PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 1, 1981

CORRECTION TO "PEANO MODELS WITH MANY GENERIC CLASSES"

J. H. SCHMERL

A theorem in the paper referred to in the title had a faulty proof which is repaired herein by employing the Halpern Läuchli-Laver-Pincus Partition Theorem.

In [4] a generalization of the MacDowell-Specker Theorem on extensions of models of Peano arithmetic was stated as Theorem 3.1. Unfortunately, the proof presented there turned out to be faulty. The purpose of this note is, first, to repair that proof, and, second, to present a slight strengthening of the theorem.

All unexplained notation and terminology is taken from [4]. If $\mathscr{N} \subseteq \mathscr{M}$ are such that whenever $X \subseteq M$ is definable in $\mathscr{M}$, then $X \cap N$ is definable in $\mathscr{N}$, then we say that the extension is *conservative*.

THEOREM. *Let $\mathscr{N}$ be a model of Peano arithmetic and $\langle X_i : i \in I\rangle$ be a collection of mutually generic classes of $\mathscr{N}$. Then $(\mathscr{N}, \langle X_i : i \in I\rangle)$ has a proper, conservative, elementary end-extension $(\mathscr{M}, \langle Y_i : i \in I\rangle)$. In addition, for each $J \subseteq I$ there is a unique $\mathscr{M}_J \preceq \mathscr{M}$ which is a proper elementary extension of $\mathscr{N}$ such that*

$$J = \{i \in I\text{: there is a class } Y \text{ of } \mathscr{M}_J \text{ such that } X_i = Y \cap N\}\ .$$

Each $(\mathscr{M}_J, \langle Y_j \cap M_J : j \in J\rangle)$ is conservative extension of $(\mathscr{N}, \langle X_j : j \in J\rangle)$.

Theorem 3.1 of [4] consists just of the first two sentences of the above theorem. This will be proved first. The latter two sentences make up the strengthening which will be established by noting the appropriate changes to make in the first proof.

The error in [4] was that a too weak combinatorial theorem was used, the correct theorem being the Halpern-Läuchli-Laver-Pincus partition theorem for trees. Proofs of this theorem can be found in [3] or [2], these proofs being easily formalizable in Peano arithmetic. Harrington has a slick proof which, while apparently not formalizable in Peano arithmetic, nevertheless, can be applied in all countable models of Peano arithmetic, and thus shows that the theorem actually is a consequence of the axioms of Peano arithmetic.

Some definitions are necessary in order to state this theorem in a form which best suits our purposes.

Let $\lhd$ be the binary relation on ω which is defined by:

$$x \trianglelefteq y \longleftrightarrow \exists z[(x+1)2^z \leqq y+1 < (x+2)2^z]\ .$$

With this definition, $(\omega, \lhd)$ is the full binary ω-tree. The rank of an element x of this tree, denoted by $rk\, x$, is just the largest z such that $2^z \leqq x + 1$. For $X \subseteq \omega$ and $n < \omega$ define

$$(X)^{<\omega} = \{A \in [X]^{<\omega}\colon \text{ if } a, b \in A, \text{ then } rk\, a = rk\, b\},$$

and

$$(X)^n = \{A \in (X)^{<\omega}\colon \operatorname{card}(A) = n\}.$$

If $A, B \in (\omega)^{<\omega}$ then write $A \trianglelefteq B$ iff for each $a \in A$ there is $b \in B$ such that $a \trianglelefteq b$. A subset $X \subseteq \omega$ is *large* iff for each $A \in (\omega)^{<\omega}$ there is some $B \in (X)^{<\omega}$ such that $A \trianglelefteq B$. If $n < \omega$, $A \in (\omega)^n$, $f\colon (\omega)^n \to \omega$ and $X \subseteq \omega$, then X is *homogeneous above* A *for* f iff whenever $B, C \in (X)^n$ are such that $A \trianglelefteq B, C$, then $f(B) = f(C)$.

THEOREM 1. (*Halpern-Läuchli-Laver-Pincus.*) *Suppose* $k, n < \omega$, $A \in (\omega)^n$ $f\colon (\omega)^n \to k$ *and* $X \subseteq \omega$, *where* X *is large. Then there is* $B \in (\omega)^n$ *and large* $Y \subseteq X$ *such that* $A \trianglelefteq B$ *and* Y *is homogeneous above* B *for* f.

This theorem has been stated only for the case in which the tree is the full binary ω-tree. However, this version implies the same result for finite-branching ω-trees, and even for countable ω-trees. This same remark applies to Corollaries 2, 3 and 4.

Theorem 1 easily implies a generalization of itself. If $n < \omega$, $f\colon (\omega)^n \to \omega$ and $X \subseteq \omega$, then X is *eventually homogeneous for* f iff for each $A \in (\omega)^n$ there is $B \in (\omega)^n$ such that $A \trianglelefteq B$ and X is homogeneous above B for f.

COROLLARY 2. *Suppose* $n < \omega$, $f_j\colon (\omega)^n \to k_j < \omega$ *for each* $j < \omega$, *and* $X \subseteq \omega$, *where* X *is large. Then there is a large* $Y \subseteq X$ *which is eventually homogeneous for each* f_j.

The above corollary can be formalized as a schema and proved in Peano arithmetic. In $\mathscr{N}$ a model of Peano arithmetic we will use the definition of $\lhd$ given above. (This is technically different from the definition given in [4], but because of the remark following Theorem 1, this difference is not substantial.) All of the previously given definitions have obvious (and uniform in models of Peano arithmetic) generalizations to $(N, \lhd)$. As a convenience we introduce the following notation for the model $\mathscr{N}$: If $\psi(u, y, x_0, \cdots, x_{n-1})$ is an $(n+2)$-ary formula and $a, b \in N$, then let $f_{\psi,a,b}\colon (N)^n \to N$ be the function such that whenever $A = \{a_0, \cdots, a_{n-1}\} \in (N)^n$, where $a_0 < \cdots < a_{n-1}$, then

$$f_{\psi,a,b}(A) = \mu c[c < b \longrightarrow \psi(a, c, a_0, \cdots, a_{n-1})] .$$

The formalization of Corollary 2 becomes the following which we state only for the particular model $\mathscr{N}$:

COROLLARY 3. *Suppose $\varphi(x)$ is a formula which defines the large subset $X \subseteq N$, and suppose $\psi(u, y, x_0, \cdots, x_{n-1})$ is an $(n+2)$-ary formula. Then there is a formula $\theta(x)$ which defines a large subset $Y \subseteq X$ such that Y is eventually homogeneous for each $f_{\psi,a,b}$.*

To proceed with the proof of the first part of the Theorem, we use the countability of the language of $\mathscr{N}$ to produce a set $\Sigma(x)$ of unary formulas which has the following properties:

(1) $\mathrm{Th}(\mathscr{N}) \subseteq \Sigma(x)$;

(2) $\Sigma(x)$ is deductively closed;

(3) if $\phi(x)$ and $\psi(x)$ are as in Corollary 3, where $\phi(x) \in \Sigma(x)$, then there is a $\theta(x) \in \Sigma(x)$ as in Corollary 3;

(4) if $\theta(x) \in \Sigma(x)$, then $\theta(x)$ defines a large subset.

From $\Sigma(x)$ we construct a set Γ of formulas in the language of $(\mathscr{N}, \langle X_i: i \in I\rangle)$ which involve only the free variables v_i for $i \in I$. Let Γ be the set of all formulas $\varphi(v_{i_0}, \cdots, v_{i_{n-1}})$ such that for some $\theta(x) \in \Sigma(x)$ the sentence

$$\exists y \forall z > y \forall x_0, \cdots, x_{n-1}\left[\left(\bigwedge_{k<n} \theta(x_k) \wedge rk\, x_k = z \wedge X_{i_k}(x_k)\right) \longrightarrow \varphi(\bar{x})\right]$$

is true in $(\mathscr{N}, \langle X_i: i \in I\rangle)$. By the genericity of $\langle X_i: i \in I\rangle$ and the conditions (1)-(4) above, the set Γ is a complete, nonprincipal I-type. Furthermore, it follows that Γ is a definable type (in the sense of Definition 1.1 of [1]), and therefore induces an extension $(\mathscr{M}, \langle Y_i: i \in I\rangle)$ of $(\mathscr{N}, \langle X_i: i \in I\rangle)$. Such extensions, induced by definable types, are always conservative end-extensions. (See [1].) This completes the proof of the first part of the theorem.

The remainder of theorem will be proved by modifying the previous proof by replacing Theorem 1 with a canonical version of itself. Some more definitions are needed.

Suppose $n < \omega$, $A \in (\omega)^n$, $f: (\omega)^n \to \omega$ and $X \subseteq \omega$. Then f is *canonical on X above A* iff either X is homogeneous above A for f or else there is $A_0 \subseteq A$ such that whenever $A \trianglelefteq B$, $C \in (X)^n$, then

(i) $rk\, B < rk\, C \Rightarrow f(B) < f(C)$, and

(ii) $rk\, B = rk\, C \Rightarrow (f(B) = f(C)$ iff $A_0 \trianglelefteq B \cap C)$.

COROLLARY 4. *Suppose $n < \omega$, $A \in (\omega)^n$, $f: (\omega)^n \to \omega$ and $X \subseteq \omega$, where X is large. Then there is $B \in (\omega)^n$ and a large $Y \subseteq X$ such that $A \trianglelefteq B$ and f is canonical on Y above B.*

There is now an obvious definition of eventually canonical with its corresponding improvements to Corollaries 2 and 3. This stronger version of Corollary 3 gives rise to a new set $\Sigma(x)$ of formulas, which yields the corresponding I-type Γ. Notice that if $i \in I$ and $\Gamma_i \subseteqq \Gamma$ consists of those formulas in Γ which have v_i as the only free variable, then Γ is a minimal type ([1]). Then Γ_i induces a minimal conservative extension $(\mathscr{M}_i, Y_i \cap M_i)$ of $(\mathscr{N}, X_i)$. More generally, for $J \subseteqq I$ let $\Gamma_J \subseteqq \Gamma$ consist of those formulas in Γ which have v_j $(j \in J)$ as their only free variables. Γ_J is a minimal J-type which induces the extension $(\mathscr{M}_J, \langle Y_j \cap M_J: j \in J\rangle)$ of $(\mathscr{N}, \langle X_j: j \in J\rangle)$. The conditions on Γ imposed by Corollary 4 guarantee not only that the type Γ_J is definable but that it has the other properties required by the theorem. The one exception occurs when $J = \phi$, and in this case construct Γ_ϕ as follows. Let Γ' be the deductive closure of $\Gamma \cup \{v = rk\, v_i: i \in I\}$. Then let Γ_ϕ be the set of formulas in Γ' whose only free variable is v. It is easily seen that Γ_ϕ is a minimal type, and that the induced extension $\mathscr{M}_\phi$ has the described properties. This completes the proof of the theorem.

References

1. H. Gaifman, *Models and types of Peano's arithmetic,* Annals of Math. Logic, **9** (1976), 223-306.

2. J. D. Halpern, *On a theorem of Laver and Pincus,* (preprint).

3. K. R. Milliken, *A Ramsey theorem for trees,* J. Combinatorial Theory (A), **26** (1979), 215-237.

4. J. H. Schmerl, *Peano models with many generic classes,* Pacific J. Math., **46** (1973), 523-536.

Received June 25, 1979 and in revised form June 11, 1980.

University of California, San Diego
La Jolla, CA 92093