If $X^*$ has the Radon-Nikodym property, then for every compact operator $T: L_1(\mu, X) \to Y$ there is a bounded function $g: \mathcal{O} \to L(X, Y)$ that is measurable for the uniform operator topology on $L(X, Y)$ such that

$$T(f) = \int_\mathcal{O} fg d\mu$$

for all $f$ in $L_1(\mu, X)$. The same result holds for weakly compact operators if $X^*$ is separable Schur space. These representations yield Radon-Nikodym theorems for operator valued measures and a generalization of a theorem of D. R. Lewis.

The representation of linear operators on the Banach space $L_1(\mu, X)$ of Bochner integrable functions, has been the object of much study for the past forty years. Dunford and Pettis began this investigation in 1940 [6] with the representation of weakly compact and norm compact operators on $L_1(\mu)$ by a Bochner integral. Their work was based on an earlier paper of Pettis [9] and was complemented by the work of Phillips [11]. More recently, the theory of liftings has been used by Dinculeanu [5] and others to obtain a representation for the general linear operator on $L_1(\mu, X)$. In this paper we will use methods in the spirit of Dunford, Pettis, and Phillips to show that if $X^*$ has the Radon-Nikodym property, then the compact operators on $L_1(\mu, X)$ are representable by measurable kernels and if $X^*$ is a separable Schur space (i.e., weakly convergent sequences converge in norm) then the weakly compact operators on $L_1(\mu, X)$ are representable by measurable kernels. As corollaries, we obtain a Radon-Nikodym theorem for operator-valued measures and a generalization of a theorem of D. R. Lewis [4, p. 88] on weakly measurable functions that are equivalent to norm measurable functions.

Throughout this paper $(\mathcal{Q}, \Sigma, \mu)$ is a finite measure space and $X, Y$ and $Z$ are Banach spaces with duals $X^*, Y^*$, and $Z^*$ respectively. The space of all bounded linear operators from $X$ to $Y$ will be denoted by $L(X, Y)$. The subspaces of $L(X, Y)$ consisting of all the weakly compact and norm compact operators from $X$ to $Y$ will be denoted by $W(X, Y)$ and $K(X, Y)$. The space $L_1(\mu, X)$ is the space of $\mu$-Bochner integrable functions on $\mathcal{Q}$ with values in $X$ and

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\(L_\infty(\mu, X)\) is the space of \(X\)-valued \(\mu\)-Bochner integrable functions on \(\Omega\) that are essentially bounded. An operator \(T: L_1(\mu, X) \to Y\) is representable by a measurable kernel if there is a bounded measurable \(g: \Omega \to L(X, Y)\) such that

\[
T(f) = \text{Bochner} - \int_\Omega f g d\mu.
\]

From this, it follows that \(\|T\| = \|g\|_\infty\) [5, p. 283]. Recall that a Banach space is weakly compactly generated if it is the closed linear span of one of its weakly compact sets. Finally, note that if \(\pi\) is a partition of \(\Omega\) into a countable number of disjoint elements of \(\Sigma\) and if \(f\) is in \(L_1(\mu, X)\), then the function \(E_\pi: L_1(\mu, X) \to L_1(\mu, X)\) defined by

\[
E_\pi(f) = \sum_{E \in \pi} \int_E f d\mu \chi_E
\]

(here the convention \(0/0 = 0\) is observed) is a linear operator.

Most of the first lemma is well-known so we omit the proof.

**Lemma 1.** For each countable partition \(\pi\), the operator \(E_\pi\) is a contraction on \(L_1(\mu, X)\) and \(L_\infty(\mu, X)\). Moreover, if the partitions are directed by refinement, then

\[
\lim_\pi \|E_\pi(f) - f\|_1 = 0 \quad \text{for all } f \text{ in } L_1(\mu, X)
\]

\[
\lim_\pi \|E_\pi(f) - f\|_\infty = 0 \quad \text{for all } f \text{ in } L_\infty(\mu, X).
\]

Before stating the main theorem we require a preliminary definition. A function \(g\) in \(L_\infty(\mu, L(X, Y))\) is said to have its essential range in the uniformly (weakly) compact operators if there is a (weakly) compact set \(C\) in \(Y\) such that \(g(\omega)x \in C\) for almost all \(\omega\) in \(\Omega\) and \(x\) in \(X\) with \(\|x\| \leq 1\).

**Theorem 2.** Let \(X^*\) have the Radon-Nikodym property. Then there is an isometric isomorphism between the space of compact operators \(K(L_1(\mu, X), Y)\) and the subspace of \(L_\infty(\mu, K(X, Y))\) consisting of those functions whose essential range is in the uniformly compact operators. In fact, \(T\) in \(K(L_1(\mu, X), Y)\) and \(g\) in \(L_\infty(\mu, K(X, Y))\) are in correspondence if and only if

\[
T(f) = \int_\Omega f g d\mu \quad \text{for all } f \in L_1(\mu, X).
\]

**Proof.** Let \(T\) be in \(K(L_1(\mu, X), Y)\). Notice that for any par-
partition $\pi$, $f$ in $L_1(\mu, X)$, and $g$ in $L_\infty(\mu, X^*) = (L_1(\mu, X))^*$, we have that

$$
\int_D E_\pi(f)gd\mu = \int_D fE_\pi(g)d\mu .
$$

It follows from this that the adjoint of $TE_\pi$ is $E_\pi T^*$. Now, if the partitions $\pi$ are countable, we have that

$$
\lim_{\pi} E_\pi f = f \quad \text{for all } f \in L_\infty(\mu, X^*)
$$

by Lemma 1. Since $\|E_\pi\|_\infty \leq 1$, this limit is uniform on compact sets. By Schauder’s theorem, $T^*: Y^* \to L_\infty(\mu, X^*)$ is compact and so

$$
\lim_{\pi} E_\pi T^* y^* = Ty^*
$$

uniformly for $\|y^*\| \leq 1$. Therefore,

$$
\lim_{\pi} E_\pi T^* = T^*
$$

in the operator norm. Since $E_\pi T^* = (TE_\pi)^*$, it follows that

$$
\lim_{\pi} TE_\pi = T
$$

in operator norm.

Now, for each countable partition $\pi$, define $g_\pi: \Omega \to L(X, Y)$ by

$$
g_\pi(\cdot)x = \sum A \frac{T(x\chi_A)}{\mu A} \chi_A(\cdot) .
$$

Then for each partition $\pi$, $\omega$ in $\Omega$, and $x$ in $X$ with $\|x\| \leq 1$, we have that $g_\pi(\omega)x \subseteq T(f: f$ in $L_1(\mu, X), \|f\| \leq 1)$. Since $T$ is compact, it follows that $g_\pi(\omega)$ is in $K(X, Y)$ for each partition $\pi$ and $\omega$ in $\Omega$. Moreover, one easily sees that

$$
TE_\pi(f) = \int_D f g_\pi d\mu
$$

for all simple functions $f$ in $L_1(\mu, X)$ and thus for all functions $f$ in $L_1(\mu, X)$. Hence if $\pi_1$ and $\pi_2$ are two partitions, then

$$
(TE_{\pi_1} - TE_{\pi_2})(f) = \int_D f(g_{\pi_1} - g_{\pi_2})d\mu .
$$

Since

$$
\lim_{\pi_1, \pi_2} \|TE_{\pi_1} - TE_{\pi_2}\| = 0 ,
$$

an appeal to [5, p. 283] establishes that
Thus the net \((g_\pi)\) is Cauchy in the norm of \(L_\infty(\mu, K(X, Y))\). It follows that there is a \(g\) in \(L_\infty(\mu, K(X, Y))\) such that

\[
\lim_\pi \|g_\pi - g\|_\infty = 0
\]

and so

\[
\lim_\pi \int_\Omega f g_\pi d\mu = \int_\Omega f g d\mu
\]

for all \(f\) in \(L_i(\mu, X)\). We also have, for almost all \(\omega\), that

\[
g(\omega) x \subseteq T\{f: f \in L_i(\mu, X), \|f\| \leq 1\}
\]

for all \(x\) in \(X\) with \(\|x\| \leq 1\). Hence the essential range of \(g\) consists of uniformly compact operators. Finally, Lemma 1 ensures that

\[
T(f) = \lim_\pi T E_\pi(f) = \lim_\pi \int_\Omega f g_\pi d\mu = \int_\Omega f g d\mu.
\]

Conversely, suppose that \(g: \Omega \rightarrow K(X, Y)\) is a bounded measurable function such that there is a compact set \(C \subset Y\) with \(g(\omega) x \in C\) for almost all \(\omega\) in \(\Omega\) and all \(x\) in \(X\) with \(\|x\| \leq 1\). Without loss of generality, we may assume \(g(\omega) x\) is in \(C\) for all \(\omega\) in \(\Omega\). Define

\[
T(f) = \int_\Omega f g d\mu
\]

for \(f \in L_i(\mu, X)\). Another appeal to [5, p. 283] shows \(\|T\| = \|g\|_\infty\).

Let

\[
f = \sum_{i=1}^n x_i \mathcal{E}_i
\]

be a simple function in \(L_i(\mu, X)\) with \(\|f\| \leq 1\) i.e.,

\[
\sum_{i=1}^n \|x_i\| \mu E_i \leq 1.
\]

Then

\[
T(f) = \int_\Omega f g d\mu = \sum_{i=1}^n \int_{E_i} g(\omega) x_i d\mu(\omega)
\]

\[
= \sum_{i=1}^n \|x_i\| \mu E_i \cdot \frac{1}{\mu E_i} \int_{E_i} g(\omega) \frac{x_i}{\|x_i\|} d\mu
\]

is in \(\text{co} C\) by [4, p. 48]. Since \(\text{co} C\) is compact by Mazur's theorem, the operator \(T\) is compact. This completes the proof.
That $X^*$ has the Radon-Nikodym property is necessary as well as sufficient for the first part of the above proof. Indeed, if each $T$ in $K(L_1(\mu, X), Y)$ is representable by a Bochner integrable $g$ in $L_\infty(\mu, K(X, Y))$, then taking $Y$ to be the scalars shows that $L_1(\mu, X)^* = L_\infty(\mu, X^*)$ which implies [4, p. 98] that $X^*$ has the RNP. An immediate consequence of Theorem 2 is a Radon-Nikodym theorem for certain operator valued measures.

**Corollary 3.** Let $X^*$ have the RNP and let $G: \Sigma \rightarrow K(X, Y)$ be a $\mu$-continuous vector measure of bounded variation. If, for each $E_1$ in $\Sigma$ with $\mu E_1 > 0$, there exists $E_2$ in $\Sigma$ with $E_2 \subseteq E_1$ and $\mu(E_2) > 0$ such that

$$\left\{ \frac{G(E)x}{\mu(E)} : x \in X, E \in \Sigma, E \subseteq E_2, \mu(E) > 0, \|x\| \leq 1 \right\}$$

is relatively norm compact, then there exists a Bochner integrable $g: \Omega \rightarrow K(X, Y)$ such that

$$G(E) = \int_E g d\mu$$

for each $E$ in $\Sigma$.

**Proof.** By exhaustion [4, p. 70], the corollary is established if for each $E_1$ in $\Sigma$ with $\mu E_1 > 0$ we can find $E_2$ in $\Sigma$ with $E_2 \subseteq E_1$ and $\mu(E_2) > 0$ and a Bochner integrable $g$ such that

$$G(E) = \int_E g d\mu$$

for all $E$ in $\Sigma$ with $E \subseteq E_2$. So let $E_1 \in \Sigma$ with $\mu(E_1) > 0$ and select the $E_2 \subseteq E_1$ guaranteed by the hypothesis. Define an operator $T$ on the simple functions in $L_1(\mu, X)$ by

$$T(f) = \sum_{i=1}^n G(A_1 \cap E_2) x_i \quad \text{if} \quad f = \sum_{i=1}^n x_i \chi_{A_i}, A_i \in \Sigma, A_i \cap A_j = \phi$$

if $i \neq j$. Notice that if $\|f\| \leq 1$

$$\sum_{i=1}^n \|x_i\| \mu A_i \leq 1,$$

then

$$\sum_{i=1}^n \|x_i\| \mu(A_i \cap E_2) \leq 1$$

and so
\[ T(f) = \sum_{i=1}^{n} \frac{G(A_i \cap E_i)\frac{x_i}{\mu(A_i \cap E_i)}}{\mu(A_i \cap E_i)} \|x_i\| \|x_i\| \mu(A_i \cap E_i) \]

is in
\[ \overline{\text{co}} \{ \frac{G(E)x}{\mu E} : x \in X, E \in \Sigma, E \subseteq E, \mu(E) > 0, \|x\| \leq 1 \}, \]
a set which is compact by Mazur's theorem. Thus \( T \) has a compact linear extension to all of \( L_1(\mu, X) \). Hence, by Theorem 2, there exists a Bochner integrable \( g : \Omega \rightarrow K(X, Y) \) such that
\[ T(f) = \int gfd\mu \]
for all \( f \in L_1(\mu, X) \). In particular, if \( E \) is in \( \Sigma \) and \( E \subseteq E_\epsilon \), then
\[ G(E)x = T(x|E) = \int gxd\mu. \]
Since \( g \) is Bochner integrable, we have, by \([4, p. 47] \), that
\[ G(E) = \int g d\mu \]
as required.

Our next result is a generalization of a theorem of D. R. Lewis \([4, p. 88] \) dealing with the equivalence of weakly measurable and measurable functions. The proof uses the following result of Amir and Lindenstrauss \([1, p. 43] \): If \( X \) is a weakly compactly generated space and \( X_0 \subseteq X \) and \( Y_0 \subseteq X^* \) are separable subspaces, then there is a bounded projection \( P : X \rightarrow X \) with separable range such that \( X_0 \subseteq P(X) \) and \( Y_0 \subseteq P^*(X^*) \).

**Proposition 4.** Let \( X^* \) and \( Y \) be weakly compactly generated Banach spaces. If \( f : \Omega \rightarrow K(X, Y) \) is a bounded function such that for each \( y^* \) in \( Y^* \) the function \( y^*f(\cdot) : \Omega \rightarrow X^* \) is measurable, then there is a bounded measurable function \( g : \Omega \rightarrow K(X, Y) \) such that for each \( y^* \) in \( Y^* \), \( y^*f(\cdot) = y^*g(\cdot) \mu\text{-a.e.}, \) (the exceptional set may depend on \( y^* \)).

**Proof.** We claim that the set \( A = \{ y^*f(\cdot) : y^* \in Y^*, \|y^*\| \leq 1 \} \) is compact in \( L_1(\mu, X^*) \). If not, then there is a sequence \( y^*_m \) in the unit ball of \( Y^* \) and \( \delta > 0 \) such that
\[ \|y^*_m f(\cdot) - y^*_n f(\cdot)\|_{L_1(\mu, X^*)} > \delta \]
for \( m \neq n \). Choose a bounded projection \( P_1 : Y \rightarrow Y \) with separable
range such that \( P_i^* y^*_n = y^*_n \) for all \( n \). Since each \( y^*_n f(\cdot): \Omega \to \mathbb{X}^* \) is measurable and hence essentially separably valued, there is a bounded projection \( P_i: \mathbb{X}^* \to \mathbb{X}^* \) with separable range and sets \( \Omega_n \) in \( \Sigma \) with \( \mu(\Omega \setminus \Omega_n) = 0 \) and \( g^*_n f(\Omega_n) \subseteq P_i(\mathbb{X}^*) \) for every \( n \). Now, since each \( f(\omega) \) is a compact operator we have, for all \( x^{**} \) in \( \mathbb{X}^{**} \), that \( f(\omega)^{**} x^{**} \) is in the natural image of \( Y \) in \( Y^{**} \) and so we may define \( h: \Omega \to K(\mathbb{X}^{**}, Y) \) by \( h(\omega)x^{**} = P_i f(\omega)^{**} P_i^* x^{**} \). We claim that for each \( x^{**} \) in \( \mathbb{X}^{**} \), the function \( h(\cdot)x^{**}: \Omega \to Y \) is measurable. To see this, note that since \( P_i \) has separable range, the functions \( h(\cdot)x^{**} \) are separably valued and since

\[ y^* h(\cdot)x^{**} = y^* P_i f(\cdot)^{**} P_i^* x^{**} = x^{**} P_i f(\cdot)^{**} P_i^* y^* \]

and each \( f(\cdot) P_i y^*: \Omega \to \mathbb{X}^* \) is measurable, the functions \( h(\cdot)x^{**} \) are weakly measurable. An appeal to the Pettis measurability theorem [4, p. 42] establishes the measurability of \( h(\cdot)x^{**} \). Now if \( Y_o \) is the Banach space obtained by taking the closed linear span of \( P_i Y \) in \( Y \), then \( Y_o \) is separable and \( h \) can be viewed as taking its values in \( K(\mathbb{X}^{**}, Y_o) \). Moreover, if we define \( S:Y \to Y_o \) by \( S y = P_i y \), then \( h(\omega)x^{**} = S P_i f(\omega)^{**} P_i^* x^{**} \). Thus, if \( y^*_o \) is in \( Y_o^{**} \), then \( h(\omega)^{**} y^*_o = P_i^* f(\omega)^{**} P_i^* S^* y^*_o \) is in \( P_i^* \mathbb{X}^* \), since the range of \( f(\omega)^{**} \) is in \( \mathbb{X}^* \) and \( P_i^* \) extends \( P_i \). Let \( Z = P_i \mathbb{X}^* \) and \( B = \{ T: T \in K(\mathbb{X}^{**}, Y_o), \, T^* Y_o^{**} \subseteq Z \} \). We claim that \( B \) is separable. To see this, let \( U \) and \( V \) denote the closed unit balls of \( \mathbb{X}^* \) and \( Y_o^{**} \) endowed with the weak* topologies. Since \( Y_o \) and \( Z \) are separable, \( U \) and \( V \) are compact metric spaces, and thus, so is \( U \times V \). For each \( T \) in \( B \), define a function \( JT \) on \( U \times V \) by \( JT(u, v) = u T^* v \). Then the map \( T \to JT \) is a linear isometry of \( B \) into \( C(U \times V) \) [8] and so, by [7, p. 437], \( B \) is separable. Since the values of \( h \) in \( K(\mathbb{X}^{**}, Y_o) \) lie in \( B \) and \( \| h(\omega) - h(\omega_2) \| \leq \| h(\omega) - h(\omega_2) \|_{K(\mathbb{X}^{**}, Y)} \) for all \( \omega, \omega_2 \) in \( \Omega \), the values of \( h \) in \( K(\mathbb{X}^{**}, Y) \) form a separable set. Now because \( h(\cdot)x^{**} \) is measurable for each \( x^{**} \) in \( \mathbb{X}^{**} \), an appeal to [5, p. 102] establishes that \( h \) is measurable. Since \( h \) is bounded, \( h \) is Bochner integrable and so we may choose a sequence \( h_n \) of \( K(\mathbb{X}^{**}, Y) \)-valued simple functions such that

\[ \lim_n \int_\Omega \| h - h_n \| d\mu = 0 \]
for all \( g \) in \( L_\infty(\mu, X^{**}) \). It follows immediately that the operator \( S \) is compact. The adjoint of \( S \) is the operator \( y^* \to y^*h(\cdot) \) and hence by Schauder’s theorem is also compact. But \( y^*h(\cdot) = y^*f(\cdot) \) a.e. This contradicts

\[
\| y^*_m f(\cdot) - y^*_n f(\cdot) \|_{L_1(\nu, X)} > \delta
\]

for \( m \neq n \) and establishes that the set \( A \) is compact.

Now choose \( y^*_m \) in \( Y^* \) such that \( y^*_m(\cdot) \) is dense in \( A \). If \( h \) is constructed as above for this sequence \( (y^*_m) \), then \( h \) is measurable and so, by Egoroff’s theorem, for all \( \delta > 0 \) there is a set \( E \) in \( \Sigma \) with \( \mu(\Omega \setminus E) < \delta \) such that \( h\chi_E \) can be approximated uniformly by simple functions. Fix \( \delta > 0 \) and choose such a set \( E \). It follows that the sequence \( y^*_m f(\cdot) \chi_E = y^*_m h(\cdot) \chi_E \) is relatively compact in \( L_\infty(\mu, X^*) \). Since this sequence is \( L_\infty(\mu, X^*) \)-dense in \( \{ y^* f(\cdot) \chi_E : \| y^* \| \leq 1 \} \), this set is relatively compact in \( L_\infty(\mu, X^*) \).

Now define \( T: Y^* \to L_\infty(\mu, X^*) \) by \( Ty^* = y^* f(\cdot) \chi_E \). Then \( T \) is compact and as an operator on \( L_1(\mu, X) \), \( T^*: L_1(\mu, X) \to Y^{**} \) is compact. Notice that the dominated convergence theorem ensures that \( T \) is \( w^* \) sequentially continuous. Thus, if \( y^{**} \) is in \( T^*(L_1(\mu, X)) \), then \( y^{**} \) is a \( w^* \) sequentially continuous functional on \( Y^* \). But since \( Y \) is weakly compactly generated, this means \( y^{**} \) is a \( w^* \) continuous functional on \( Y^* \) [3, p. 148]. Hence, \( T^*(L_1(\mu, X)) \) is contained in \( Y \).

Theorem 2 now produces a Bochner integrable \( g: E \to K(X, Y) \) such that

\[
T^*(k) = \int_E kgd\mu
\]

for all \( k \) in \( L_1(\mu, X) \). But, if \( y^* \) is in \( Y^* \), then \( T^{**}y^* = y^*g \). It follows that \( y^*g = y^*f \) a.e. on \( E \). Since \( \mu(\Omega \setminus E) < \delta \), this completes the proof.

Theorem 2 does not hold for weakly compact operators. To see this, let \( \Omega \) be the unit interval endowed with Lebesgue measure and let \( r_n(\cdot) \) be the \( n \)th Rademacher function i.e., \( r_n(\omega) = \text{signum}(\sin 2^n \pi \omega) \). Consider the function \( g: [0, 1] \to L(\epsilon_2, \epsilon_2) \) defined by \( g(\omega)(\alpha_n) = (r_n(\omega)\alpha_n) \) for all \( (\alpha_n) \in \epsilon_2 \). The function \( g \) is not essentially separably valued, since if \( \omega_1 \) and \( \omega_2 \) are different numbers in \( [0, 1] \) there exists a Rademacher function \( r_n \) with \( |r_n(\omega_1) - r_n(\omega_2)| = 2 \) and hence, \( \| g(\omega_1) - g(\omega_2) \|_{L(\epsilon_2, \epsilon_2)} \geq 2 \). Thus, \( g \) is not measurable. Define an operator \( T: L_1(\mu, \epsilon_2) \to \epsilon_2 \) by

\[
T(f) = \int_{[0,1]} fgd\mu
\]

and note that \( T \) is weakly compact. If \( T \) were representable by a kernel, then that kernel would be equal to \( g \) a.e. and so \( g \) would be
measurable, which is a contradiction. However, we can use Proposition 4 to obtain a representation theorem for weakly compact operators by imposing further conditions on $X^*$.

**Theorem 5.** Let $X^*$ be a separable Schur space. Then there is an isometric isomorphism between the space of weakly compact operators $W(L_1(\mu, X), Y)$ and the subspace of $L_\infty(\mu, W(X, Y))$ consisting of those functions whose essential range is in the uniformly weakly compact operators. In fact, $T$ in $W(L_1(\mu, X, Y))$ and $g$ in $L_\infty(\mu, W(X, Y))$ are in correspondence if, and only if,

$$T(f) = \int_\Omega f g d\mu$$

for all $f$ in $L_1(\mu, X)$.

**Proof.** Let $T$ be in $W(L_1(\mu, X), Y)$. By the Factorization Lemma [2, p. 314], there is a reflexive space $R$ and operators $S: L_1(\mu, X) \to R$ and $J: R \to Y$ such that $T = JS$. Suppose $S$ is representable by a measurable kernel $h: \Omega \to L(X, R)$. Then $T$ is representable by the measurable kernel $g: \Omega \to L(X, Y)$ given by $g(\omega)x = Jh(\omega)x$ for all $x$ in $X$ and $\omega$ in $\Omega$. Hence, without loss of generality, we may assume that $Y$ is reflexive.

Let $G: \Sigma \to L(X, Y)$ be the representing measure of $T$ i.e.,

(i) $G(E)x = T(ax_E)$ for all $x$ in $X$ and $E$ in $\Sigma$

(ii) $T(f) = \int_\Omega f dG$ for all $f$ in $L_1(\mu, X)$ and

(iii) $\|T\| = \sup_{\mu E > 0} \frac{\|G(E)\|}{\mu E}$.

An appeal to [10, p. 345] produces a bounded function $g: \Omega \to L(X, Y)$ such that

(1) $g(\cdot)x: \Omega \to Y$ is Bochner integrable for all $x$ in $X$ and

(2) $G(E)x = \int_E g(\omega)xd\mu(\omega)$ for all $x$ in $X$ and $E$ in $\Sigma$.

It follows quickly from the density of simple functions in $L_1(\mu, X)$ that

$$T(f) = \int_\Omega f g d\mu$$

for all $f$ in $L_1(\mu, X)$. Consider, for each $y^*$ in $Y^*$, the functions $y^*g(\cdot): \Omega \to X^*$. Since these functions are separably valued and weak* measurable, they are measurable by [4, p. 42]. Now $L(X, Y) = K(X, Y)$, since $X^*$ is a Schur space and $Y$ is reflexive. Consequently, Proposition 4 now produces a bounded measurable $h: \Omega \to K(X, Y)$.
such that, for each \( y^* \) in \( Y^* \), \( y^* g(\cdot) = y^* h(\cdot) \mu \text{-a.e.} \). Thus, for all \( y^* \) in \( Y^* \) and \( f \) in \( L_1(\mu, X) \) we have that

\[
\langle y^*, Tf \rangle = \int_\Omega \langle y^*, g(\omega)f(\omega) \rangle d\mu(\omega)
= \int_\Omega \langle y^*, h(\omega)f(\omega) \rangle d\mu
= y^* \left( \int_\Omega h f d\mu \right)
\]

and so

\[
T(f) = \int_\Omega h f d\mu.
\]

It follows easily that \( h(x) \leq T(f) : f \in L_1(\mu, X), \|f\|_1 \leq 1 \)

for almost all \( \omega \) in \( \Omega \) and all \( x \) in \( X \) with \( \|x\| \leq 1 \). Hence, the essential range of \( h \) consists of uniformly weakly compact operators.

The converse is proved in the same way as in Theorem 2 so we omit the proof.

Our final result follows from Theorem 5 in the same way that Corollary 3 follows from Theorem 2 so the proof is omitted.

**Corollary 6.** Let \( X^* \) be a separable Schur space and let \( G: \Sigma \to K(X; Y) \) be a \( \mu \)-continuous vector measure of bounded variation. If, for each \( E_i \) in \( \Sigma \) with \( \mu E_i > 0 \), there exists an \( E_2 \) in \( \Sigma \) with \( E_2 \subseteq E_i \) and \( \mu(E_2) > 0 \) such that

\[
\left\{ \frac{G(E)x}{\mu E} : x \in X, E \in \Sigma, E \subseteq E_2, \mu E > 0, \|x\| \leq 1 \right\}
\]

is relatively weakly compact, then there exists a Bochner integrable \( g: \Omega \to K(X, Y) \) such that

\[
G(E) = \int_E g d\mu
\]

for each \( E \) in \( \Sigma \).

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