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DIMENSION MODULES

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M is called a dimension module if $d(A+B)=d(A)+d(B)-d(A\cap B)$ holds for all submodules A and B of M, where $d(M)$ denotes the Goldie (uniform) dimension of a module M. We characterize these modules as the modules which have no submodules of the form $X\oplus X/Y$ with Y an essential submodule of X. As a test, the structure of a completely decomposable injective dimension module is determined.

A sum $A + B$ of submodules of a module M need not satisfy the usual vector space dimension formula $d(A + B) = d(A) + d(B) - d(A \cap B)$, where $d(M)$ denotes the Goldie dimension of M (that is, $d(M)$ is the number of components in a longest direct sum of submodules contained in M, and is ∞ if no such direct sum exists). This was noted by the authors in [1], where the following substitute formula was proved for arbitrary modules.

THEOREM (*Dimension Formula I*). *Let A and B be submodules of a module M. Let $C = A \cap B$ and let 1_C denote the identity map on C. Let g be a maximal monic extension of 1_C considered as a partial homomorphism from A to B, and let D be the domain of g. Then*

$$d(A + B) = d(A) + d(B) - d(D) + d(D/C)\ .$$

In this paper, we study modules whose submodules satisfy the usual vector space dimension formula itself; these we call *dimension modules*. This class turns out to be somewhat larger than we had originally anticipated. It includes, for instance, all nonsingular modules and all modules whose lattice of submodules is distributive. These examples are obtained in § 1 from a characterization of dimension modules which arises, in turn, from a revision of the dimension formula. In § 2 we show that maximal essential dimension extensions of dimension modules exist. The article concludes in § 3 with a study of injective dimension modules and direct sums of dimension modules. In an appendix, $d(M)$ is compared with the reduced rank $\rho(M)$ of a module M over a right noetherian ring (ρ does satisfy the classical dimension formula).

1. Dimension modules.

We begin with some notation and definitions. All symbols $A, B, M, N, X, Y, \cdots$ indicate modules over an arbitrary ring R. $A \leq B$ means that A is a submodule of B, and $A \trianglelefteq B$ indicates that A is an essential submodule of B. A *partial homomorphism* from A to B is a homomorphism from a submodule of A to B. By a *uniform* module we means a module of dimension equal to 1; alternatively, a uniform module is one in which any two nonzero submodules have nonzero intersection. For N a submodule of M, we let $\bar{N}$ denote a maximal essential extension of N in M. Although $\bar{N}$ is not necessarily unique, this will cause no ambiguity in the sequel. When $N = \bar{N}$, we say N is *closed* in M.

Perhaps the simplest example of a module which is not a dimension module is the abelian group $M = \boldsymbol{Z}_{p^2} \oplus \boldsymbol{Z}_p$. The subgroups $A = \boldsymbol{Z}_{p^2}$ and $B = \{(m, m) \mid 0 \leqq m < p^2\} \cong \boldsymbol{Z}_{p^2}$ are both uniform, and $A + B = M$, $A \cap B = p\boldsymbol{Z}_{p^2}$. But $d(A + B) = 2$ while $d(A) + d(B) - d(A \cap B) = 1 + 1 - 1 = 1$. The surprising fact is that this example is generic, as the next proposition reveals.

PROPOSITION 1. *A module M is a dimension module if and only if for every partial endomorphism $f\colon A \to M$ with $fA \cap A = 0$, kernel f is closed in A.*

In the example $M = \boldsymbol{Z}_{p^2} \oplus \boldsymbol{Z}_p$ above, the partial endomorphism $f\colon \boldsymbol{Z}_{p^2} \to M$ via $f(a, 0) = (0, a)$ is the culprit that causes M to fail to be a dimension module. The following rephrasal of the proposition puts the situation in a clearer perspective.

COROLLARY 2. *M fails to be a dimension module precisely when it has a submodule isomorphic to $X \oplus X/Y$ for some $Y \ntrianglelefteq X$.*

Before proving Proposition 1 we state a revision of the dimension formula, which contains a more explicit error term, as compared to the dimension formula for vector spaces. Our proof of the revised formula utilizes the original version.

THEOREM. (*Dimension Formula II.*) *Let A and B be submodules of a module M. Let $1_{A \cap B}$ denote the identity map on $A \cap B$ considered as a partial homomorphism from A to B, and let $f\colon E \to B$ be a maximal extension of $1_{A \cap B}$ such that $A \cap B \trianglelefteq E \leqq A$. Then*

$$d(A + B) = d(A) + d(B) - d(A \cap B) + d(E/A \cap B)\ .$$

Proof. Such a pair f, E exists by a standard application of Zorn's lemma; and f must be a monomorphism because $A \cap B \trianglelefteq E$. Now choose a maximal monic extension g of f considered as a partial homomorphism from A to B, and let $D \geqq E$ be the domain of g. From the original dimension formula we have

(1) $d(A + B) = d(A) + d(B) - d(D) + d(D/A \cap B)$.

Next, Lemma 3 of [1] asserts that for any modules $X \leqq Y$

(2) $d(Y/X) = d(Y) - d(X) + d(\bar{X}/X)$,

where $\bar{X}$ is any maximal essential extension of X in Y. In our situation, E is a maximal essential extension of $A \cap B$ in D; else we would violate the choice of E. Hence applying the preceding formula we get

(3) $d(D/A \cap B) = d(D) - d(A \cap B) + d(E/A \cap B)$.

Finally, putting (1) and (3) together yields

$$d(A + B) = d(A) + d(B) - d(A \cap B) + d(E/A \cap B) .$$

Proof of Proposition 1. In view of the revised dimension formula, M will be a dimension module if and only if for every A, $B \leqq M$, $A \cap B$ is closed in the domain of any monic extension of $1_{A \cap B}$ regarded as a partial homomorphism from A to B.

Now suppose that M is a dimension module and let $f: A \to M$ be a partial endomorphism of M with $fA \cap A = 0$. Setting $g = 1 - f: A \to M$, we have $A \cap gA =$ kernel f; so $g = 1$ on $A \cap gA$, and g is monic because $fA \cap A = 0$. g is therefore a maximal monic extension in A of $1_{A \cap gA}: A \cap gA \to gA$. By hypothesis then, $A \cap gA =$ kernel f is closed in A.

Conversely, assume the stated condition holds, and let f be a maximal monic extension in A of $1_{A \cap B}: A \cap B \to B$ with $D =$ domain f, A and B submodules of M. We must show that $A \cap B$ is closed in D. Set $g = 1 - f: D \to M$. Now $gD \cap D = 0$. (For if $gd = d'$ for some $d, d' \in D$, then $fd = d - d' \in A \cap B$, so $fd = 1_{A \cap B}(d - d') = f(d - d')$. Then $d = d - d'$ because f is monic, and so $d' = 0$.) By hypothesis kernel g is closed in D, and we are done since kernel $g = A \cap B$.

COROLLARY 3. *For R a ring, every R-module is a dimension module if and only if R is semisimple artinian.*

Proof. It suffices to prove that no module has a proper essential submodule. So let $B \trianglelefteq A$ and consider $M = A \oplus A/B$. By Corollary 2, M cannot be a dimension module unless $B = A$.

We remark that the above proof actually shows that if $A \oplus A/B$ is a dimension module for every $B \trianglelefteq A$ then A is a semisimple module.

The rest of this section will be devoted to producing examples of dimension modules. Recall that an R-module is called *nonsingular* if the annihilator of each nonzero element is not essential in R.

PROPOSITION 4. *A nonsingular module is a dimension module.*

Proof. We use Corollary 2. Suppose that M is nonsingular and contains a submodule $A \oplus A/B$ with $B \trianglelefteq A$. Then each element of A/B has an essential annihilator in R. So $A/B = 0$, and M is therefore a dimension module.

PROPOSITION 5. *An abelian group is a dimension group if and only if it is one of the following types: torsion-free, or a torsion group whose p-primary component for each prime p is isomorphic to either a subgroup of $\boldsymbol{Z}_{p^\infty}$, or a direct sum of copies of $\boldsymbol{Z}_p$.*

Proof. The question is, which abelian groups do not contain subgroups of the form $A \oplus A/B$ with $B \trianglelefteq A$? Now any mixed group contains copies of $\boldsymbol{Z}$ and $\boldsymbol{Z}/n\boldsymbol{Z}$ for some $n > 1$, and these have zero intersection. Hence a dimension group must be either torsion or torsion-free. Since the torsion-free groups are precisely the nonsingular groups, we need only discover the torsion dimension groups.

So assume that $A = \bigoplus_p A_p$ is a torsion group with p-primary component A_p. Since $B = \bigoplus_p (B \cap A_p)$ for any subgroup B of A we may assume that $A = A_p$ is p-primary. Let $A(p)$ be the subgroup of elements of order p in A; $A(p)$ is an essential subgroup of A, if $|A(p)| = p$, then A is isomorphic to a subgroup of $\boldsymbol{Z}_{p^\infty}$, a uniform group, and so is trivially a dimension group. If $A(p) = A$ then A is a dimension group since $A(p)$ is a vector space over $\boldsymbol{Z}_p$. So we are left with the possibility that $|A(p)| > p$ but $A(p) \neq A$. Choose any element $a \in A \backslash A(p)$; a has order p^n for some $n > 1$. Then $\boldsymbol{Z}a \cap A(p)$ has order p, so we may choose an element $b \in A(p)$, $b \notin \boldsymbol{Z}a$. Then $\boldsymbol{Z}a \cap \boldsymbol{Z}b = 0$ and $\boldsymbol{Z}b \cong \boldsymbol{Z}a/p^{n-1}\boldsymbol{Z}a$, so A fails to be a dimension group by Corollary 2.

Our final source of examples is the class of distributive modules, where a module is called *distributive* if for any trio of submodules A, B, C, $A \cap (B + C) = (A \cap B) + (A \cap C)$.

PROPOSITION 6. *A distributive module is a dimension module.*

Proof. Without loss of generality we may assume that we have a distributive module of the form $M = A \oplus A/B$ with $B \trianglelefteq A$, and our task is to prove that then necessarily $B = A$. Set $N = A/$

$B \oplus A/B$, which being a homomorphic image of M is also a distributive module. On the other hand, define the following submodules of N: $L = \{(x + B, x + B) | x \in A\}$, $A_1 = A/B \oplus 0$, $A_2 = 0 \oplus A/B$. Then $L = L \cap (A_1 + A_2) = (L \cap A_1) + (L \cap A_2) = 0$, and so $B = A$.

2. Maximal dimension modules.

Our basic result here shows that an ascending union of dimension modules is a dimension module. Thus every module contains maximal dimension submodules.

PROPOSITION 7. *Let M_i be an ascending chain of dimension modules. Then $M = \cup M_i$ is also a dimension module.*

Proof. Let $f: A \to M$ be a partial endomorphism of M with $fA \cap A = 0$, and suppose kernel $f \trianglelefteq B$ for some submodule B of A; we must show that then kernel $f = B$. For any $b \in B$, choose M_i so that b and fb are in M_i. Then set $A_i = (\text{kernel}\, f \cap M_i) + Rb \subseteq M_i$. $f_i = f|_{A_i}$ is a partial endomorphism of M_i, and kernel $f_i \trianglelefteq A_i$. So by hypothesis, kernel $f_i = A_i$. It follows that $fb = 0$, and since $b \in B$ was arbitrary, kernel $f = B$.

We can now give a characterization of the essential extensions of a given module which are dimension modules. For a module M we let $E(M)$ denote the injective hull of M, and we define $\mathscr{S}(M) = \{X \leqq E(M) | \text{if } Y \leqq X,\ f \in \text{End}\, E(M),\ fY \cap Y = 0 \text{ and } f(Y \cap M) = 0, \text{ then } fY = 0\}$.

THEOREM 8. (1) $M \in \mathscr{S}(M)$.

(2) *$\mathscr{S}(M)$ has maximal elements.*

(3) *If M is a dimension module and $X \in \mathscr{S}(M)$ then X is a dimension module.*

(4) *If $X \trianglelefteq E(M)$ and $X \notin \mathscr{S}(M)$ then X is not a dimension module.*

Proof. (1) is trivial.

For (2), suppose $X_1 \leqq X_2 \leqq \cdots$ is an ascending chain of elements of $\mathscr{S}$. Set $X = \cup X_i$ and let $f \in \text{End}\, E(M)$, $Y \leqq X$, with $fY \cap Y = 0$ and $f(Y \cap M) = 0$. Let $Y_i = Y \cap X_i$. Then $fY_i \cap Y_i = 0$ and $f(Y_i \cap M) = 0$, so $f(Y_i) = 0$ since $X_i \in \mathscr{S}$. But $Y = \cup Y_i$, so $f(Y) = 0$.

Suppose that $X \in \mathscr{S}(M)$ is not a dimension module. Then there is a submodule Z of X and a homomorphism $f: Z \to X$ with $fZ \cap Z = 0$ and kernel f not closed in Z. Without loss of generality we may replace Z by $\overline{\text{kernel}\, f}$ in Z, and assume that kernel $f \trianglelefteq Z$. Now $fZ \cap M$ is an essential submodule of fZ. So the restriction of

$f: f^{-1}(fZ \cap M) \to fZ \cap M$ is a nonzero map. It then follows that $f(f^{-1}(fZ \cap M) \cap M) \neq 0$ because $f^{-1}(fZ \cap M) \leqq X$ and $X \in \mathscr{S}(M)$. Setting g equal to the restriction of f to $N = f^{-1}(fZ \cap M) \cap M$, we have that $gN \cap N = 0$ and kernel $g \trianglelefteq N$. So M is not a dimension module. This proves (3).

Now let X be as in (4). Since $X \notin \mathscr{S}(M)$ we may choose $f \in \operatorname{End} E(M)$ and $Y \leqq X$ with $fY \cap Y = 0$, $f(Y \cap M) = 0$, but $fY \neq 0$. Now consider f as a homomorphism: $Y \to fY$, and let f_1 be the restriction of f to $X_1 = f^{-1}(fY \cap M)$. $f_1 \neq 0$ because $fY \cap M \neq 0$. Now $X_1 \cap M \leqq Y \cap M \leqq$ kernel f, and $X_1 \cap M \trianglelefteq X_1$, so kernel $f_1 \trianglelefteq X_1$. Since $f_1 X_1 \cap X_1 = 0$ we learn that X is not a dimension module.

We now present an example to show that maximal essential dimension extensions of a module need not be unique; that is, the sum of two essential dimension extensions need not be a dimension module.

EXAMPLE 9. Choose R to be a commutative local ring with radical J, where $J^2 = 0$ and dimension $J = 2$. Then $J = \operatorname{Socle} R$, so we may write $J = S_1 \oplus S_2$ where each S_i is a simple module. R is itself a dimension module because each proper ideal of R is semisimple. Observe that for any $x \in J$ and $r \in R$, $rx = 0$ if and only if $x = 0$ or $r \in J$; and from this it is clear that S_1 and S_2 are isomorphic R-modules. We let $f: S_1 \to S_2$ and $g: S_2 \to S_1$ denote a pair of mutually inverse isomorphisms.

Set $E = E(J) = E(S_1) \oplus E(S_2)$. Now consider the monomorphisms $i_1, i_2: J \to E$ defined by $i_1 = 1$, $i_2(s_1 + s_2) = gs_2 + fs_1 (s_i \in S_i)$, and extend these to monomorphisms $i_1, i_2: R \to E$. Let $x = i_1(1)$, $y = i_2(1)$. Then $Rx \cong Ry \cong R$ are both dimension essential extensions of J in E. We claim, however, that their sum is not a dimension module.

First note that in $Rx + Ry$, $R(x + y) \cap S_1 = 0$. (For if $R(x + y) \cap S_1 \neq 0$ then $J(x + y) \cap S_1 \neq 0$ because $J(x + y) \trianglelefteq R(x + y)$. Choose $0 \neq s = (s_1 + s_2)(x + y)$ with $s, s_1 \in S_1$, $s_2 \in S_2$. Then $s = s_1 + s_2 + fs_1 + gs_2$, so $s - s_1 - gs_2 = s_2 + fs_1 \in S_1 \cap S_2 = 0$. Hence $fs_1 = -s_2$ and so $gs_2 = -s_1$. But then $s = s_1 + gs_2 = 0$, a contradiction.) Next, $Rx + Ry$ has dimension 2 so $R(x + y)$ must be uniform. We have a nonzero homomorphism $h: R(x+y) \to Rx + Ry$ defined by $h(r(x+y)) = rs_1$ for s_1 fixed in S_1. Also $h(R(x + y)) \cap R(x + y) \subseteqq S_1 \cap R(x+y) = 0$, and kernel $h \trianglelefteq R(x+y)$ because $J(x + y) \subseteqq$ kernel h. Hence by Proposition 1, $Rx + Ry$ fails to be a dimension module.

An interesting question, for which we do not have the answer, is whether a maximal essential dimension extension is unique up to isomorphism.

3. **Injective modules.** It is our intention in this section to develop conditions which force an injective module to be a dimension module. The first step is to give a condition based on possible direct sum decompositions of the injective as a sum of two submodules. We state an elementary lemma that will be used for this characterization.

LEMMA 10. *If* $A \trianglelefteq B = X \oplus Y$ *then* $X \cap A$ *is closed in* A.

Proof. If $X \cap A \trianglelefteq A_1 \leqq A$, then $X + A_1 = X \oplus Y_1$ where $Y_1 = (X + A_1) \cap Y$. Next $Y_1 \cap A \subseteqq (X + A_1) \cap{}' A = (X \cap A) + A_1 = A_1$. If $Y_1 \neq 0$, then $0 \neq Y_1 \cap A \subseteqq A_1$, so $0 \neq (Y_1 \cap A) \cap (X \cap A)$ because $X \cap A \trianglelefteq A_1$. But $(Y_1 \cap A) \cap (X \cap A) \subseteqq Y \cap X = 0$, a contradiction. Hence $Y_1 = 0$, and $A_1 \subseteqq X$. So $X \cap A = A_1$ and $X \cap A$ is closed in A.

THEOREM 11. *Let* E *be an injective module.* E *is a dimension module if and only if, whenever* $E = X \oplus Y$ *and* $h: X \to Y$ *then* h *splits.*

Proof. If E is a dimension module then by Proposition 1, kernel h is closed in X, hence is injective. So kernel h is itself a direct summand.

Conversely, let a partial endomorphism of E be given, $f: A \to E$ with $fA \cap A = 0$. Choose $B, C \leqq E$ injective hulls of A and fA, respectively. Then also $B \cap C = 0$, so $E = B \oplus C \oplus D$ for some submodule D. Let $\bar{f}: B \to C$ be an extension of f. Regarding $\bar{f}$ as a map: $B \to C \oplus D$, $\bar{f}$ must split by hypothesis; write $B =$ kernel $\bar{f} \oplus B_1$. Now apply Lemma 10 to learn that kernel $\bar{f} \cap A =$ kernel f is closed in A. By Proposition 1, E is a dimension module.

COROLLARY 12. *If* E *is an injective module and* Rad End $E = 0$ *then* E *is a dimension module.*

Proof. As is well known [3; 19.27], Rad End $E = \{f \in \text{End}\, E \mid \text{kernel}\, f \trianglelefteq E\}$. Suppose $E = X \oplus Y$ and $h: X \to Y$. Write $X = \overline{\text{kernel}\, h} \oplus X_1$ and define $\bar{h} \in \text{End}\, E$ by $\bar{h} = h$ on $\overline{\text{kernel}\, h}$ and $\bar{h}|_{X_1 \oplus Y} = 0$. Then kernel $\bar{h} \trianglelefteq E$, so $\bar{h} = 0$; that is, kernel $h = \overline{\text{kernel}\, h}$, and we are done by Theorem 11.

The previous corollary generalizes Proposition 4 in view of [3; 19.29].

Now over a Noetherian ring, every injective module is a direct sum of indecomposable modules. So we ask, what conditions must be placed on indecomposable injective summands in order that their direct sum be a dimension module? We can in fact answer this question for direct sums of arbitrary modules.

THEOREM 13. *Let* $M = \sum_{\alpha \in I} \oplus M_\alpha$. *Then* M *is a dimension module if and only if each* M_α *is a dimension module and every partial homomorphism between two distinct* M_α *has a closed kernel.*

The proof of this theorem is in fact quite complicated. We have therefore divided its proof among three lemmas.

LEMMA 14. *The following conditions are equivalent.*

(a) *Partial homomorphisms from* X *to* Y *have closed kernels.*

(b) *If* $A \cap X \leqq K \trianglelefteq A \leqq X \oplus Y$, *then* $K/A \cap X \trianglelefteq A/A \cap X$.

Proof. (a) $\Rightarrow$ (b) Assume that partial homomorphisms from X to Y have closed kernels and that there exist submodules $A \cap X \leqq K \trianglelefteq A \leqq X \oplus Y$. Suppose further that there is a submodule $A \cap X \leqq A_0 \leqq A$ with $A_0/(A \cap X) \cap K/(A \cap X) = 0$; that is, $A_0 \cap K = A \cap X$. We prove that $K/(A \cap X) \trianglelefteq A/(A \cap X)$ by showing that necessarily $A_0 = A \cap X$.

Set $X_0 = \{x \in X \mid$ there exists $y_x \in Y$ with $(x, y_x) \in A_0\}$. We first claim that the assignment $f(x) = y_x$ is a homomorphism from X_0 to Y. To see this it clearly suffices to prove that f is well defined; and this, in turn, is established provided we can show that $A_0 \cap Y = 0$. But if $A_0 \cap Y \neq 0$, then $K \cap A_0 \cap Y \neq 0$ because $K \trianglelefteq A$; and then using the fact that $A_0 \cap K = A \cap X$, we would have $0 \neq K \cap A_0 \cap Y \leqq X \cap Y = 0$, a contradiction.

Next observe that $\text{kernel} f = A_0 \cap X = A_0 \cap K \trianglelefteq A_0$ because $K \trianglelefteq A$; and it then follows that $\text{kernel} f \trianglelefteq X_0$. Since partial homomorphisms from X to Y have closed kernels by hypothesis, we have that kernel $f = X_0$. Hence $A_0 = A_0 \cap X \leqq A \cap X$ and therefore $A_0 = A \cap X$.

(b) $\Rightarrow$ (a) Assume (b), and let f be any partial homomorphism of X into Y. Choose X_0 to be a maximal essential extension of kernel f in the domain of f. Our task is to prove that $X_0 = \text{kernel} f$.

Set $A = \{(x, f(x)) \mid x \in X_0\}$, a submodule of $X \oplus Y$. Then $A \cap X = \text{kernel} f \leqq A \leqq X \oplus Y$, with $\text{kernel} f \trianglelefteq A$ because $\text{kernel} f \trianglelefteq X_0$. So from (b), with $K = A \cap X$, we learn that $0 \trianglelefteq A/(A \cap X)$. But this means that $A = A \cap X$, and hence that $f(X_0) = 0$. That is, $X_0 = \text{kernel } f$. This completes the proof of this lemma.

For notational convenience, let us write $X \mapsto Y$ to mean that partial homomorphisms from X to Y have closed kernels.

LEMMA 15. *If $X_i \mapsto X_j$ for any $i \neq j$ and $X_i \mapsto Y$ for all i, then $\sum_{i=1}^{n} \oplus X_i \mapsto Y$.*

Proof. Let $f \in \mathrm{Hom}_R(A, Y)$ where $A \leqq \sum_{i=1}^{n} \oplus X_i$. By replacing A with a maximal essential extension of kernel f in A, we may assume without loss of generality that kernel $f \trianglelefteq A$. We proceed by induction on n, the case $n = 1$ being trivially true.

For $n > 1$, set $Z = \sum_{i=2}^{n} \oplus X_i$. Then the restriction of f to $A \cap Z$ has an essential kernel, so by the inductive hypothesis we have that $f(A \cap Z) = 0$.

Let π_1 be the natural projection of $\sum_{i=1}^{n} \oplus X_i$ onto X_1. Then $A \cap \mathrm{kernel}\, \pi_1 = A \cap Z$ and $f(A \cap Z) = 0$, so π_1 induces a monomorphism $\pi: A/(A \cap Z) \to X_1$. Consider now the canonical epimorphism $p: A/(A \cap Z) \to A/\mathrm{kernel}\, f$. $A/(A \cap Z) \cong \mathrm{image}\ \pi \leqq X_1$ and $A/\mathrm{kernel}\, f \cong \mathrm{image}\ f \leqq Y$, so by hypothesis the kernel of p is closed in $A/(A \cap Z)$; that is, kernel $f/(A \cap Z)$ is closed in $A/(A \cap Z)$. On the other hand, $A \cap Z \leqq \mathrm{kernel}\, f \trianglelefteq A \leqq Z \oplus X_1$, and by our induction hypothesis partial homomorphisms from Z to X_1 have closed kernels. Hence by the previous lemma, kernel $f/(A \cap Z) \trianglelefteq A/(A \cap Z)$. From this it follows that kernel $f = A$, and the proof is complete.

LEMMA 16. *If X and Y are dimension modules with $X \mapsto Y$ and $Y \mapsto X$, then $X \oplus Y$ is a dimension module.*

Proof. Let $A \leqq X \oplus Y$ and let f be a homomorphism from A to $X \oplus Y$ with $A \cap fA = 0$. We must prove that kernel f is closed in A. As usual we can assume without loss of generality that kernel $f \trianglelefteq A$, and we must show that $f = 0$. We let π_X and π_Y denote the canonical projections of $X \oplus Y$ onto X and Y, respectively.

If $\pi_X f(A) = 0$ then $f(A) \subseteqq Y$, and we would be done by the previous lemma. So we may assume without loss of generality that $\pi_X f(A) \neq 0$, and also that $\pi_Y f(A) \neq 0$. Then either $A \cap \pi_X f(A) \ntrianglelefteq \pi_X f(A)$ or $A \cap \pi_Y f(A) \ntrianglelefteq \pi_Y f(A)$; else $A \cap (\pi_X f(A) \oplus \pi_Y f(A))$ would be essential in $\pi_X f(A) \oplus \pi_Y f(A)$, and would therefore have nonzero intersection with $f(A)$, contradicting the fact that $A \cap fA = 0$.

Thus we may suppose that $A \cap \pi_Y f(A) \leqq \pi_Y f(A)$, so that there exists $0 \neq Y_0 \leqq \pi_Y f(A)$ with $A \cap Y_0 = 0$. Setting $A_0 = (\pi_Y f)^{-1}(Y_0)$, $\pi_Y f|_{A_0}$ is a partial homomorphism from $X \oplus Y$ to Y. Hence by the preceding lemma, kernel $\pi_Y f|_{A_0}$ is closed in A_0. But kernel $\pi_Y f|_{A_0} \trianglelefteq A_0$ because kernel $f \trianglelefteq A$, and so $\pi_Y f(A_0) = 0$, contradicting $\pi_Y f(A_0) =$

$Y_0 \neq 0$. This contradiction establishes the fact that $f = 0$.

Proof of Theorem 13. From Corollary 2, it is clear that if M fails to be a dimension module then so does some finite direct sum of the M_α, $\alpha \in A$. Thus it suffices to prove that $\sum_{i=1}^{n} \oplus M_i$ is a dimension module whenever each M_i is a dimension module and partial homomorphisms between distinct M_i have closed kernels.

We proceed by induction on n. For $n \geqq 2$, we claim that $M_1 \mapsto \sum_{i=2}^{n} \oplus M_i$. For if f is a partial homomorphism from M_1 to $\sum_{i=2}^{n} \oplus M_i$ with an essential kernel, then f composed with the projection π_i onto M_i, $2 \leqq i \leqq n$, is a partial homomorphism from M_1 to M_i with an essential kernel. By hypothesis then, each $\pi_i f = 0$, from which it follows that $f = 0$.

Next, $\sum_{i=2}^{n} \oplus M_i \mapsto M_1$ from Lemma 15, and $\sum_{i=2}^{n} \oplus M_i$ is a dimension module by the induction hypothesis. One may now apply Lemma 16 to complete the proof.

COROLLARY 17. *Let $U = \sum_{\alpha \in I} \oplus U_\alpha$ where each U_α is a uniform module. Then U is a dimension module if and only if every nonzero partial homomorphism between two distinct U_α is a monomorphism.*

A module U is called *monoform* if each nonzero partial endomorphism of U is a monomorphism. We have the following immediate consequence of the previous corollary.

COROLLARY 18. *For a uniform module U the following conditions are equivalent.*

(a) *U is monoform.*

(b) *$U \oplus U$ is a dimension module.*

(c) *$U^{(I)} = \sum_I \oplus U$ is a dimension module for every index set I.*

One can actually show a somewhat stronger result for a monoform module U. Namely, that if $f: A \to U^{(t)}$ is a partial homomorphism of $U^{(s)}$ into $U^{(t)}$ then kernel f is closed in $U^{(s)}$. In particular, when $s = t$ one need not assume that $fA \cap A = 0$ to reach this conclusion. Our proof of this is lengthy and will therefore not be exhibited here.

We can now apply the previous results to a completely decomposable injective module.

THEOREM 19. *Let $E = \sum_{\alpha \in I} \oplus E_\alpha$ be an injective module with each E_α indecomposable. E is a dimension module if and only if,* *whenever there exists a nonzero homomorphism* $f: E_\alpha \to E_\beta$ *with* $\alpha \neq \beta$ *then* $E_\alpha \cong E_\beta$ *and* $\operatorname{End} E_\alpha$ *is a division ring.*

Proof. Suppose E is a dimension module and $0 \neq f: E_\alpha \to E_\beta$ with $\alpha \neq \beta$. By Corollary 17, f is a monomorphism. Since fE_α is an injective submodule of the indecomposable module E_β, f must be an isomorphism. Now suppose $0 \neq g \in \operatorname{End} E_\alpha$. If $\operatorname{kernel} g \neq 0$ we would obtain a nonzero homomorphism $E_\alpha \to E_\beta$ which is not an isomorphism. Hence $\operatorname{kernel} g = 0$ and it follows as above that g is an automorphism of E_α.

For the converse we use Theorem 11. Let a homomorphism $h: X \to Y$ be given where $E = X \oplus Y$. It suffices to show that $\operatorname{kernel} h$ is closed in X when $\operatorname{kernel} h \neq 0$. Write $X = \overline{\operatorname{kernel} h} \oplus X_1$. Now $\overline{\operatorname{kernel} h}$ and Y are themselves direct sums of indecomposable injective modules, and by the Krull-Schmidt-Azumaya Theorem [3; 21.14] their indecomposable summands are isomorphic to the E_α's. If we restrict h to an indecomposable summand of $\overline{\operatorname{kernel} h}$, the restriction $h|$ is not monic since $\operatorname{kernel} h \trianglelefteq \overline{\operatorname{kernel} h}$. By hypothesis, $h|$ followed by projection onto any indecomposable summand of Y must be zero, hence $h = 0$ on $\overline{\operatorname{kernel} h}$; that is, $\operatorname{kernel} h = \overline{\operatorname{kernel} h}$ and we are done.

Since we have determined when a finite direct sum of uniform modules is a dimension module, it would be interesting to solve the corresponding problem for finite dimensional modules; that is, for essential extensions of the class of modules already known. Although this problem seems quite difficult we are able to determine a certain extension of a direct sum of uniform modules, similar to the maximal rational extension, which is a dimension module if the direct sum of uniforms is a dimension module.

PROPOSITION 20. *Let* $U = \sum_{\alpha \in I} \oplus U_\alpha$ *be a dimension module where the* U_α *are uniform. Let* $E(U_\alpha)$ *denote the injective hull of* U_α *and set* $X_\alpha = \bigcap \operatorname{kernel} f$, *the intersection taken over all* $f \in \operatorname{Hom}(E(U_\alpha), E(U_\beta))$ *with* $\alpha \neq \beta$ *and* $fU_\alpha = 0$. *Then* $\sum_{\alpha \in I} \oplus X_\alpha$ *is a dimension module.*

Proof. The proof follows easily using Corollary 17. For let f be a partial homomorphism from X_α to X_β, $\alpha \neq \beta$. Extend f to $\bar{f} \in \operatorname{Hom}(E(U_\alpha), E(U_\beta))$. If $\bar{f}U_\alpha \neq 0$, then $\bar{f}$ gives rise by restriction to a nonzero partial endomorphism from U_α to U_β, which must perforce be a monomorphism. In this case $\bar{f}$ and therefore f must also be monic. If, on the other hand, $\bar{f}U_\alpha = 0$, then by the definition of

X_α, $fX_\alpha = \bar{f}X_\alpha = 0$. Thus f must be zero or monic.

4. Appendix. This appendix on reduced rank was added at the suggestion of the referee.

Let R be a right Noetherian ring with nilpotent radical N. The reduced rank, $\rho(M)$, of an R-module M was defined in [4] as follows.

If $MN = 0$, $\rho(M)$ is the Q-composition length of $M \otimes_{R/N} Q$ where Q is the quotient ring of R/N; while if M is an arbitrary module, $\rho(M) = \sum_{i \geq 0} \rho(MN^i/MN^{i+1})$. It is well known that

$$(1) \qquad \rho(X/Y) = \rho(X) - \rho(Y)\,,$$

so that in this respect ρ behaves like composition length. This fact is applied quite cleverly in [2] to several aspects of noncommutative ring theory.

It is a folklore result that if $N=0$ then $\rho(M)=d(M)-d(Z(M))$, where $Z(M)$ is the singular submodule of M. The question is, what is the relationship between reduced rank and the results in this paper?

First observe that

$$(2) \qquad \rho(A + B) = \rho(A) + \rho(B) - \rho(A \cap B)\,.$$

In fact, if ρ is any function satisfying (1) and $\rho(X \oplus Y) = \rho(X) + \rho(Y)$, then

$$\begin{aligned}\rho(A + B) &= \rho\Big(\frac{A+B}{A \cap B}\Big) + \rho(A \cap B) = \rho\Big(\frac{A}{A \cap B} \oplus \frac{B}{A \cap B}\Big) \\ &+ \rho(A \cap B) = \rho\Big(\frac{A}{A \cap B}\Big) + \rho\Big(\frac{B}{A \cap B}\Big) + \rho(A \cap B) = \rho(A) \\ &+ \rho(B) - 2\rho(A \cap B) + \rho(A \cap B) = \rho(A) + \rho(B) - \rho(A \cap B)\,.\end{aligned}$$

Next define $\tau(M) = d(Z(M))$, and set $\sigma = d - \tau$. Then

$$(3) \qquad \sigma(X + Y) = \sigma(X) + \sigma(Y) - \sigma(X \cap Y)\,.$$

This is true because τ satisfies the conditions listed in the remark at the end of [1] and so satisfies $\tau(X/Y) = \tau(X) - \tau(Y) + \tau(\bar{Y}/Y)$. Moreover, $\bar{Y}/Y$ is singular, so $\tau(\bar{Y}/Y) = d(\bar{Y}/Y)$ and therefore σ satisfies (1). Hence by the discussion in the preceding paragraph, σ satisfies (3).

We can therefore conclude that a module is a dimension module with respect to d if and only if it is a dimension module with respect to τ. On the other hand it is not true that a module is a dimension module if its singular submodule is. For an example, consider $M = Z/4Z \oplus Z/2Z$ as a module over $Z/4Z$. Let A be the submodule generated by $(\bar{1}, \bar{1})$ and let B be the submodule generated by $(\bar{1}, \bar{0})$. Then $M = A + B$ and $d(A) = d(B) = d(A \cap B) = 1$. Since $d(M) = 2$, M is not a dimension module; but $Z(M)$ equals the socle of M and is therefore a dimension module.

Finally, observe that for a module M over an artinian ring R, $\rho(M)$ is the composition length of M while $d(M)$ is the length of the socle of M. Thus it is apparent that there is no useful relationship between these two invariants, other than the fact that they happen to coincide if $\text{Rad}\, R = 0$. In fact the three dimension functions $\rho(M)$, $d(M)$, $\sigma(M)$ are equal in the event that $\text{Rad}\, R = 0$, and are different otherwise. In fact, whenever R_R has no simple summands, $\sigma(M) = 0$ because any maximal right ideal is then large so $\text{Socle}\, M \subseteqq Z(M)$.

In summary then, the main points of this appendix are

(1) if $N = 0$, then $\rho(M) = d(M) - d(Z(M))$;

(2) ρ satisfies the classical dimension formula (2);

(3) d satisfies the classical dimension formula if and only if τ does.

REFERENCES

1. V. Camillo and J. Zelmanowitz, *On the dimension of a sum of modules*, Comm. in Algebra, **6** (1978), 345-352.

2. A. W. Chatters, A. W. Goldie, C. R. Hajarnavis, and T. H. Lenagan, *Reduced rank in Noetherian rings*, (preprint).

3. C. Faith, *Algebra II, Ring Theory*, Springer-Verlag, New York, 1976.

4. A. W. Goldie, *Torsion-free modules and rings*, J. Algebra, **1** (1964), 268-287.

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