(\mathscr{A}, \mathscr{D})$ and $\mathscr{B}_0$ be as above.

It is possible to consider various topologies on $\mathscr{A}$ which are all "natural" in the sense that they generalize the uniform topology for bounded operators. These topologies have been studied in detail by Lassner et al [9] and the topology we shall consider here is related to the so-called "quasi-uniform" topologies of [9]. In most cases, e.g., when $\mathscr{A}$ by is barrelled all these natural topologies coincide [9], [10].

Consider on $\mathscr{A}$ the topology τ defined by the set of norms:

$$q_{f,A_0}(B) = \max\{\|Bf\|_{A_0}, \|B^+f\|_{A_0}\}, \tag{6}$$

where f runs in $\mathscr{D}$ and A_0 in $\mathscr{B}_0$. We shall also have to use another topology τ', weaker than τ, defined by the countable set of norms:

$$q_{e_i,A_0}(B) = \max\{\|Be_i\|_{A_0}, \|B^+e_i\|_{A_0}\} \tag{7}$$

where $A_0 \in \mathscr{B}_0$ as above and $\{e_i\}_{i=1,2}$ is a countable dense set in $\mathscr{D}$ for the $\mathscr{B}_0$-graph topology. $\mathscr{A}$ provided with this topology τ' is a metrizable (in general not complete) *-algebra. From now on, we shall assume that *$\mathscr{A}$ is separable in the topology τ* (hence also in τ') i.e., there exists a countable set $\mathscr{A}_0 \subset \mathscr{A}$, τ-dense in $\mathscr{A}$. (This set can be completed in a *-subalgebra of $\mathscr{A}$ on the complex rational field.)

This assumption of separability of $\mathscr{A}$ in this particular topology can seem a bit artificial at this point, but in fact when we shall consider in § III, strongly continuous representations of abstract *-algebras (assumed to be separable in their own topology, whatever it is), this separability in the topology τ will be automatically satisfied.

2.2. Decomposition of the dense subset $\mathscr{A}_0$.

2.2.1. Since $\mathscr{A}_0$ is a countable Op^*-algebra we can decompose it in a direct integral, using the result of [3] recalled at the end of § I. So there exists a direct integral decomposition of $\mathscr{H} \cong \int_\Lambda \mathscr{H}(\lambda)d\mu(\lambda)$, there exist for almost every $\lambda \in \Lambda$, a dense domain $\mathscr{D}(\lambda)$ in $\mathscr{H}(\lambda)$ and a countable Op^*-algebra $(\mathscr{A}_0(\lambda), \mathscr{D}(\lambda))$ such that:

$$\mathscr{A}_0 \cong \int_\Lambda \mathscr{A}_0(\lambda)d\mu(\lambda).$$

We shall now prove that the $\mathscr{A}_0(\lambda)$ obtained in this way are irreducible i.e., we need to prove:

$$\mathscr{M} \subset (\mathscr{A}_0)'_s \quad \text{and} \quad \mathscr{M} = \mathscr{M}' \cap (\mathscr{A}_0)'_w .$$

In fact we shall see that the choice of the topology τ is just the right one giving that result.

LEMMA 2.2.2. *$(\mathscr{A}_0)'_s = \mathscr{A}_s{}'$ and $(\mathscr{A}_0)'_w = \mathscr{A}_w{}'$.*

Proof. (a) $\mathscr{A}_s{}' \subseteq (\mathscr{A}_0)'_s$ and $\mathscr{A}_w{}' \subseteq (\mathscr{A}_0)'_w$ are obvious.

(b) Take $C \in (\mathscr{A}_0)'_s$. We have $C\mathscr{D} \subseteq \mathscr{D}$ by definition. For any $B \in \mathscr{A}$, then exists a net $\{B^\alpha\} \subset \mathscr{A}_0$ such that $q_{f,A_0}(B^\alpha - B)$ tends to zero, for every $f \in \mathscr{D}$, $A_0 \in \mathscr{B}_0$. Particularizing to $A_0 = 1$ we have $\|(B^\alpha - B)f\| \to 0$, for every $f \in \mathscr{D}$. Since C is bounded we have also $\|CB^\alpha f - CBf\| \to 0$. On the other hand, $f \in \mathscr{D}$ implies $Cf \in \mathscr{D}$ and thus $\|B^\alpha Cf - BCf\| \to 0$. Finally $CBf = \lim_\alpha CB^\alpha f = \lim_\alpha B^\alpha Cf = BCf$, $\forall f \in \mathscr{D}$ and $\forall B \in \mathscr{A}$ which means $C \in \mathscr{A}_s{}'$.

(c) Take $C \in (\mathscr{A}_0)'_w$. For every $B \in \mathscr{A}$, there exists $\{B^\alpha\} \subset \mathscr{A}_0$ such that $\|(B^\alpha - B)f\| \to 0$ and $\|(B^{\alpha+} - B^+)f\| \to 0$ for every $f \in \mathscr{D}$. So

$$|(f, C(B^\alpha - B)f)| \leqq \|f\|\,\|C\|\,\|(B^\alpha - B)f\| \longrightarrow 0$$

and similarly $|((B^{\alpha+} - B^+)f, Cf)| \leqq \|(B^{\alpha+} - B^+)f\|\,\|C\|\,\|f\| \to 0$. We have thus:

$$(f, CBf) = \lim_\alpha (f, CB^\alpha f) = \lim_\alpha (B^{\alpha+}f, Cf) = (B^+f, Cf)$$

and by polarization $(g, CBf) = (B^+g, Cf)$, $\forall f, g \in \mathscr{D}$, $\forall B \in \mathscr{A}$ which means $C \in \mathscr{A}_w{}'$.

COROLLARY 2.2.3. *$(\mathscr{A}_0(\lambda), \mathscr{D}(\lambda))$ is irreducible* a.e.

2.3. **Construction of $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$.** On the various Op^*-algebras $(\mathscr{A}_0(\lambda), \mathscr{D}(\lambda))$ we can consider a topology τ'_λ defined in analogy with τ' by the set of semi-norms.

$$(8) \qquad q_{\lambda,e_i,A_0}(B(\lambda)) = \max\{\|B(\lambda)e_i(\lambda)\|_{A_0(\lambda)}, \|B^+(\lambda)e_i(\lambda)\|_{A_0(\lambda)}\}$$

where $\{e_i\}$ is as above and $A_0 \in \mathscr{B}_0$($\mathscr{B}_0$ being a countable set can also be decomposed); the set $\{e_i(\lambda)\}$ is a.e. dense in $\mathscr{D}(\lambda)$ for the $\mathscr{B}_0(\lambda)$-graph topology). $\mathscr{A}_0(\lambda)$ provided with the topology τ'_λ is a metric algebra. What can be said about its completion? Consider a cauchy sequence $\{B^k(\lambda)\}$ in $\mathscr{A}_0(\lambda)$. For every $e_i(\lambda)$, $i = 1, 2 \cdots$ and for every $A_0 \in \mathscr{B}_0$ we have

$$\|(B^k(\lambda) - B^j(\lambda))e_i(\lambda)\|_{A_0(\lambda)} \leqq \varepsilon$$

and

$$\|(B^{k+}(\lambda) - B^{j+}(\lambda))e_i(\lambda)\|_{A_0(\lambda)} \leqq \varepsilon \quad \text{for } k, j \text{ big enough .}$$

Since $\mathscr{D}(\lambda)$ is complete in the $\mathscr{B}_0(\lambda)$-graph topology [3], the following limits:

$$\mathscr{B}_0(\lambda) - \lim_k B^k(\lambda)e_i(\lambda) = \psi_i(\lambda)$$

and

$$\mathscr{B}_0(\lambda) - \lim_k B^{k+}(\lambda)e_i(\lambda) = \phi_i(\lambda) \quad \text{exist in } \mathscr{D}(\lambda) .$$

These two last relations define two operators $B(\lambda)$ and $T(\lambda)$ such that

$$B(\lambda)e_i(\lambda) = \psi_i(\lambda) \quad \text{and} \quad T(\lambda)e_i(\lambda) = \phi_i(\lambda) , \quad \forall i .$$

These operators are linear from $\{e_i(\lambda)\}$ into $\mathscr{D}(\lambda)$ and moreover $T(\lambda) = B(\lambda)^* \restriction_{\{e_i(\lambda)\}}$.

We thus see that completing $\mathscr{A}_0(\lambda)$ in the topology τ'_λ we get out of $\mathscr{L}_+(\mathscr{D}(\lambda))$ and we get linear operators from $\{e_i(\lambda)\}$ into $\mathscr{D}(\lambda)$. In particular, an element of this completion will not necessarily be continuous in the $\mathscr{B}_0(\lambda)$-graph topology, but those which are, can be extended to a linear continuous operator from $\mathscr{D}(\lambda)$ into itself.

Since we are trying to define some Op^*-algebras on $\mathscr{D}(\lambda)$, we shall restrict ourselves to those $B(\lambda)$ in the completion of $\mathscr{A}_0(\lambda)$ which are, as well as their "adjoint" $T(\lambda)$, continuous in the $\mathscr{B}_0(\lambda)$-graph topology. Then $B(\lambda)$ and $T(\lambda)$ can both be extended to elements adjoints to each other in $\mathscr{L}_+(\mathscr{D}(\lambda))$. By this procedure we get a linear subset of $\mathscr{L}_+(\mathscr{D}(\lambda))$ and we shall call $\mathscr{A}(\lambda)$ the *Op^*-algebra* on $\mathscr{D}(\lambda)$ *generated* by this subset.

Thus beginning from $\mathscr{A}_0(\lambda)$ we have been able to construct in this way an Op^*-algebra $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$. A generic element of $\mathscr{A}(\lambda)$ is obtained by algebraic operations from elements of $\mathscr{L}_+(\mathscr{D}(\lambda))$ which, on the dense subset $\{e_i(\lambda)\}$ are approximated by elements of $\mathscr{A}_0(\lambda)$.

THEOREM 2.4. *With the same notations and hypothesis as above, let $B \in \mathscr{A}$ be decomposed by the method of* [3] i.e., $B = \int B(\lambda)d\mu(\lambda)$ *where $B(\lambda)$ is* a.e. *a continuous operator on $\mathscr{D}(\lambda)$ (with the $\mathscr{B}_0(\lambda)$-graph topology).*
Then $B(\lambda) \in \mathscr{A}(\lambda)$ almost everywhere.

Proof. Let $B \in \mathscr{A}$. Since $\mathscr{A}$ is separable for the topology τ'

(which is metrizable) there exists a sequence $\{B^k\}$, $k = 1, 2\cdots$ in $\mathscr{A}_0$ such that $\forall\varepsilon > 0$ and for every norm q_{e_i,A_0}, $q_{e_i,A_0}(B^k - B) < \varepsilon$ for k big enough, which means

$$||(B^k - B)e_i||_{A_0} < \varepsilon \quad \text{and} \quad ||(B^{k+} - B^+)e_i||_{A_0} < \varepsilon \ .$$

But $||(B^k - B)e_i|| = \int ||(B^k(\lambda) - B(\lambda))e_i(\lambda)||_{A_0(\lambda)}d\mu(\lambda) < \varepsilon$ implies that there exists a subsequence $\{B^{kj}\}$ (which might depend on A_0 and e_i) such that

$$||(B^{kj}(\lambda) - B(\lambda))e_i(\lambda)||_{A_0(\lambda)} < \varepsilon \quad \text{for almost every} \quad \lambda\ .$$

(See [2] Chapter II § 1 Prop. 5.)

In fact, because there is only a countable set of $A_0 \in \mathscr{B}_0$ and e_i, we can extract, by a diagonal procedure, a subsequence independent of A_0 and e_i and suitable for B^+ as well. So for every $A_0 \in \mathscr{B}_0$, and every e_i, $i = 1, 2\cdots$ and for almost every $\lambda \in \Lambda$, we have:

$$q_{\lambda,e_i,A_0}(B^{kj}(\lambda) - B(\lambda)) = \max\{||(B^{kj}(\lambda) - B(\lambda))e_i(\lambda)||_{A_0(\lambda)}\ , \\ ||(B^{kj+}(\lambda) - B^+(\lambda))e_i(\lambda)||_{A_0(\lambda)}\} < \varepsilon$$

i.e., $B(\lambda)$ belongs to the completion of $\mathscr{A}_0(\lambda)$ with respect to τ'_λ. Since we already know that $B(\lambda)$ is continuous in the $\mathscr{B}_0(\lambda)$-graph topology, we get $B(\lambda) \in \mathscr{A}(\lambda)$ a.e.

THEOREM 2.5. *Let $(\mathscr{A}, \mathscr{D})$ be an Op*-algebra in a separable Hilbert space $\mathscr{H}$. Assume that the graph topology on $\mathscr{D}$ is given by a countable set of graph-norms $||\ ||_{A_0}$, $A_0 \in \mathscr{B}_0$ (countable subset of $\mathscr{A}$). Assume that $\mathscr{A}$ is separable in the topology τ (described in* 2.1) *and that there exists an Abelian von Neumann algebra $\mathscr{M} \subset \mathscr{A}'_s$*

Then, there exists an integral decomposition $\mathscr{H} \cong \int \mathscr{H}(\lambda)d\mu(\lambda)$, there exist domains $\mathscr{D}(\lambda)$ dense in $\mathscr{H}(\lambda)$ and Op-algebras $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ such that for every $A \in \mathscr{A}$, $f \in \mathscr{D}$*

$$Af = \int_\Lambda A(\lambda)f(\lambda)d\mu(\lambda)$$

where $f(\lambda) \in \mathscr{D}(\lambda)$ and $A(\lambda) \in \mathscr{A}(\lambda)$ a.e. *If moreover $\mathscr{M}$ is maximal in the sense that $\mathscr{M} = \mathscr{M}' \cap \mathscr{A}_w'$, then $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ is irreducible* a.e.

Proof. Only the last point need to be proved. But $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))'_w \subseteq (\mathscr{A}_0(\lambda), \mathscr{D}(\lambda))'_w$ and we have seen that this last commutant is trivial (Corollary 2.2.3).

III. Representations of separable local convex *-algebras.

3.1. From now on $\mathscr{A}$ will denote a separable locally convex topological *-algebra with identity. By "topological" we mean that the product of two elements is separately continuous and the involution is continuous. We shall consider *-representations π of $\mathscr{A}$ by Op*-algebras $(\pi(\mathscr{A}), \mathscr{D})$ of unbounded operators acting in a separable Hilbert space. For every $A \in \mathscr{A}$, we have $\pi(A^*) = \pi(A)^+ = (\pi(A))^* \restriction_{\mathscr{D}}$. The representation π if said to be *weakly continuous* if $(f, \pi(A)g)$ is continuous in A, for every $f, g \in \mathscr{D}$. It is *strongly continuous* if $A \to \|\pi(A)f\|$ is continuous, for every $f \in \mathscr{D}$. A strongly continuous representation is automatically weakly continuous. On the other hand, if $\mathscr{A}$ is either barrelled or if the multiplication is jointly continuous, then a weakly continuous representation is also strongly continuous [8] [1]. In order to exploit the results of the previous sections we shall assume that $\mathscr{A}$ is dominated by a countable subset $\mathscr{B}_0 \subset \mathscr{A}$ [12], i.e., for every $A \in \mathscr{A}$ and for any representation $(\pi(\mathscr{A}), \mathscr{D}_\pi)$ of $\mathscr{A}$ there exists $B \in \mathscr{B}_0$ such that for any $f \in \mathscr{D}_\pi$

$$\|\pi(A)f\| \leqq k \ \|\pi(B)f\| \quad \text{(for some constant } k) .$$

If we do not ask this condition for any representation π but only for a particular one, we say that "$\mathscr{B}_0$ dominates $\mathscr{A}$ in the representation π".

If $\mathscr{A}$ is separable (in its own topology) and if we consider strongly continuous representations π, $\pi(\mathscr{A})$ is automatically separable in the topology defined by the norms:

$$q_{f,A_0}(\pi(B)) = \max\{\|\pi(B)f\|_{\pi(A_0)}, \|\pi(B)^+f\|_{\pi(A_0)}\}$$
$$f \in \mathscr{D} \ , \quad A_0 \in \mathscr{B}_0 \quad (B \in \mathscr{A}) \ ,$$

which corresponds to the topology τ of the § II. So applying our previous result we get:

THEOREM 3.2. *Let $\mathscr{A}$ be a separable locally convex *-algebra, with identity and dominated by a countable subset $\mathscr{B}_0$. Let π be a strongly continuous *-representation of $\mathscr{A}$ by unbounded operators defined on a dense domain $\mathscr{D}$ of a separable Hilbert space $\mathscr{H}$. There exists a separable Hilbert space $\hat{\mathscr{H}}$ containing $\mathscr{H}$ as a closed subspace and a direct integral decomposition $\hat{\mathscr{H}} = \int_\Lambda \mathscr{H}(\lambda)d\mu(\lambda)$ where μ is a positive Borel measure on a compact space Λ. There exist* a.e. *-representations π_λ of $\mathscr{A}$ by unbounded operators defined on dense subspaces $\mathscr{D}(\lambda)$ of $\mathscr{H}(\lambda)$ such that:*

(a) *For every $f \in \mathscr{D}$, $A \in \mathscr{A}$:*

$$\pi(A)f = \int_{\Lambda} \pi_\lambda(A)f(\lambda)d\mu(\lambda) .$$

(b) $(\pi_\lambda(\mathscr{A}))'_w$ *is trivial* a.e.

(c) *If* $A = \lim_\alpha A^\alpha$ *in* $\mathscr{A}$, *then there exists a sequence* $\{A^i\} \subset \{A^\alpha\}$ *and a subsequence* $\{A^{i_k}\}$ *such that* $\|(\pi_\lambda(A^{i_k}) - \pi_\lambda(A))e_j(\lambda)\|_{\pi_\lambda(A_0)}$ *tends to zero* a.e. *when* $i_k \to \infty$, *for every* e_j *belonging to any countable dense subset of* $\mathscr{D}$ *for the* $\pi(\mathscr{B}_0)$*-graph topology, and for every* $A_0 \in \mathscr{B}_0$.

Proof. The existence of the bigger space $\hat{\mathscr{H}}$ comes from the extension theory developed in [1]. If in our representation $(\pi(\mathscr{A}), \mathscr{D})$ we cannot find a $\mathscr{M}$ in $(\pi(\mathscr{A}))'_s$ such that $\mathscr{M} = \mathscr{M}' \cap (\pi(\mathscr{A}))'_w$ we have to use [1] in order to get another representation $(\hat{\pi}(\mathscr{A}), \hat{\mathscr{D}})$ in a bigger space $\hat{\mathscr{H}}$ such that there exists $\mathscr{M} \subset (\hat{\pi}(\mathscr{A}))'_s$ satisfying $\mathscr{M} = \mathscr{M}' \cap (\hat{\pi}(\mathscr{A}))'_w$. The domain $\hat{\mathscr{D}}$ of this new representation of $\mathscr{A}$ is related to $\mathscr{D}$ by: $\hat{\mathscr{D}}$ = finite linear space of $\mathscr{M}\mathscr{D}$ and $\mathscr{H}$ is a closed subspace of $\hat{\mathscr{H}}$ (we refer to [1] for the details). In order to be able to apply our Theorem 2.5 to $(\hat{\pi}(\mathscr{A}), \hat{\mathscr{D}})$ we must check the two following points:

(1) $\hat{\mathscr{D}}$ with the graph topology is a metrizable space. This is the case because $\mathscr{B}_0$ is a dominating subset.

(2) $\hat{\pi}(\mathscr{A})$ is separable in the topology $\hat{\pi}$ defined by the set of norms

$$\hat{q}_{g,A_0}(\hat{\pi}(B)) = \max\{\|\hat{\pi}(B)g\|_{\hat{\pi}(A_0)}, \|\hat{\pi}(B)^+g\|_{\hat{\pi}(A_0)}\}$$

where $g \in \hat{\mathscr{D}}$ and $A_0 \in \mathscr{B}_0$ $(B \in \mathscr{A})$.

But this follows from the separability of $\mathscr{A}$ which implies the separability of $\pi(\mathscr{A})$ in the topology τ. If $\mathscr{A}_0$ is a countable dense set in $\mathscr{A}$, $\pi(\mathscr{A}_0)$ will be a countable dense set in $\pi(\mathscr{A})$ (for τ). Then for every $B \in \mathscr{A}$, $\exists\{B^\alpha\} \subset \mathscr{A}_0$ such that

$$\|\pi(B^\alpha - B)f\|_{\pi(A_0)} \longrightarrow 0 \quad \text{and} \quad \|\pi(B^{\alpha+} - B^+)f\|_{\pi(A_0)} \longrightarrow 0 \quad \text{for any}$$

f in $\mathscr{D}$ and any $A_0 \in \mathscr{B}_0$.

Consider now $g \in \hat{\mathscr{D}}$ (it suffices to consider g of the form $g = Mf$ with $M \in \mathscr{M}$, $f \in \mathscr{D}$). We have:

$$\begin{aligned}\|\hat{\pi}(B^\alpha - B)g\|^2_{\hat{\pi}(A_0)} &= \|\hat{\pi}(B^\alpha - B)Mf\|^2 + \|\hat{\pi}(A_0)\hat{\pi}(B^\alpha - B)Mf\|^2 \\ &= \|M\pi(B^\alpha - B)f\|^2 + \|M\pi(A_0)\pi(B^\alpha - B)f\|^2 \\ &\leqq \|M\|^2\|\pi(B^\alpha - B)f\|^2_{\pi(A_0)} \longrightarrow 0\end{aligned}$$

and similarly for the adjoints.

It follows that $\hat{\pi}(\mathscr{A}_0)$ is a countable dense set $\hat{\pi}(\mathscr{A})$ for $\hat{\tau}$.

As a conclusion of this we can apply Theorem 2.5 to $(\hat{\pi}(\mathscr{A}),$

$\hat{\mathscr{D}}$) and get a decomposition of $\hat{\pi}(\mathscr{A})$ which will induce a decomposition of $\pi(\mathscr{A})$. Part (a) and (b) of Theorem 3.2 follows immediately.

(c) We do not know in general if the π_λ obtained in the integral decomposition are strongly continuous representations of $\mathscr{A}$. Nevertheless they satisfy a weaker condition mentioned in (c).
If $A = \lim_\alpha A^\alpha$ in $\mathscr{A}$, we have that $\forall f \in \mathscr{D}$ and $\forall A_0 \in \mathscr{B}_0$:

$$\|(\pi(A^\alpha) - \pi(A))f\|_{\pi(A_0)} \longrightarrow 0 .$$

If we restrict ourselves to the f's contained in a $\pi(\mathscr{B}_0)$-dense subset $\{e_j\}_{j=1,2\ldots}$ of $\mathscr{D}$ we get a metrizable topology on $\pi(\mathscr{A})$ and so there exists a sequence $\{A^i\} \subset \{A^\alpha\}$ such that

$$\|(\pi(A^i) - \pi(A))e_j\|_{\pi(A_0)} \longrightarrow 0 \quad \text{for any} \quad e_j,\ j = 1, 2\ldots \quad \text{and} \quad A_0 \in \mathscr{B}_0 .$$

But this means that there exists a subsequence $\{A^{i_k}\}$ (independent of e_j and A_0 by a diagonal procedure) such that for almost every $\lambda \in \Lambda$;

$$\|(\pi_\lambda(A^{i_k}) - \pi_\lambda(A))e_j(\lambda)\|_{\pi_\lambda(A_0)} \longrightarrow 0 .$$

3.3. **Symmetric and self-adjoint algebras.** There is a class of *-algebras for which we do not need to use the extension theory of [1] because it is easy to find a $\mathscr{M}$ satisfying our conditions: the symmetric and the self-adjoint algebras.

DEFINITION. A *-algebra $\mathscr{A}$ with identity 1 is called *symmetric* if for each $A \in \mathscr{A}$, the element $(1 + A^*A)^{-1}$ exists in $\mathscr{A}$.

Any representation of a symmetric *-algebra is a *symmetric Op*-algebra* i.e., for every $A \in \mathscr{A}$, $(1 + \pi(A)^+\pi(A))^{-1}$ exists and belongs to the bounded part $(\pi(\mathscr{A}))_b$ of $\pi(\mathscr{A})$. $(\pi(\mathscr{A})_b) \equiv \{\pi(A) \in \pi(\mathscr{A}) \mid \overline{\pi(A)} \in B(\mathscr{H})\}$ where the bar denotes the operator closure). If moreover $\pi(\mathscr{A})_b$ is a C^*-algebra or a von Neumann algebra then $\pi(\mathscr{A})$ is called an EC^*-algebra or a EW^*-algebra. This kind of algebra has been studied in detail in [6] [7].

DEFINITION. An Op^*-algebra $(\pi(\mathscr{A}), \mathscr{D})$ is called self-adjoint if

$$\mathscr{D} = \mathscr{D}^* \equiv \bigcap_{A \in \mathscr{A}} D((\pi(A))^*) .$$

If $\pi(\mathscr{A})$ is a symmetric Op^*-algebra, its "closure"

$$\overline{\pi(\mathscr{A})} \equiv \overline{\{\pi(A)}\restriction_{\mathscr{D}} \mid A \in \mathscr{A}\} \text{ is self-adjoint [6] .}$$

Self-adjoint representations of *-algebras have been studied by [12] and [4]. The authors of [4] characterize the states on *-algebras which give rise to representations by self-adjoint Op^*-algebras, by the GNS construction [12]. Those states are the so-called Riesz states [4]. In the same paper, they exhibit a necessary and sufficient condition for an Op^*-algebra to be self-adjoint namely:

$$\pi(\mathscr{A}) \text{ is self-adjoint iff } (\pi(\mathscr{A}))'_s = (\pi(\mathscr{A}))'_w$$

(denoted therefore by $\pi(\mathscr{A})'$) and

$$\mathscr{D}^* = \bigcup\{Cf \mid f \in \mathscr{D}\ , \quad C \in \pi(\mathscr{A})')\ .$$

The important property for us is that if $\pi(\mathscr{A})$ is self-adjoint, the strong and the weak commutant coincide and is a von Neumann algebra. In that case it is easy to find a maximal Abelian von Neumann algebra $\mathscr{M} \subset \pi(\mathscr{A})'$ for instance the one generated by 1 and a simple hermitian element in $\pi(\mathscr{A})'$.

We can then immediately apply Theorem 2.5 to decompose $\pi(\mathscr{A})$ without need to build an extension $\hat{\pi}$. From this it follows that if $\mathscr{A}$ is symmetric *-algebra, Theorem 3.2 applies to it without the assumption that $\mathscr{B}_0$ is a dominating subset of $\mathscr{A}$ but with the hypothesis that $\mathscr{B}_0$ dominates $\mathscr{A}$ in the considered representation π only.

IV. Decomposition of states.

THEOREM 4.1. *Let $\mathscr{A}$ be a separable topological *-algebra with identity either barrelled or such that the multiplication is jointly continuous, and with a countable dominating subset $\mathscr{B}_0$. Then every positive continuous linear functional ω on $\mathscr{A}$ admits the decomposition*

$$\omega = \int_{\Lambda} \omega_\lambda d\mu(\lambda) \quad (\textit{in the weak sense})$$

*where $d\mu$ is a regular Borel measure on a compact space Λ and where ω_λ is an extremal state almost everywhere. (If $\mathscr{A}$ is a symmetric *-algebra, we just ask that $\mathscr{B}_0$ dominates $\mathscr{A}$ in the* GNS *representation associated to ω).*

Proof. A positive continuous linear functional ω on $\mathscr{A}$ defines by the GNS construction [12] a weakly continuous cyclic representation π of $\mathscr{A}$ on a dense domian $\mathscr{D} = \pi(\mathscr{A})\Omega$ of a separable Hilbert space (Ω = cyclic vector). But if $\mathscr{A}$ is either barrelled or such that the multiplication is jointly continuous, this representation π is also strongly continuous so that we can apply Theorem 3.2 to it. We get $\forall A \in \mathscr{A}$:

$$\omega(A) = (\Omega, \pi(A)\Omega) = \int_{\Lambda} (\Omega(\lambda) , \quad \pi_{\lambda}(A)\Omega(\lambda)) d\mu(\lambda) .$$

Defining $\omega_{\lambda}(A) \equiv (\Omega(\lambda), \pi_{\lambda}(A)\Omega(\lambda))$, we get a positive linear functional ω_{λ} on $\mathscr{A}$. By Theorem 3.2, π_{λ} is irreducible a.e., i.e., $(\pi_{\lambda}(\mathscr{A}))'_{w}$ is trivial a.e. which means that ω_{λ} is extremal a.e.

(4.2). Here again, we do not know if the states ω_{λ} obtained in the above decomposition are continuous or not. All we know is that if $A = \lim_{\alpha} A^{\alpha}$ in $\mathscr{A}$, there exists a sequence $\{A^{i}\} \subset \{A^{\alpha}\}$ and a subsequence $\{A^{i_k}\}$ such that $\omega_{\lambda}(A^{i_k *} A^{i_k}) \to \omega_{\lambda}(A^{*}A)$ almost everywhere, which implies $\omega_{\lambda}(A^{i_k}) \to \omega_{\lambda}(A)$ almost everywhere.

Although this last properly is weaker than the continuity, this is sufficient to insure that $\omega_{\lambda}(A)$ is positive for all A belonging to the *closed* positive cone $\mathscr{A}_{+}$ of $\mathscr{A}$.

Then we can check in each individual case if we are not dealing with an algebra such that every state on it is automatically continuous (as for C^{*}-algebras). Several sufficient conditions for that are given in [15] p. 228 namely:

(a) The set $\mathscr{A}_{+}$ of positive elements of $\mathscr{A}$ has a nonempty interior.

(b) $\mathscr{A}$ is metrizable and complete and $\mathscr{A}_{h} = \mathscr{A}_{+} - \mathscr{A}_{+}$ (where $\mathscr{A}_{h}$ = set of hermitian elements of $\mathscr{A}$).

(c) $\mathscr{A}_{h}$ is bornological and $\mathscr{A}_{+}$ is a sequentially complete strict $\mathscr{B}$-cone ($\mathscr{B}$ denotes the set of bounded sets of $\mathscr{A}_{h}$. $\mathscr{A}_{+}$ is a strict $\mathscr{B}$-cone if $\{(S \cap \mathscr{A}_{+}) - (S \cap \mathscr{A}_{+}) \mid S \in \mathscr{B}\}$ is a fundamental family of $\mathscr{B}$).

As noticed in [11], condition (b) is not applicable to locally convex *-algebras in general. Condition (a) has at least one important counterexample: the field algebra. But this algebra satisfies condition (c) as shown in [16] so that every state on it is continuous. The author of [11] proved also that any locally convex *-algebra satisfies part of (c): $\mathscr{A}_{+}$ is always a strict $\mathscr{B}$-cone in $\mathscr{A}_{h}$.

If we are in a case where the ω_{λ} are continuous we can prove further results: defining: $L(\omega) = \{A \in \mathscr{A} \mid \omega(A^{*}A) = 0\}$ and $L(\omega_{\lambda}) = \{A \in \mathscr{A} \mid \omega_{\lambda}(A^{*}A) = 0\}$ we have $L(\omega) \subset L(\omega_{\lambda})$ a.e. and for the associated GNS representation: $\ker \pi \subset \ker \pi_{\lambda}$ a.e.

The proof comes from the fact that:

$$0 = \omega(A^{*}A) = \int \omega_{\lambda}(A^{*}A) d\mu(\lambda)$$

implies $\omega_{\lambda}(A^{*}A) = 0$, for every λ outside a null set depending on A. We get a common null set by considering the union of the null sets associated to all $A \in \mathscr{A}_{0}$. (This is still a null set since $\mathscr{A}_{0}$ is countable). For $A \in \mathscr{A} \backslash \mathscr{A}_0$, $\omega_\lambda (A^*A) = 0$ outside the same null set since ω_λ is continuous.

The proof is similar for the kernel of the representations.

ACKNOWLEDGMENTS. It is a pleasure to acknowledge here the useful help of M. Debacker about very subtle technical details of topology. I am also indebted to Dr. K. Schmüdgen whose comments about a previous version improved the present manuscript.

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Received July 12, 1979.

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