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TOPOLOGICAL ALGEBRAS WITH ORTHOGONAL SCHAUDER BASES

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Topological algebras with Schauder orthogonal bases are studied. Radicals, closed ideals and closed maximal ideals of such algebras are described. It turns out that a locally m-convex algebra with identity and having an orthogonal basis is metrizable. This implies that a complete locally m-convex algebra with an orthogonal basis and identity is algebraically and topologically isomorphic with the Fréchet algebra of all complex sequences.

Introduction. Let A be a topological algebra. A (Schauder) basis $\{x_n\}$ in A is called an *orthogonal* (*Schauder*) *basis* if $x_n x_m = \delta_{nm} x_n$, $n, m = 1, 2, \cdots$ where δ_{nm} denotes the Kronecker delta. Algebras with such bases (actually a variation of this definition which we will discuss below) were first studied by Husain. In [3] Husain and Liang proved that every multiplicative linear functional on a Fréchet algebra (i.e., complete metrizable locally m-convex algebra) with an unconditional orthogonal basis is continuous. This result answers Michael's question [5] (as to whether every multiplicative linear functional on a Fréchet algebra is continuous) in the affirmative for such Fréchet algebras.

In this paper we study the structure of topological algebras having an orthogonal Schauder basis. In § 1 we discuss some properties of bases in topological algebras which we will use later. In § 2 we describe the closed ideals and show that each closed ideal is the closure of the linear span of the basis elements it contains. In § 3 we give a characterization of complete locally m-convex algebras with identity having an orthogonal basis. In another paper [4] we study topological algebras having unconditional orthogonal bases.

For definitions and results concerning bases in Banach spaces see [1], [7]. For general notions regarding topological algebras see Michael [5] and Zelazko [8]. A sequence $\{x_n\}$ in a topological vector space E is a basis if for each $x \in E$ there is a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. Each linear functional $x_n^*(x) = \alpha_n$ is called a coefficient functional. If each x_n^* is continuous then $\{x_n\}$ is called a Schauder basis. It is well known that each basis in a complete metrizable vector space is a Schauder basis. We show that each orthogonal basis in a locally m-convex algebra is a Schauder basis (Prop. 3.1) and each unital locally m-convex algebra A with an orthogonal basis is metrizable (Theorem 3.3) and if, in addition

A is complete, then it is isomorphic and homeomorphic with the Fréchet algebra s of all complex sequences (Theorem 3.4). These results generalize results in [3] proved for Fréchet algebras.

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1. Orthogonal bases. In this section we consider the following conditions on a topological algebra A with a Schauder basis $\{x_n\}$:

(i) $x_n x_m = 0$ for $n \neq m$;

(ii) $x_n x_m = 0$ for $n \neq m$ and $x_n^2 \neq 0$;

(iii) $x_n x_m = 0$ for $n \neq m$ and $x_n^2 = c_n x_n$, $c_n \neq 0$;

(iv) $x_n x_m = \delta_{nm} x_n$.

In the sequel a basis satisfying the condition (iv) will be called an *orthogonal basis*. We start with some elementary results.

It is obvious that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). (ii) and (iii) are trivially equivalent. (iii) implies that one can replace $\{x_n\}$ by another base $\{y_n\}$ satisfying (iv). If A has an identity, then (i) $\Rightarrow$ (ii). Thus for a topological algebra with an identity for a base $\{x_n\}$ to be orthogonal, it is enough to assume that $\{x_n\}$ satisfies (i) because we can always replace $\{x_n\}$ by another basis $\{y_n\}$ which satisfies (iv). The proofs of these statements as well as that of the following are easy and therefore omitted.

LEMMA 1.1. *Let A be a topological algebra satisfying* (i).

(a) *If $x = \sum \alpha_i x_i$, $y = \sum \beta_i x_i$, then $xy = \sum \alpha_i \beta_i x_i^2$. Hence A is commutative.*

(b) *If $x_n^2 = x_n$ then the corresponding coefficient functional x_n^* is multiplicative.*

To describe the radical of a topological algebra with a basis satisfying (i), we first have the following:

LEMMA 1.2. *Let A be a topological algebra with a Schauder basis $\{x_n\}$ satisfying* (i) *and let D be any subset of $\{x_n\}$. Then, $x = \sum \alpha_i x_i$ belongs to $\overline{SpD}$ (closure of the linear span SpD of D) iff $\alpha_n = 0$ whenever $x_n \notin D$.*

Proof. Suppose that for some $n \in N$, $\alpha_n \neq 0$ and $x_n \notin D$. Let $x \in \overline{SpD}$, then there is a net $\{x_\lambda\}$ in SpD such that $x_\lambda \to x$. Since each coordinate functional x_n^* is continuous, we have $0 = \lim_\lambda x_n^*(x_\lambda) = x_n^*(x) = \alpha_n \neq 0$, a contradiction. Conversely, if $x = \sum \alpha_i x_i$ and $\alpha_n = 0$ whenever $x_n \notin D$, then clearly $S_k(x) = \sum_{i=1}^k \alpha_i x_i \in SpD$ for all $k \in N$. Whence we have $x = \lim_{k\to\infty} S_k(x) \in \overline{SpD}$.

LEMMA 1.3. *Let A be a topological algebra with a Schauder basis $\{x_n\}$ satisfying* (i). *The following are equivalent*:

(a) $x_n \in \operatorname{Rad} A$.

(b) $x_n^2 \in \overline{Sp}\{x_k : x_k^2 = 0\}$.

(c) $x_n^*(x_n^2) = 0$.

(d) $x_n^3 = 0$.

Proof. Let $x_n^2 = \sum_{k=1}^{\infty} \alpha_k x_k$, then $0 = x_k x_n^2 = \alpha_k x_k^2$ and so $\alpha_k = 0$ whenever $x_k^2 \neq 0$. Thus, by Lemma 1.2, $\alpha_n = x_n^*(x_n^2) = 0$ iff $x_n^2 \in \overline{Sp}\{x_k : x_k^2 = 0\}$. This proves (b) $\Leftrightarrow$ (c). For (c) $\Rightarrow$ (d), note that

$$x_n^3 = x_n x_n^2 = x_n\left(\sum_{k=1}^{\infty} x_k^*(x_n^2) x_k\right) = x_n^*(x_n^2) x_n^2 = 0 .$$

Since (d) $\Rightarrow$ (a) is obvious, it remains to show that (a) $\Rightarrow$ (b). Suppose $x_n^2 \notin \overline{Sp}\{x_k : x_k^2 = 0\}$, then by Lemma 1.2, there exists k_0 such that $\alpha_{k_0} \neq 0$ and $x_{k_0}^2 \neq 0$. By the first sentence in this proof $k_0 = n$. Thus $\alpha_n \neq 0$, whence the sequence $\{y_k\}$ where $y_k = x_k$, $k \neq n$, and $y_n = x_n/\alpha_n$, is a basis for A, and $y_n^2 = x_n^2/\alpha_n^2 = \sum_{k=1}^{\infty} (\alpha_k/\alpha_n^2) x_k$, so $y_n^*(y_n^2) = 1$ and $y_n^*(y_k^2) = 0$ for $k \neq n$. Now, for

$$\begin{aligned} x, y \in A,\ & y_n^*(xy) \\ &= y_n^*\left(\sum_{k=1}^{\infty} y_k^*(x) y_k^*(y) y_k^2\right) = y_n^*(x) y_n^*(y) y_n^*(y_n^2) = y_n^*(x) y_n^*(y) . \end{aligned}$$

Thus y_n^* is a continuous multiplicative linear functional and $y_n^*(y_n) = 1 \neq 0$. It follows that $y_n \notin \operatorname{Rad} A$, hence the same is true for x_n.

REMARK. From the above proof we note that in general, for a basis element x_n, either $x_n^3 = 0$ or $x_n^3 = c_n x_n^2$, $\alpha \in \boldsymbol{C}$. By a suitable transformation $\{x_n\}$ can be "normalized" to a basis $\{y_n\}$ so that $y_n^3 = y_n^2$ and by the above proof y_n^* is then multiplicative.

THEOREM 1.4. $\operatorname{Rad} A = Sp\{x_n : x_n$ *satisfies one of the equivalent conditions in Lemma* 1.3$\}$. *In particular,* $\operatorname{Rad} A = \{x \in A : x^3 = 0\}$.

Proof. Let $D = \{x_n : x_n^3 = 0\}$. We show $\operatorname{Rad} A = \overline{SpD}$. To this end let $x \in \overline{SpD}$, $x = \sum \alpha_k x_k$. By Lemma 1.2, $\alpha_k \neq 0$ iff $x_k^3 = 0$, so $x^3 = \sum_{k=1}^{\infty} \alpha_k^3 x_k^3 = 0$, hence $x \in \operatorname{Rad} A$. Conversely, if $x \notin \overline{SpD}$, then there exists k_0 such that $\alpha_{k_0} \neq 0$ and $x_{k_0}^3 \neq 0$. By choosing a basis as in (a) $\Rightarrow$ (b) of Lemma 1.3, we show that $x \notin \operatorname{Rad} A$.

COROLLARY 1.5. *A topological algebra with an orthogonal Schauder basis is semisimple.*

COROLLARY 1.6. *An F-algebra with an unconditional (see* [1]*) orthogonal basis has unique F-algebra topology.*

Proof. By Theorem 4 of [3], A is functionally continuous, commutative (Lemma 1.1) and semisimple (Corollary 1.5). By a Theorem of Michael [5, p. 62], A has unique F-algebra topology.

PROPOSITION 1.7. *If A is a topological algebra with a Schauder basis satisfying* (i), *then $A/\mathrm{Rad}\, A$ has an orthogonal basis.*

Proof. If D is any subset of $\{x_i\}$ and $\{x_{i_k}\}$ is the sequence of basis elements complementary to D, then the sequence $\{\eta(x_{i_k})\}$, where $\eta: A \to A/\overline{SpD}$ is the canonical map, is a basis for $A/\overline{SpD}$. (This is proved in [7, Prop. 4.1] for Banach spaces but the theorem is true for Schauder bases in any TVS by a slight modification of the proof there). Now, if D is as in the proof of Theorem 1.4, then $\mathrm{Rad}\, A = \overline{SpD}$ and a simple verification shows that the basis $\{\eta(x_{i_k})\}$ can be modified to yield an orthogonal basis for $A/\mathrm{Rad}\, A$.

We end this section by showing that an orthogonal basis in a topological algebra A is "essentially unique". Precisely we have:

THEOREM 1.8. *If $\{x_i\}$ and $\{y_i\}$ are orthogonal bases in a topological algebra A, then $\{x_i\} = \{y_i\}$.*

Proof. Let $x_n \in \{x_i\}$. There exists $y_m \in \{y_i\}$ such that $x_n y_m \neq 0$. For, otherwise it follows that $x_n = 0$, which is impossible. Now writing $x_n = \sum \alpha_i y_i$ and multiplying it by y_m, we obtain $x_n y_m = \alpha_m y_m$ which, if multiplied by x_n, yields $x_n y_m = \alpha_m x_n y_m$. This implies that $\alpha_m = 1$ and so $x_n y_m = y_m$. Now writing $y_m = \sum \beta_i x_i$, by similar arguments we get $x_n y_m = x_n$, whence $x_n = y_m$. This proves that $\{x_i\} \subset \{y_i\}$ and the result follows by symmetry.

2. Closed ideals. Throughout this section A will denote a topological algebra with an orthogonal Schauder basis $\{x_n\}$. Also for each x_n^*, let $M_n = \{x \in A: x_n^*(x) = 0\}$ be its kernel.

THEOREM 2.1. *If I is a closed ideal in A, then there exists $n \in N$ such that $I \subseteq M_n$. In particular, $\{M_n: n \in N\}$ is the set of all closed maximal ideals of A and this set with the Gelfand Topology is homeomorphic to N.*

Proof. If $I \subseteq M_n$, $n \geqq 1$, then for each $n \in N$, there exists $x \in I$ with $x_n^*(x) \neq 0$. Since $x_n^*(x)^{-1} x_n x = x_n$ we have $x_n \in I$ for all $n \geqq 1$. Thus I is dense in A, contradicting the assumption that I is closed.

To see that this set is discrete, consider the subbasic neighborhood V of x_n^*,

$$V = V\left(\frac{1}{2}, x_n, x_n^*\right) = \left\{ x_k^* : |x_k^*(x_n) - x_n^*(x_n)| \leqq \frac{1}{2} \right\} = \{x_n^*\} ,$$

the last equality being true because $x_k^*(x_n) = \delta_{kn}$.

For $I \subset A$, set $Z(I) = \{n \in N : x_n^*(x) = 0 \text{ for all } x \in I\}$ and write $Z(x)$ for $Z(\{x\})$. Also let $K = \{n : x_n \in I\}$. With this notation Lemma 1.2 says that $x \in \overline{Sp}\{x_n : n \in K\}$ iff $N \backslash K \subseteqq Z(x)$.

THEOREM 2.2. *Let I be a closed ideal in A. Then*

(a) $Z(I) = N \backslash K$.

(b) $I = \overline{Sp}\{x_n : n \in K\}$
$= \cap \{M_n : n \in Z(I)\}$
$= \{x \in A : Z(I) \subseteqq Z(x)\}$.

Proof. (a) If $n \in Z(I)$, then $x_n^*(x) = 0$ for all $x \in I$. Thus $x_n \notin I$ and so $n \notin K$. Conversely, if $n \notin Z(I)$, then $x_n^*(x) \neq 0$ for some $x \in I$. Now $x_n x = x_n^*(x) x_n$, so $x_n \in I$, whence $n \in K$.

(b) Since I is closed, $\overline{Sp}\{x_n : n \in K\} \subset I$. If $x \in I$, then by (a), $N \backslash K \subseteqq Z(x)$ and so by Lemma 1.2, $x \in \overline{Sp}\{x_n : n \in K\}$. The other two equalities follow from this and Lemma 1.2.

REMARK. In view of the proof of Proposition 1.7, the first equality of part (b) of the above proposition shows that for any closed ideal I of A, A/I has an orthogonal basis.

COROLLARY 2.3. *Let $x \in A$ and let $I = \overline{\langle x \rangle}$, the closure of the principle ideal generated by x. Then*

(a) $Z(I) = \{n : x_n^*(x) = 0\}$.

(b) $I = \overline{Sp}\{x_n : n \in N \backslash Z(I)\}$
$= \cap \{M_n : n \in Z(I)\}$
$= \{y \in A : Z(x) \subset Z(y)\}$.

Proof. The ideal $\langle x \rangle$ contains exactly those basis elements x_n for which $x_n^*(x) \neq 0$. Whence $\overline{\langle x \rangle}$ contains exactly those same basis elements also. Now (a) follows from part (a) of Theorem 2.2. Part (b) follows directly from Theorem 2.2(b).

3. Locally *M*-convex algebras. In this section we generalize some results in [3]. In particular we give a characterization of complete locally m-convex algebras with an orthogonal basis.

PROPOSITION 3.1. *Each orthogonal basis in a locally m-convex algebra (cf:* [5]) *A is a Schauder basis.*

Proof. Let $\{x_n\}$ be an orthogonal basis in A and let $\{p_\alpha\}$ be a family of submultiplicative seminorms generating the topology of A. For $x \in A$, $x = \sum x_n^*(x)x_n$ we have $xx_n = x_n^*(x)x_n^2 = x_n^*(x)x_n (n \geqq 1)$ and so for each p_α and n, $|x_n^*(x)|p_\alpha(x_n) \leqq p_\alpha(x)p_\alpha(x_n)$. Since A is Hausdorff, there exists p_β such that $p_\beta(x_n) \neq 0$, with this p_β from the above inequality we get $|x_n^*(x)| \leqq p_\beta(x)$, $x \in A$, which proves the continuity of x_n^* for each $n \geqq 1$.

REMARK. Note that if f is any multiplicative linear functional on A with $f(x_n) \neq 0$ for some $n \geqq 1$, then by the arguments used in the proof of Theorem 2.1 we get that $f = x_n^*$. Hence f is continuous by the above proposition. This is known for Fréchet algebras [3].

Let E be a topological vector space with a basis $\{x_n\}$. We define a map σ from E into the space s of all complex sequences by $\sigma(x) = \{x_n^*(x)\}_{n=1}^\infty$, $x \in A$.

LEMMA 3.2. *Let A be a locally m-convex algebra with an orthogonal basis $\{x_n\}$ and let P be a family of submultiplicative seminorms generating the topology of A. Consider the following statements*:

(a) *$\sigma: A \to s$ is surjective.*

(b) *A has an identity.*

(c) *For each $p \in P$ there exists $N \in N$ such that $p(x_n) = 0$ whenever $n > N$.*

Then (1): (a) *implies each of* (b) *and* (c), *and* (b) *implies* (c); (2): *if A is complete, these statements are equivalent.*

Proof. The proof of this lemma follows from the proofs of Propositions 1 and 3 of [3] if one replaces the sequence of seminorms by a family of seminorms.

THEOREM 3.3. *Let A be a locally m-convex algebra with an orthogonal basis $\{x_n\}$. If A has an identity, then A is metrizable.*

Proof. For $p \in P$, let $K_p = \{n: p(x_n) = 0\}$ and for $p, q \in P$ define pRq iff $K_p = K_q$. Note that R is an equivalence relation and since each set K_p is cofinite in N (Lemma 3.2) there can be at most a countable number of R-classes. Let pRq. Clearly $\ker p = \{x \in A: p(x) = 0\}$ is a closed ideal in A and so by Theorem 2.2(b), $\ker p = \overline{Sp}\{x_n: n \in K_p\}$ from which it follows that $\ker p = \ker q$ and is of finite codimension in A. Thus $A/\ker p = A/\ker q$ is finite dimensional so p and q induce equivalent norms on it. Hence p and q are equivalent seminorms on A.

We have the following theorem which generalizes Theorem 1 of [3].

THEOREM 3.4. *Let A be a complete locally m-convex algebra with an orthogonal basis $\{x_n\}$ and let P be a family of seminorms generating the topology of A. The following are equivalent:*

(a) *A has an identity.*

(b) *σ is onto s.*

(c) *for every $p \in P$, $p(x_n) = 0$ for all sufficiently large n.*

(d) *A is algebraically and topologically isomorphic to s.*

4. Examples. We conclude this paper by giving several examples.

EXAMPLE 1. The Banach algebras $l^p(N) = \{\{\alpha_i\} \in s: \sum |\alpha_i|^p < \infty\}$ $1 \leqq p < \infty$; c_0, the algebra of complex sequences converging to 0, and the Fréchet algebra s of all complex sequences (all with pointwise operations) have the sequence $e_n = (\delta_{nm})_{m=0}^{\infty}$, $n \geqq 1$ as a basis. Clearly this basis is orthogonal in our sense.

EXAMPLE 2. The space $L^p(T)$, $1 < p < \infty$ is a Banach algebra with convolution multiplication (see [8]). The sequence of trigonometric polynomials $e_n(t) = t^n$, $t \in T$, $n \geqq 1$, is an orthogonal basis for $L^p(T)$, where T is the circle group.

EXAMPLE 3. The Hardy spaces $H^p(D)$, $1 < p < \infty$, where D is the open unit disc in $\boldsymbol{C}$ are Banach algebras with the product

$$(f*g)(x) = \frac{1}{2\pi i}\int_{|z|=r} f(z)g(xz^{-1})z^{-1}dz ,$$

where $f, g \in H^p$ and $|x| < r < 1$ [6]. The sequence $e_n(x) = x^n$, $x \in D$, is a basis for H^p and a simple computation shows that it is an orthogonal basis with respect to the above product.

Let E be a Banach space with an unconditional basis $\{x_i\}$. For $x, y \in E$, $x = \sum \alpha_i x_i$, $y = \sum \beta_i x_i$, define $x * y = \sum \alpha_i \beta_i x_i$. This definition makes sense because, without loss of generality, assume that the basis $\{x_i\}$ is normalized (i.e., $\|x_i\| = 1$, $i \geqq 1$). Then $\lim_{i\to\infty} \alpha_i = 0$ [6]. Thus the sequence $\{\alpha_i\}$ is bounded and therefore, since $\sum \beta_i y_i$ converges unconditionally (hence is bounded multiplier convergent [1]), it follows that $\sum \alpha_i \beta_i x_i$ converges in E. Thus x^*y is a well defined element of E for $x, y \in E$. Moreover, it is clear that A is an algebra with this product. More is true:

PROPOSITION 4.1. *If E is a Banach space with an unconditional basis, then E is a Banach algebra (in an equivalent norm) with $*$product and the basis is orthogonal.*

Proof. Without loss of generality assume that the basis is normalized and let $||\cdot\cdot||_0$ be the norm on E given by $||x||_0 = \sup_{n\in N}|x_n^*(x)|$. Then with the map $\sigma: E \to m$ (the Banach space of bounded sequence with the sup norm $||\cdots||_\infty$) defined by $\sigma(x) = \{x_n^*(x)\}_{n=1}^\infty$, we have for each f_k, the kth coefficient functional on m,

$$(f_k\circ\sigma)(x) = f_k(\sigma(x)) = f_k(\{x_n^*(x)\}_{n=1}^\infty) = x_k^*(x) .$$

Since E is a Banach space, the functionals x_n^* are continuous [7]. Hence each $f_k\circ\sigma$ is continuous on E for each $k \geqq 1$. Since the family $\{f_k\}$ is a separating family of continuous linear functionals on m, it follows by the closed graph theorem [2] that σ is continuous. Therefore, there exists $c_1 > 0$ such that $||\sigma(x)||_\infty \leqq c_1||x||$, $x \in E$. Since $||x||_0 = ||\sigma(x)||_\infty$, we have $||x||_0 \leqq c_1||x||$. Now define

$$|||x||| = \sup_{f\in E', ||f||\leqq 1}\left[\sum_{n=1}^\infty |x_n^*(x)||f(x_n)|\right],$$

where E' is the topological dual of E. It is easy to show [7, p. 463] that $|||\cdots|||$ is a norm on E equivalent to the original norm of E. Hence, there is a constant $c_2 > 0$ such that $||x|| \leqq c_2|||x|||$, $x \in E$. Now

$$\begin{aligned} |||x*y||| &= \sup_{f\in E', ||f||\leqq 1}\left[\sum_{n=1}^\infty |x_n^*(x)x_n^*(y)||f(x_n)|\right] \\ &\leqq ||x||_0 \sup_{f\in E', ||f||\leqq 1}\left[\sum_{n=1}^\infty |x_n^*(y)||f(x_n)|\right] \\ &\leqq c_1||x||\ |||y||| \\ &\leqq c_1c_2|||x|||\,|||y||| . \end{aligned}$$

This shows that E is a Banach algebra in a norm equivalent to $|||\cdots|||$ [8]. Finally, it is clear that the basis $\{x_n\}$ has the property: $x_n*x_m = \delta_{nm}x_n$, $n, m \geqq 1$.

REMARK. We note here that an infinite dimensional normed algebra A with an orthogonal basis $\{x_n\}$ cannot have an identity e. For, if $e \in A$, $e = \sum_{k=1}^\infty x_k$ converges, hence $||x_n|| < 1$ for sufficiently large n. Thus, $||x_n|| = ||x_n^k|| \leqq ||x_n||^k$ for all $k \geqq 1$. So, $||x_n|| = 0$ which is impossible since $x_n \neq 0$.

We now give an example of a topological algebra with an orthogonal basis which is not a Banach algebra.

EXAMPLE 4. Let $H(D)$ be the F-space (with the compact-open topology) of all functions holomorphic in the open unit disc D. $H(D)$ is a topological algebra with identity 1 with the product

$$(f*g)(x) = \frac{1}{2\pi i}\int_{|z|=r} f(z)g(xz^{-1})z^{-1}dz\ ,$$

where $x \in D$, and $|x| < r < 1$ [6]. The sequence of functions $\xi_n(z) = z^n$, $z \in D$, $n \geqq 0$ is a basis for $H(D)$. A simple computation shows that this basis is an orthogonal basis. Note that $H(D)$ cannot be locally m-convex in view of Theorem 3.4, since it is not s.

Finally, we note that if A and B are topological algebras with orthogonal bases then the product basis [7, p. 28] in $A \times B$ is an orthogonal basis as can easily be checked (note that the construction of product bases for Banach spaces given in [7] has a natural extension to topological vector spaces and the proofs are similar to the Banach space case). Also, if A and B are Banach algebras with orthogonal bases then it is easy to check that the tensor product of these bases [7, p. 173] in $A \otimes_p B$, the projective tensor product of A and B, is an orthogonal basis for the Banach algebra $A \otimes_p B$.

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