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BOUNDS FOR THE PERRON ROOT OF A NONNEGATIVE IRREDUCIBLE PARTITIONED MATRIX

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It is well-known that the Perron root of a nonnegative irreducible matrix lies between the smallest and the largest row sum of A. This result is generalized to the case when the matrix A is partitioned into blocks.

1. Introduction and notations. If $A = (a_{ij})$ is a nonnegative irreducible $n \times n$ matrix, then the Perron root $r(A)$ of A satisfies the classical inequalities of Frobenius [1, p. 37; 9; 10, p. 63; 21, p. 31]

$$(1) \qquad \min_i S_i \leqq r(A) \leqq \max_i S_i ,$$

where S_i denotes the ith row sum of A, i.e., $S_i = \sum_{j=1}^n a_{ij}$ $(i = 1, \cdots, n)$. Moreover, we have strict inequalities in (1) unless all the S_i's are equal.

Other bounds for $r(A)$ have been found by Ledermann [13], Ostrowski [15], Brauer [2], Ostrowski and Schneider [17], Hall and Porsching [11], Brauer and Gentry [3; 4], and Deutsch [8]. (In some of these papers one has assumed that A is a positive matrix.)

The purpose of this paper is to give some simple generalizations of the inequalities (1), by considering certain partitionings of A.

We introduce a few notations. By $\boldsymbol{R}^m$ we denote the vector space of all column m-tuples of real numbers and $(x)_i$ denotes the ith (scalar) component of the vector $x \in \boldsymbol{R}^m$. By $\boldsymbol{R}^{m \times m}$ we denote the algebra of all $m \times m$ real matrices and $(A)_{ij}$ denotes the (scalar) (i, j)-entry of the matrix $A \in \boldsymbol{R}^{m \times m}$. For two vectors $x, y \in \boldsymbol{R}^m$, the inequality $x \leqq y$ $(x < y)$ means $(x_i) \leqq (y)_i$ $((x)_i < (y)_i)$ for all $i = 1, \cdots, m$. If $X_1, \cdots, X_t \in \boldsymbol{R}^{m \times m}$, then $\bigwedge_{s=1}^t X_s$ $(\bigvee_{s=1}^t X_s)$ denotes the greatest lower bound (least upper bound) of the matrices $X_1, \cdots, X_t$ in the natural (i.e., componentwise) partial ordering of $\boldsymbol{R}^{m \times m}$. In other words,

$$\left(\bigwedge_{s=1}^t X_s\right)_{ij} = \min_{s=1,\cdots,t} (X_s)_{ij} , \quad \left(\bigvee_{s=1}^t X_s\right)_{ij} = \max_{s=1,\cdots,t} (X_s)_{ij} ,$$

for all $i, j = 1, \cdots, m$.

The transpose of a matrix A (vector u) will be denoted by $A^\top$ $(u^\top)$ and the Perron root of a nonnegative matrix $A \in \boldsymbol{R}^{m \times m}$ will be denoted by $r(A)$.

2. Let

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix} \tag{2}$$

be a nonnegative irreducible $n \times n$ matrix, where A_{ij} is an $n_i \times n_j$ submatrix $(i, j = 1, \cdots, k)$. Clearly, $n_1 + \cdots + n_k = n$.

Let p_{ij} denote the smallest row sum of A_{ij}, let q_{ij} denote the largest row sum of A_{ij} $(i, j = 1, \cdots, k)$ and consider the $k \times k$ matrices

$$P(A) = (p_{ij})_{i,j=1,\cdots,k}\ , \quad Q(A) = (q_{ij})_{i,j=1,\cdots,k}\ . \tag{3}$$

PROPOSITION 1. *We have*

$$r(P(A)) \leqq r(A) \leqq r(Q(A))\ . \tag{4}$$

Proof. Let $x \in \boldsymbol{R}^n$ be a Perron eigenvector of A, i.e.,

$$Ax = \rho x \quad (x > 0)\ , \tag{5}$$

where $\rho = r(A)$. We partition x as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \boldsymbol{R}^n\ ,$$

where $x_j \in \boldsymbol{R}^{n_j}$ $(j = 1, \cdots, k)$. Now, equation (5) can be written

$$A_{i1}x_1 + \cdots + A_{ik}x_k = \rho x_i \quad (i = 1, \cdots, k)\ . \tag{6}$$

We assume that $(x_i)_{M_i}$ is the smallest (scalar) component of x_i, i.e.,

$$(x_i)_{M_i} = \min\{(x_i)_1, (x_i)_2, \cdots, (x_i)_{n_i}\}\ .$$

Equating the M_ith components of both sides of (6), we obtain

$$\rho(x_i)_{M_i} = (A_{i1}x_1)_{M_i} + \cdots + (A_{ik}x_k)_{M_i}\ ,$$

or

$$\rho(x_i)_{M_i} = \sum_{s=1}^{n_1} (A_{i1})_{M_i,s}(x_1)_s + \cdots + \sum_{s=1}^{n_k} (A_{ik})_{M_i,s}(x_k)_s\ ,$$

whence, replacing $(x_j)_s$ by $(x_j)_{M_j}$ and then replacing the row sums of A_{ij} by p_{ij}, we have

$$\rho(x_i)_{M_i} \geqq p_{i1}(x_1)_{M_1} + \cdots + p_{ik}(x_k)_{M_k} \quad (i = 1, \cdots, k)\ . \tag{7}$$

Introducing the vector

$$v = ((x_1)_{M_1}, \cdots, (x_k)_{M_k})^{\top} \in \boldsymbol{R}^k ,$$

inequalities (7) can be written as

$$\rho v \geqq P(A)v \quad (v > 0) ,$$

which implies [1, p. 28; 5; 22, p. 33] $r(P(A)) \leqq \rho = r(A)$.

The right-hand inequality of (4) is proved in an entirely similar manner.

REMARK 1. Since q_{ij} is the row-sum norm [20, p.180] of A_{ij}, the right-hand inequality of (4) follows at once also from the theory of matricial norms [6; 7], (see also [16; 18; 19]).

PROPOSITION 2. *Either* $P(A) = Q(A)$, *or*

$$r(P(A) < r(A) < r(Q(A)) .$$

Proof. Assume $P(A) \neq Q(A)$. We construct a nonnegative irreducible matrix $B \in \boldsymbol{R}^{n\times n}$ by decreasing certain entries of A so that $P(B) = Q(B) = P(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 1, $r(B) = r(P(B))$. Consequently, $r(P(A)) < r(A)$. Similarly, we construct a nonnegative irreducible matrix $C \in \boldsymbol{R}^{n\times n}$ by increasing certain entries of A so that $P(C) = Q(C) = Q(A)$. Then $r(A) < r(C)$ and, by Proposition 1, $r(C) = r(P(C))$. Consequently, $r(A) < r(Q(A))$.

COROLLARY 1. *The following statements are equivalent*:
(a) $P(A) = Q(A)$;
(b) $r(A) = r(P(A))$;
(c) $r(A) = r(Q(A))$;
(d) $r(P(A)) = r(Q(A))$.

REMARK 2. If a nonnegative irreducible matrix $A \in \boldsymbol{R}^{n\times n}$, partitioned as in (1), satisfies the equivalent conditions of Corollary 1, then it follows from condition (a) that, for each fixed pair $i, j \in \{1, \cdots, k\}$, all the row sums of A_{ij} are equal to $p_{ij}(=q_{ij})$. Thus, A is a so-called *block-stochastic* matrix [12]. In this case, every eigenvalue of $P(A) \in \boldsymbol{R}^{k\times k}$ is an eigenvalue of $A \in \boldsymbol{R}^{n\times n}$ (see [12, Theorem 2]).

EXAMPLE 1. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ \hline 2 & 3 & 5 \end{array}\right) .$$

We have

$$P(A) = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}, \quad Q(A) = \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix},$$

and $r(P(A)) = 7$, $r(Q(A)) = 8$. Thus, $7 < r(A) < 8$. This result is better than those obtained by several other methods [4; 14, p. 158].

EXAMPLE 2. We consider the partitioned matrix

$$A = \left(\begin{array}{ccc:cc} 3 & 1 & 5 & 1 & 4 \\ 2 & 2 & 5 & 2 & 3 \\ 1 & 5 & 3 & 1 & 4 \\ \hdashline 1 & 1 & 3 & 4 & 1 \\ 0 & 2 & 3 & 3 & 2 \end{array}\right).$$

We have

$$P(A) = Q(A) = \begin{pmatrix} 9 & 5 \\ 5 & 5 \end{pmatrix}$$

and thus, in this case Proposition 1 yields the exact value of the Perron root of A: $r(A) = r(P(A)) = r(Q(A)) = 7 + \sqrt{29} \approx 12.38$. The matrix A is block-stochastic (see Remark 2).

3. Let

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \tag{8}$$

be a nonnegative irreducible $n \times n$ matrix, where each A_{ij} is a square $k \times k$ matrix. Clearly, $n = kN$.

Denote

$$R_i(A) = \sum_{j=1}^{N} A_{ij} \in \boldsymbol{R}^{k \times k} \quad (i = 1, \cdots, N). \tag{9}$$

PROPOSITION 3. *We have*

$$r\Big(\bigwedge_{j=1}^{N} R_j(A)\Big) \leqq r(A) \leqq r\Big(\bigvee_{j=1}^{N} R_j(A)\Big). \tag{10}$$

Proof. Let $y \in \boldsymbol{R}^n$ be a Perron eigenvector of $G = A^{\mathrm{T}}$, i.e.,

$$Gy = \rho y \quad (y > 0), \tag{11}$$

where $\rho = r(A)$. We partition y as

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \in \boldsymbol{R}^n ,$$

where $y_j \in \boldsymbol{R}^k$ for all $j=1, \cdots, N$. Denoting $G_{ij}=A_{ji}^{\top}$ $(i, j=1, \cdots, N)$, equation (11) can be written

$$\sum_{j=1}^{N} G_{ij} y_j = \rho y_i \quad (i = 1, \cdots, N) . \tag{12}$$

Summing the equations (12) with respect to i, we obtain

$$\sum_{i=1}^{N} \sum_{j=1}^{N} G_{ij} y_j = \rho w , \tag{13}$$

where $w = \sum_{i=1}^{N} y_i \in \boldsymbol{R}^k$. Interchanging the order of summation in the left-hand side of (13), we have

$$\rho w = \sum_{j=1}^{N} (R_j(A))^{\top} y_j ,$$

from where one has

$$\left[\bigwedge_{j=1}^{N} (R_j(A))^{\top}\right] w \leqq \rho w \leqq \left[\bigvee_{j=1}^{N} (R_j(A))^{\top}\right] w .$$

This, in turn, implies the inequalities (10) [1, p. 28; 5; 22, p. 33].

PROPOSITION 4. *Either* $R_1(A) = \cdots = R_N(A)$, *or*

$$r\Big(\bigwedge_{j=1}^{N} R_j(A)\Big) < r(A) < r\Big(\bigvee_{j=1}^{N} R_j(A)\Big) .$$

Proof. Assume that $R_1(A), \cdots, R_N(A)$ are not equal. We construct a nonnegative irreducible $n \times n$ matrix B by decreasing certain entries of A so that $R_1(B) = \cdots = R_N(B) = \bigwedge_{j=1}^{N} R_j(A)$. Then $r(B) < r(A)$ [1, p. 27; 21, p. 30] and, by Proposition 3, $r(B) = r(\bigwedge_{j=1}^{N} R_j(B))$. Consequently, $r(\bigwedge_{j=1}^{N} R_j(A)) < r(A)$. Similarly, we construct a nonnegative irreducible $n \times n$ matrix C by increasing certain entries of A so that $R_1(C) = \cdots = R_N(C) = \bigvee_{j=1}^{N} R_j(A)$. Then $r(A) < r(C)$ and, by Proposition 3, $r(C) = r(\bigvee_{j=1}^{N} R_j(C))$. Consequently, $r(A) < r(\bigvee_{j=1}^{N} R_j(A))$.

COROLLARY 2. *The following statements are equivalent*:
(a) $R_1(A) = \cdots = R_N(A)$;
(b) $r(A) = r(\bigwedge_{j=1}^{N} R_j(A))$;
(c) $r(A) = r(\bigvee_{j=1}^{N} R_j(A))$;

(d) $r(\bigwedge_{j=1}^{N} R_j(A)) = r(\bigvee_{j=1}^{N} R_j(A))$.

EXAMPLE 3. We consider the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ \hline 1 & 4 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right).$$

We have

$$R_1(A) = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix},$$

and

$$\bigwedge_{j=1}^{2} R_j(A) = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^{2} R_j(A) = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix},$$

the last two matrices having Perron roots $2 + \sqrt{5}$ and 5, respectively. Thus, $4.236 < r(A) < 5$. The classical inequalities (1) yield only $3 < r(A) < 8$.

REMARK 3. The results of § 3 can be obtained from those of § 2. Indeed, if A is the $n \times n$ matrix given in (8) and if we arrange the rows and columns of A in the following positions;

$$\begin{array}{l} 1,\ N+1,\ 2N+1,\ \cdots,\ (k-1)N+1\ , \\ 2,\ N+2,\ 2N+2,\ \cdots,\ (k-1)N+2\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ N,\ \ 2N,\ \ 3N,\ \cdots,\ kN\ , \end{array}$$

then we obtain a matrix $A' = (A'_{ij})_{i,j=1,\cdots,k} \in \boldsymbol{R}^{n \times n}$, where each A'_{ij} is an $N \times N$ submatrix. It can be easily seen that

$$P(A') = \bigwedge_{i=1}^{N} R_i(A)\ , \quad Q(A') = \bigvee_{i=1}^{N} R_i(A)\ .$$

Since $r(A) = r(A')$, Propositions 3, 4 and Corollary 2 follow at once from Propositions 1, 2 and Corollary 1, respectively.

REMARK 4. It should be noted that the bounds given by Proposition 3 (or Proposition 1) are not always better than those given by the classical bounds (1). For example, considering the partitioned matrix

$$A = \left(\begin{array}{cc|cc} 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right),$$

we have

$$R_1(A) = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}, \quad R_2(A) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\bigwedge_{j=1}^{2} R_j(A) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \bigvee_{j=1}^{2} R_j(A) = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix},$$

the last two matrices having Perron roots 2 and $\frac{1}{2}(7 + \sqrt{17})$, respectively. Thus, $2 < r(A) < 5.562$. However, all row-sums of A are equal to 4 and thus $r(A) = 4$.

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