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SUPERCUSPIDAL COMPONENTS OF THE QUATERNION WEIL REPRESENTATION OF $\mathrm{SL}_2(\mathfrak{k})$

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Let $\mathfrak{k}$ be a p-adic field of odd residual characteristic. It is known that all but one summand of the quaternion Weil representation are supercuspidal. These summands are precisely identified in terms of corresponding summands of quadratic extension Weil representations.

1. Let $\mathfrak{k}$ be a p-adic field of odd residual characteristic. From [6] we know that all supercuspidal representations of $G = \mathrm{SL}_2(\mathfrak{k})$ occur as summands of various Weil representations associated with quadratic extensions of $\mathfrak{k}$. It is also known that the Weil representation associated to the unique quaternion division algebra over $\mathfrak{k}$ decomposes into a direct sum of irreducible representations, all but one of which are supercuspidal. The object of this paper is to show just how these representations correspond to summands of quadratic extension Weil representations. The methods depend heavily on [3]. The primary motivation for this paper was the problem of decomposing tensor products of certain supercuspidal representations of G. The authors have been told that some similar computations have been worked out by J. Shalika and W. Casselman.

2. In this paper, the ring of integers in $\mathfrak{k}$ and its prime ideal are denoted respectively by $\mathfrak{o}$ and $\mathfrak{p}$. We choose a generator π of $\mathfrak{p}$ and a non square unit ε in $\mathfrak{o}$. The order of the residue class field will be denoted by q.

For $\theta \in \{\pi, \varepsilon, \varepsilon\pi\}$, we let $\mathfrak{o}_\theta$ and $\mathfrak{p}_\theta$ denote the ring of integers in $\mathfrak{k}(\sqrt{\theta})$ and its prime ideal respectively. Trace and norm of $\mathfrak{k}(\sqrt{\theta})$ over $\mathfrak{k}$ are written τ_θ and ν_θ respectively.

The quaternion division algebra over $\mathfrak{k}$ will be denoted by D. Its integers will be denoted by A and the prime ideal of A will be P. The reduced norm and trace of D over $\mathfrak{k}$ are written respectively ν_D and τ_D. The set $\{1, i, j, k\}$ is a basis of D over $\mathfrak{k}$ where $i^2 = \varepsilon$, $j^2 = \pi$, and $ij = -ji = k$. There are convenient imbeddings of $\mathfrak{k}(\sqrt{\varepsilon})$ and $\mathfrak{k}(\sqrt{\pi})$ in D where $\mathfrak{k}(\sqrt{\varepsilon}) = \{a + bi : a, b \in \mathfrak{k}\}$ and $\mathfrak{k}(\sqrt{\pi}) = \{a + bj : a, b \in \mathfrak{k}\}$. Let S be a complete set of residues of $\mathfrak{o}_\varepsilon/\mathfrak{p}_\varepsilon$ (and thus of A/P) consisting of zero and roots of unity. Then for any $z \in D$ we may write $z = \sum_{n=N}^{\infty} \alpha_n j^n$ where each $\alpha_n \in S$.

Since π can be chosen to be any element in $\mathfrak{k}$ that generates $\mathfrak{p}$, we will generally consider only the cases $\theta = \varepsilon$ and $\theta = \pi$.

3. From [3], we recall some information on the representations

of $\Gamma = \{\gamma \in D: \nu_D(\gamma) = 1\}$. Let $C^\theta = \{z \in \mathfrak{k}(\sqrt{\theta}): \nu_\theta(z) = 1\}$. Given U in $\hat{\Gamma}$, there is a natural way of choosing a character $\psi \in \hat{C}^\theta$ for some θ. We then say that U is of type θ and that U corresponds to ψ. Let U and, consequently θ and ψ, be fixed. Let m be the smallest integer such that U is trivial on $\Gamma_m = \Gamma \cap (1 + P^m)$. If m is even, U is of type $\varepsilon\pi$ or π. If m is odd, U is of type ε. It happens that under this correspondence, no $U \in \hat{\Gamma}$ matches a square trivial character of C^π or $C^{\varepsilon\pi}$.

We will describe U as an induced representation from a subgroup $B(U)$. The inducing representation will have a character χ_U whose degree is either 1 or q. $B(U)$ will always be of the form $C^\theta H$ where H is a subgroup of Γ depending on m. Let M be the smallest odd integer not less than m. We shall describe H by giving its elements modulo Γ_M.

First let $\theta = \pi$. Then m is even. If U corresponds to $\psi \in \hat{C}^\pi$, then the conductor of ψ is $C_s^\pi = C^\pi \cap (1 + \mathfrak{p}_\pi^s)$ where $s = m$. Then H is given by the set of elements $\nu_D(\gamma)^{-1/2}\gamma$ where $\gamma = 1 + \sum_{n=s/2}^{s+1} \alpha_n j^n$. Each $\alpha_n \in S$ and also $\alpha_n \in S \cap \mathfrak{k}i$ if n is odd. Thus we may write (modulo Γ_M)

$$H = \left\{1 + bij^{s/2} + i\sum_{n=s/2+1}^{s+1} b_n j^n - (-1)^{s/2}\frac{\varepsilon}{2} b^2\pi^{s/2} : b, b_n \in S \cap \mathfrak{k}\right\} .$$

(When $-1 \notin (\mathfrak{k}^\times)^2$, the choice of U may force j and k to be interchanged; this has no effect on the results.) Let $\delta = \alpha h \in B(U)$ where $\alpha \in C^\pi$ and $h \in H$. Then $\chi_U(\delta) = \psi(\alpha)$ and χ_U is a character of degree one.

Now let $\theta = \varepsilon$. Then m is odd. If U corresponds to $\psi \in \hat{C}^\varepsilon$, then the conductor of ψ is $C_s^\varepsilon = C^\varepsilon \cap (1 + \mathfrak{p}_\varepsilon^s)$ where $2s - 1 = m = M$. Let H be given by those elements $\nu_D(\gamma)^{-1/2}\gamma$ where $\gamma = 1 + \sum_{n=s-1}^{2s-3} \alpha_n j^n$ and $\alpha_n \in S$ is zero when n is even.

Assume first that s is odd. Then $\nu_D(\gamma)^{-1/2} \equiv 1$ (modulo Γ_M) so that γ in the form above is in H. Here χ_U is of degree one and for $\alpha \in C^\varepsilon$, we have $\chi_U(\alpha h) = \psi(\alpha)$.

Now let s be even. Then $h \in H$ is in the form $\mu(\beta)h_0$ where $\mu(\beta) = 1 + \beta j^{s-1} + (1/2)\nu_\varepsilon(\beta)\pi^{s-1}$, $\beta \in S$, and $h_0 \in H \cap (1 + P^s)$. For $b \in S \cap \mathfrak{k}$, we define $\omega_M(b) \in C^\varepsilon / C^\varepsilon \cap \Gamma_M$ by $\omega_M(b) \equiv 1 + \pi^{s-1}bi$, modulo $1 + P^M$. Let $\alpha \in C^\varepsilon$ be written $\alpha = \alpha_0\alpha_1$ where $\alpha_0 \in C^\varepsilon \cap S$ and $\alpha_1 \in C_1^\varepsilon$. Then we have

$$\chi_U(\alpha_0\alpha_1\mu(\beta)h_0) = \begin{cases} -\psi(\alpha_0\alpha_1)\psi\left[\omega_M\left[\dfrac{r}{2t\varepsilon}\nu_\varepsilon(\beta)\right]\right], & \text{if } \alpha_0 \neq \pm 1 \\ q\psi(\alpha_0\alpha_1), & \text{if } \beta = 0 \text{ and } \alpha_0 = \pm 1 \\ 0, & \text{if } \alpha_0 = \pm 1 \text{ and } \beta \neq 0 . \end{cases}$$

Here $\alpha_0 = r + ti$. χ_U is a character of degree q.

4. In this section we give definitions for the Weil representations using formulae from [5]. We fix once and for all a character Φ of $\mathfrak{k}^+$ with conductor $\mathfrak{o}$. For $\lambda \in \mathfrak{k}$, let $\Phi_\lambda(x) = \Phi(\lambda x)$. Haar measures on the additive groups of $\mathfrak{k}(\sqrt{\theta})$ and D are normalized so that in each case the ring of integers has unit measure.

We let

$$p(\theta) = \begin{cases} 1, & \text{for } \theta = \varepsilon \\ \zeta, & \text{for } \theta = \pi \end{cases}$$

where $\zeta = \sum_{x \in \mathfrak{o}/\mathfrak{p}} \Phi(\pi^{-1}x^2)$.

The quadratic extension Weil representations of G will be denoted $T(\theta, \lambda)$. They act on $L^2(\mathfrak{k}(\sqrt{\theta}))$. We will define $T(\theta, \lambda)$ on generators of G. Let $f \in L^2(\mathfrak{k}(\sqrt{\theta}))$. Then

$$\left[T(\theta, \lambda)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f\right](z) = \Phi_\lambda(b\nu_\theta(z))f(z)$$

$$\left[T(\theta, \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right](z) = c(\theta, \lambda)p(\theta)\mathrm{sgn}_\theta(-\lambda)\int_{\mathfrak{k}(\sqrt{\theta})} f(w)\Phi_\lambda(\tau_\theta \bar{z} w)dw$$

where $c(\theta, \lambda) > 0$ is a constant chosen so as to make the second operator unitary.

The quaternion Weil representation is denoted $T(D, \lambda)$ (although all choices of λ give equivalent representations). Let $f \in L^2(D)$. Then

$$\left[T(D, \lambda)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f\right](z) = \Phi_\lambda(b\nu_D(z))f(z)$$

$$\left[T(D, \lambda)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right](z) = -c(D, \lambda)\int_D f(w)\Phi_\lambda(\tau_D(\bar{z}w))dw .$$

Again $c(D, \lambda) > 0$ is chosen so that the second operator is unitary.

We have the well known decomposition (see [2] or [7])

$$T(\theta, \lambda) = \coprod_{\psi \in \hat{C}^\theta} T(\theta, \lambda, \psi)$$

where the representation space of $H(\theta, \lambda, \psi)$ of $T(\theta, \lambda, \psi)$ is given by

$$\{f \in L^2(\mathfrak{k}(\sqrt{\theta})) : \forall z \in \mathfrak{k}(\sqrt{\theta}),\ \forall \alpha \in C^\theta,\ f(z\alpha) = f(z)\psi(\alpha)\} .$$

Similarly

$$T(D, \lambda) = \coprod_{U \in \hat{\Gamma}} T(D, \lambda, U) .$$

Let θ_U be the character of U. Then the representation space of $T(D, \lambda, U)$ is

$$H(U) = \left\{ f \in L^2(D) \colon \int_\Gamma f(z\gamma)\overline{\theta_U(\gamma)}d\gamma = f(z) \right\} .$$

For $U \neq 1$, $H(U)$ consists of supercuspidal summands. We can write the vector space sum

$$H(U) = H^1(U) \oplus H^\varepsilon(U) \oplus H^\pi(U) \oplus H^{\varepsilon\pi}(U) .$$

Here $H^\delta(U) = \{f \in H(U) \colon f(x) \neq 0 \Rightarrow \nu_D(x) \in \delta(\mathfrak{k}^\times)^2\}$. Now suppose U is of type θ for some $\theta \in \{\varepsilon, \pi, \varepsilon\pi\}$. Let $\{\theta', \theta''\} = \{\varepsilon, \pi, \varepsilon\pi\} - \{\theta\}$. From Proposition 1.5 of [4], we may conclude that $H^1(U) \oplus H^\theta(U)$ and $H^{\theta'}(U) \oplus H^{\theta''}(U)$ are both G-invariant. (See also Lemma 4.1 of [3].)

5. From [5] and [7], we know that all supercuspidal representations of G are induced from some compact open subgroup. For suitable λ, we may assume that $T(\theta, \lambda, \psi)$ is induced from $K = \mathrm{SL}_2(\mathfrak{o})$. Let the inducing representation be denoted by $S(\theta, \lambda, \psi)$. The object of this section is to give explicit formulae for matrix coefficients of $S(\theta, \lambda, \psi)$ at generators of K. To do this we must pick out an orthonormal basis of the representation space of $S(\theta, \lambda, \psi)$.

Let ψ have conductor $C_s^\theta = C^\theta \cap (1 + \mathfrak{p}_\theta^s)$. We will exclude the cases where $\psi \in \hat{C}^\pi$ (or $\hat{C}^{\varepsilon\pi}$) and $\psi^2 \equiv 1$. Then choose λ to be of order n (i.e., $\lambda = u\pi^n$ where $u \in \mathfrak{o}^\times$) where

$$n = \begin{cases} -s, & \text{if } \theta = \varepsilon \\ -\left(\dfrac{s}{2} + 1\right), & \text{if } \theta = \pi . \end{cases}$$

From [2] or [7] we see that this is the "suitable choice" of λ. Now set

$$n' = \begin{cases} s, & \text{if } \theta = \varepsilon \\ s + 1, & \text{if } \theta = \pi . \end{cases}$$

Let $H_\theta(s)$ be the space of all functions supported on $\mathfrak{o}_\theta$ and constant on cosets of $\mathfrak{p}_\theta^{n'}$. Let $H_0(\theta, \lambda, \psi) = H_\theta(s) \cap H(\theta, \lambda, \psi)$. $H_0(\theta, \lambda, \psi)$ is then the representation space of $S(\theta, \lambda, \psi)$. The action is simply the action of $T(\theta, \lambda, \psi)$ restricted to K.

We now construct an orthonormal basis of $H_0(\theta, \lambda, \psi)$. Let $J_\theta(s)$ be a complete set of orbit representatives in $\mathfrak{o}_\theta/\mathfrak{p}_\theta^{n'} - \pi\mathfrak{o}_\theta/\mathfrak{p}_\theta^{n'}$

under multiplication by elements of C^{θ}. For each $z \in J_\theta(s)$, we define

$$f_z(x) = \begin{cases} r(s, \theta)\psi(\alpha), & \text{for } x \in z\alpha(1 + \mathfrak{p}_\theta^{n'}) \\ 0, & \text{elsewhere} . \end{cases}$$

Here, $r(s, \theta) > 0$ is chosen to make f_z a unit vector. We do not need to compute it explicitly.

For x and $y \in J_\theta(s)$, let $m_{xy}^{\psi}(g) = \langle S(\theta, \lambda, \psi)(g) f_x | f_y \rangle$. We will now compute m_{xy}^{ψ} for generators of $K = \mathrm{SL}_2(\mathfrak{o})$. First let $g = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ where $b \in \mathfrak{o}$. Then

$$m_{xy}^{\psi}(g) = \begin{cases} \Phi_\lambda(b\nu_\theta(x)), & \text{if } x = y \\ 0, & \text{if } x \neq y . \end{cases}$$

Now let $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We write

$$m_{xy}^{\psi}(g) = \int_{k(\sqrt{\theta})} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f_x \right](z)\overline{f_y(z)}dz .$$

Let $v_\theta(n')$ be the measure of $\mathfrak{p}_\theta^{n'}$. Then

$$m_{xy}^{\psi}(g) = v_\theta(n')c(\theta, \lambda)p(\theta)\mathrm{sgn}_\theta(-\lambda) \sum_{\alpha \in C^\theta / C_s^\theta} \psi(\alpha)\Phi_\lambda(\tau_\theta(\bar{y}x\alpha)) .$$

Since

$$c(\theta, \lambda) = \begin{cases} q^{s/2}, & \text{if } \theta = \pi \\ q^s, & \text{if } \theta = \varepsilon \end{cases}$$

we have

PROPOSITION 5.1. *Let*

$$\phi(s) = \begin{cases} -\dfrac{s}{2} - 1, & \text{if } \theta = \pi \\ -s, & \text{if } \theta = \varepsilon . \end{cases}$$

Then

$$m_{xy}^{\psi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = q^{\phi(s)}p(\theta)\mathrm{sgn}_\theta(-\lambda) \sum_{\alpha \in C^\theta / C_s^\theta} \psi(\alpha)\Phi_\lambda(\tau_\theta(\bar{y}x\alpha)) .$$

6. To describe the representations $T(D, \lambda)$, we wish to find their irreducible subrepresentations. Since the representations $S(\theta, \lambda, \psi)$ induce irreducibly from K, to find copies of $T(\theta, \lambda, \psi)$ in a representation it suffices to look for copies of $S(\theta, \lambda, \psi)$ in its

restriction to K. This we shall do by comparing matrix coefficient functions.

We shall try, insofar as possible, to imitate for the representation $T(D, \lambda)$ the construction of § 5. We shall define vectors F_z which transform nicely under the restriction of $T(D, \lambda)$ to K and then study the corresponding coefficient functions.

Let U correspond to a character ψ of C^θ with conductor C_s^θ. Choose λ of an order determined by s as in § 5. As in § 3, let

$$M = \begin{cases} 2s - 1, & \text{if } \theta = \varepsilon \\ s + 1, & \text{if } \theta = \pi \end{cases}$$

so that Γ_M is the largest congruence subgroup of Γ contained in the kernel of U. Let the set of C^θ-orbit representatives $J_\theta(s)$ be imbedded in D in the natural way mentioned in §2. For $z \in J_\theta(s)$, set

$$F_z(x) = \begin{cases} \chi_U(\gamma)R(s)q^{-V(z)/2}, & \text{for } x \in z\gamma(1 + P^M) \\ 0, & \text{elsewhere} . \end{cases}$$

Here $R(s) > 0$ is chosen to make $\|F_z\|_2 = 1$ and $P^{V(z)}$ is the smallest power of P containing z.

It should be noted here that the F_z's do not necessarily span a K-invariant subspace of $H^1(U) \oplus H^\theta(U)$. The unfortunate case is when $\theta = \varepsilon$ and s is even. We shall disregard this problem for now and go on to compute matrix coefficients in all cases.

For $g \in K$, let $M_{xy}^U(g) = \langle T(D, \lambda)(g)F_x | F_y \rangle$. When $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ we get an easy result:

LEMMA 6.1. *Let U correspond to $\psi \in \hat{C}^\theta$ and let $\phi \in \hat{C}^\theta$ be any character with the same conductor as ψ. Then* $M_{xy}^U\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = m_{xy}^\phi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *where $b \in \mathfrak{o}$.*

Proof.

$$M_{xy}^U\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{cases} \Phi_\lambda(b\nu_D(x)), & \text{if } x = y \\ 0, & \text{if } x \neq y . \end{cases}$$

Since for $x \in J_\theta(s)$, $\nu_D(x) = \nu_\theta(x)$, the result follows. □

Thus to distinguish the various representations, we must evaluate the coefficient functions at $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Roughly speaking, this element acts as a Fourier transform, so we are obliged to compute various character sums. Recall the groups H and $B(U)$; H was defined in § 3 and then $B(U)$ was defined by $B(U) = C^\theta H$. Let $\mathscr{B}_M(U) = B(U)/(1 + P^M)$. Then

$$M_{xy}^U(g) = \int_D \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_x\right](w)\overline{F_y(w)}dw$$
$$= R(s)q^{-((1/2)V(y)+2M)} \sum_{\gamma \in \mathscr{B}_M(U)} \left[\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] F_x\right](y\gamma)\overline{\chi_U(\gamma)} .$$

Now $c(D, \lambda) = q^M$ so

$$M_{xy}^U(g) = -R(s)^2 q^{-((1/2)V(xy)+3M)} \sum_{\gamma,\delta \in \mathscr{B}_M(U)} \chi_U(\delta)\Phi_\lambda(\tau_D \bar{y} x \delta \bar{\gamma})\overline{\chi_U(\gamma)}$$
$$= -R(s)^2 q^{-((1/2)V(xy)+3M)} |\mathscr{B}_M(U)| (\deg \chi_U)^{-1} \sum_{\delta \in \mathscr{B}_M(U)} \chi_U(\delta)\Phi_\lambda(\tau_D \bar{y} x \delta) .$$

Since $R(s)^2 q^{-2M} |\mathscr{B}_M(U)| = 1$ we have

LEMMA 6.2.

$$M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-((1/2)V(xy)+M)}(deg\chi_U)^{-1} \sum_{\delta \in \mathscr{B}_M(U)} \chi_U(\gamma)\Phi_\lambda(\tau_D \bar{y} x \delta) .$$

It is this formula which we must evaluate more fully. We now consider several cases. First let $\theta = \pi$ and assume that x and y are units in $J_\pi(s)$. Then

$$M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-(s+1)} \sum_{\delta \in \mathscr{B}_M(U)} \chi_U(\gamma)\Phi_\lambda(\tau_D \bar{y} x \delta) .$$

From § 3 we see that the elements of $\mathscr{B}(U)$ can be identified with pairs (α, h) where $\alpha \in C^\pi/C_s^\pi$ and $h \in \mathscr{H} = H/(1 + P^M)$. Morever elements of $\mathscr{H}$ can be expressed in the form

$$1 + bij^{s/2} - (-1)^{s/2}\frac{\varepsilon}{2}b^2\pi^{s/2} + i \sum_{n=s/2+1} b_n j^n$$

where for all n, $b_n \in S \cap \mathfrak{k}$. Thus

$$M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-(s+1)} \sum_{\alpha \in C^\pi/C_s^\pi} \psi(\gamma)\Phi_\lambda(\tau_\pi \bar{y} x \alpha) \sum_{h \in \mathscr{H}} \Phi_\lambda(\tau_D \bar{y} x \alpha(h-1))$$

where U corresponds to $\psi \in \hat{C}^\pi$. Consider the inside sum over $\mathscr{H}$.

$$(*) = \sum_{h \in \mathscr{H}} \Phi_\lambda(\tau_D \bar{y} x \alpha(h-1)) = q^{s/2} \sum_{b \in S \cap \mathfrak{k}} \Phi_\lambda(-yx\tilde{\alpha}(-1)^{s/2}\varepsilon\pi^{s/2}b^2)$$

where $\tilde{\alpha} = \pm 1$ is the image of α modulo C_1^π. Now let σ be the character of order 2 on the group of units of $\mathfrak{o}_\pi$ whose kernel is the squares. Then

$$(*) = -\sigma(\bar{y}x)\sigma(\alpha)\mathrm{sgn}_\pi(-\lambda)p(\pi)q^{s/2} .$$

Thus we have

LEMMA 6.3. *Let $\theta = \pi$ and assume that $x, y \in \mathfrak{o}_\pi^\times \cap J_\pi(s)$. Then with σ as above and $U \in \hat{\Gamma}$ corresponding to $\psi \in \hat{C}^\pi$ we have*

$$M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma(\bar{y}x)q^{-(s/2)-1}p(\pi)\mathrm{sgn}_\pi(-\lambda) \sum_{\alpha \in C^\pi/C_s^\pi} (\psi\sigma)(\alpha)\Phi_\lambda(\tau_\pi(\bar{y}x\alpha)) .$$

LEMMA 6.4. *If x and y are both elements of $\mathfrak{p}_\pi$, then*

$$m_{xy}^{\psi\sigma}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

where U corresponds to ψ as before.

Proof. If either x or y is in $\mathfrak{p}_\pi$, the expression $(*)$ no longer depends on α. Thus

$$\sum_{\alpha \in C^\pi/C_s^\pi} (\psi\sigma)(\alpha)\Phi_\lambda(\tau_\pi(\bar{y}x\alpha))$$

is a common factor of $m_{xy}^{\psi\sigma}$ and M_{xy}^U. If both x and y are in $\mathfrak{p}_\pi$, then this factor is zero since $\Phi_\lambda(\tau_\pi(\bar{y}x\alpha))$ is constant on cosets of a subgroup of C^π on which $\psi\sigma$ is nontrivial. □

Now let $\theta = \varepsilon$. Thus we take $U \in \hat{\Gamma}$ to be of type ε corresponding to $\psi \in \hat{C}^\varepsilon$. Let ψ have conductor $C^\varepsilon \cap (1 + \mathfrak{p}_\varepsilon^s)$. Then set $M = 2s - 1$. We consider cases according to the parity of s. First let s be odd. Then $\deg \chi_U = 1$. A simple computation gives

$$\begin{aligned} M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= -q^{-(2s-1)} \sum_{\delta \in \mathscr{B}_M(U)} \chi_U(\delta)\Phi_\lambda(\tau_D\bar{y}x\delta) \\ &= -q^{-(2s-1)} \sum_{\alpha \in C^\varepsilon/C_s^\varepsilon} [\psi(\alpha)\Phi_\lambda(\tau_\varepsilon\bar{y}x\alpha) \sum_{\mathscr{H}} \Phi_\lambda(\tau_D\bar{y}x\alpha(h-1))] . \end{aligned}$$

Now λ is chosen according to the prescription in § 5. Thus for s odd, $\mathrm{sgn}_\varepsilon(-\lambda) = -1$. In this case, $\Phi_\lambda(\tau_D\bar{y}x\alpha(h-1)) = 1$ for all $h \in \mathscr{H}$. Thus we have

LEMMA 6.5. *Let s be odd. Then*

$$M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{sgn}_\varepsilon(-\lambda)p(\varepsilon)q^{-s} \sum_\alpha \psi(\alpha)\Phi_\lambda(\tau_\varepsilon\bar{y}x\alpha) .$$

The case when s is even is more involved. From § 3, we recall that $\deg \chi_U = q$. Therefore by Lemma 6.2

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-2s} \sum_{\delta \in \mathscr{B}_M(U)} \chi_U(\delta)\Phi_\lambda(\tau_D \bar{y}x\delta) .$$

We shall simplify this sum and find

LEMMA 6.6. *Let s be even. Then*

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{cases} \operatorname{sgn}_\varepsilon(-\lambda)p(\varepsilon)q^{-s} \sum_{\alpha \in C^\varepsilon/C_s^\varepsilon} \psi(\alpha)\Phi_\lambda(\tau_\varepsilon \bar{y}x\alpha), & \text{if } \bar{y}x \in \mathfrak{k} \\ -q^{-s-1} \sum_{\alpha \in C^\varepsilon/C_s^\varepsilon} \psi(\alpha)\Phi_\lambda(\tau_\varepsilon \bar{y}x\alpha), & \text{if } \bar{y}x \notin \mathfrak{k} . \end{cases}$$

Proof. We need to find an acceptable parametrization of the elements of $\mathscr{B}_M(U)$. Recall that $\delta \in B(U)$ is of the form $\delta = \alpha h$ where $\alpha \in C^\varepsilon$ and $h \in H$. For $\alpha \in C^\varepsilon$, we have $\alpha = \alpha_0\rho\omega_M(b)$ (modulo C_s^ε) where $\alpha_0 \in S \cap C^\varepsilon$, ρ is a representative of C_{s-1}^ε in C_1^ε, and $\omega_M(b) \in C_s^\varepsilon$ is as defined in § 3. For $h \in H$ we have $h = \mu(\beta)h_0$ where $\mu(\beta) = 1 + \beta j^{s-1} + (1/2)\varepsilon\nu_\varepsilon(\beta)\pi^{s-1}$, $\beta \in S$, and $h_0 \in H \cap 1 + P^s$. As before, let $\mathscr{H} = H/\Gamma_M$ and also let $\mathscr{H}_0 = [H \cap (1 + P^s)]/\Gamma_M$. $\mathscr{H}_0$ is given by elements of the form

$$1 + \sum_{n=s+1}^{2s-2} \beta_n j^n$$

where $\beta_n \in S$. Now

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-2s} \sum_{\alpha \in C^\varepsilon/C_s^\varepsilon} \sum_{h \in \mathscr{H}} \chi_U(\alpha h)\Phi_\lambda(\tau_D \bar{y}x\alpha)\Phi_y(\tau_D \bar{y}x\alpha(h-1)) .$$

We choose $v \in \mathfrak{o}^\times$ such that $\psi(\omega_M(b)) = \Phi_\lambda(\pi^{s-1}vb)$. Let $Z_s^\varepsilon = C^\varepsilon/C_s^\varepsilon - [\pm C_1^\varepsilon/C_s^\varepsilon]$. Then

$$(**) \quad \begin{aligned} M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = &-q^{-2s} \sum_{\alpha \in \pm C_1^\varepsilon / C_s^\varepsilon} q\psi(\alpha)\Phi_\lambda(\tau_D \bar{y}x\alpha) \sum_{h \in \mathscr{H}} \Phi_\lambda(\tau_D \bar{y}x\alpha(h-1)) \\ &- q^{-2s} \sum_{\alpha \in Z_s^\varepsilon} \chi_U(\alpha h)\Phi_\lambda(\tau_D \bar{y}x\alpha)\Phi_\lambda(\tau_D \bar{y}x\alpha(h-1)) . \end{aligned}$$

Let us assume for the moment that $\bar{y}x \in \mathfrak{k}$. It is easy to check that for $\alpha \in \pm C_1^\varepsilon$, the expression $\Phi_\lambda(\tau_D \bar{y}x\alpha) \sum_{h \in \mathscr{H}} \Phi_\lambda(\tau_D \bar{y}x\alpha(h-1))$ is constant on cosets of C_{s-1}^ε. Hence the first sum in (**) vanishes. Thus

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-2s} \sum_{\alpha \in Z_s^\varepsilon} \sum_{h \in \mathscr{H}} \chi_U(\alpha h)\Phi_\lambda(\tau_D \bar{y}x\alpha)\Phi_\lambda(\tau_D \bar{y}x\alpha(h-1)) .$$

Let $\alpha \in Z_s^\varepsilon$ be given in the form $\alpha_0\rho\omega_M(b)$ where $\alpha_0 \neq \pm 1$, $\rho \in C_1^\varepsilon/C_{s-1}^\varepsilon$, and $\omega_M(b) \in C_{s-1}^\varepsilon/C_s^\varepsilon$. Let $\alpha_0 = r + ti$. This yields

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sum_{\alpha_0 \neq \pm 1} \Big(\sum_{\rho, b} - \psi(\alpha_0\rho)\Phi_\lambda(\pi^{s-1}vb)\Phi_\lambda(\tau_D\bar{y}x\alpha_0\rho\omega_M(b))\Big)$$
$$\cdot\left(\sum_{\beta\in S}\sum_{h\in\mathscr{H}_0}\Phi_\lambda\left(\pi^{s-1}\frac{vr}{2t\varepsilon}\nu_\varepsilon(\beta)\right)\Phi_\lambda\left(\tau_D\bar{y}x\alpha_0\frac{1}{2}\nu_\varepsilon(\beta)\pi^{s-1}\right)\right).$$

The first factor of each term can be written

$$\sum_\rho\sum_b - \psi(\alpha_0\rho)\Phi_\lambda(\pi^{s-1}vb)\Phi_\lambda(2\pi^{s-1}\bar{y}xt\varepsilon b)\,.$$

Hence if the term corresponding to $\alpha_0 = r + ti$ is to be nonzero, we will have $-v = 2\bar{y}xt\varepsilon$. This fixes t. The second factor then becomes

$$\sum_{\beta\in S}\sum_{h\in\mathscr{H}_0}\Phi_\lambda(-\pi^{s-1}r\bar{y}x\nu_\varepsilon(\beta))\Phi_\lambda\left(\tau_D\bar{y}x(r+ti)\frac{1}{2}\nu_\varepsilon(\beta)\pi^{s-1}\right)$$

which simplifies to q^s. Hence

$$M_{xy}^{U}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -q^{-s}\sum_{\alpha\in(\pm r+ti)C_1^\varepsilon/C_s^\varepsilon}\psi(\alpha)\Phi_\lambda(\tau_\varepsilon\bar{y}x\alpha)\,.$$

But this is just

$$\operatorname{sgn}_\varepsilon(-\lambda)q^{-s}\sum_{\alpha\in C^\varepsilon/C_s^\varepsilon}\psi(\alpha)\Phi_\lambda(\tau_D\bar{y}x\alpha)$$

since for s even, $\operatorname{sgn}_\varepsilon(-\lambda) = 1$ and all terms vanish for which $\alpha\notin(\pm r+ti)C_1^\varepsilon$.

Now assume $\bar{y}x\notin\mathfrak{k}$, that is, $\bar{y}x = m + ni$ where n is a unit. Consider the formula (**). It is no longer necessarily true that the first sum vanishes. In any case it can be simplified to

$$-q^{-s-1}\sum_{\alpha\in\pm C_1^\varepsilon/C_s^\varepsilon}\psi(\alpha)\Phi_\lambda(\tau_D\bar{y}x\alpha)\,.$$

The second sum in (**) is

$$-q^{-2s}\sum_{\alpha\in Z_s^\varepsilon}\sum_{h\in\mathscr{H}}\chi_U(\alpha h)\Phi_\lambda(\tau_D\bar{y}x\alpha)\Phi_\lambda(\tau_D\bar{y}x\alpha(h-1))$$
$$= -q^{-2s}\sum_{\alpha_0\neq 1}\Big(\sum_{\rho,b}-\psi(\alpha_0\rho)\Phi_\lambda(\pi^{s-1}vb)\Phi_\lambda(\tau_D\bar{y}x\alpha)\Big)$$
$$\cdot\left(\sum_{\beta\in S}\sum_{h\in\mathscr{H}_0}\Phi_\lambda\left(\pi^{s-1}\frac{vr}{2t\varepsilon}\nu_\varepsilon(\beta)\right)\Phi_\lambda\left(\tau_D\bar{y}x\alpha\frac{1}{2}\nu_\varepsilon(\beta)\right)\right).$$

Reasoning as before, we see that the first factor in each term is zero unless $2mt\varepsilon + 2nr\varepsilon = -v$. In that event the second factor becomes

$$\sum_{h \in \mathscr{M}_0} \sum_{\beta \in S} \Phi_\lambda\left(\pi^{s-1}\left(\frac{-n}{t}\right)\nu_\varepsilon(\beta)\right) = -q^{s-1} . \qquad \square$$

7. We now state and prove the main result. Let $U \in \hat{\Gamma}$ be chosen where U is nontrivial and of degree d. Let U correspond to $\psi \in \hat{C}^\theta$ in the manner of § 3. Choose $\lambda \in \mathfrak{k}^\times$ and set

$$\lambda' = \begin{cases} \pi\lambda, & \text{for } \theta = \varepsilon \\ \varepsilon\lambda, & \text{for } \theta \neq \varepsilon . \end{cases}$$

Let $\sigma \in \hat{\mathfrak{o}}_\theta^\times$ be the character of order 2 whose kernel is the squares. Set

$$\tilde{\psi} = \begin{cases} \psi, & \text{if } \theta = \varepsilon \text{ or } -1 \in (\mathfrak{k}^\times)^2 \\ \psi\sigma, & \text{if } \theta \neq \varepsilon \text{ and } -1 \notin (\mathfrak{k}^\times)^2 . \end{cases}$$

THEOREM 7.1. $T(D, \lambda, U) \cong d^2[T(\theta, \lambda, \tilde{\psi}) \oplus T(\theta, \lambda', \tilde{\psi})]$.

LEMMA 7.2. *To prove the theorem, it suffices to consider the case where λ is determined by ψ as in* § 5.

Proof. Let λ_1 and λ_2 in $\mathfrak{k}^\times$ be given. Then there exists an element $g_0 \in \mathrm{GL}_2(\mathfrak{k})$ such that $T(\theta, \lambda_1)(g_0 g g_0^{-1}) = T(\theta, \lambda_2)(g)$ and $T(D, \lambda_1)(g_0 g g_0^{-1}) = T(D, \lambda_2)(g)$. (See Lemma 3.3 of [1].) $\square$

Now let $U \in \hat{\Gamma}$ be fixed with corresponding $\psi \in \hat{C}^\theta$. Let M be determined as in § 3. Let V_M be the space of functions on D which are supported on A and are constant on cosets of P^M. With λ chosen as before, we see that V_M is a finite-dimensional K-invariant subspace of $L^2(D)$. For $\delta \in \{1, \varepsilon, \pi, \varepsilon\pi\}$ set $H_0^\delta(U) = H^\delta(U) \cap V_M$. Thus $H_0^1(U) \oplus H_0^\theta(U)$ is also K-invariant. The following lemma can be derived from basic properties of $T(D, \lambda)$.

LEMMA 7.3. *Assume that $W \subset H^1(U) \oplus H^\theta(U)$ is isomorphic to some $H(\theta, \lambda, \rho)$. Then*

(a) $H^1(U) \oplus H^\theta(U) \cong d^2 H(\theta, \lambda, \rho)$.

(b) $H^{\theta'}(U) \oplus H^{\theta''}(U) \cong d^2 H(\theta, \lambda', \rho)$ *where* $\{\theta', \theta''\} = \{1, \pi, \varepsilon, \varepsilon\pi\} - \{1, \theta\}$.

(c) $H_0^1(U) \oplus H_0^\theta(U) \cong d^2 H_0(\theta, \lambda, \rho)$.

Since we know that for $U \neq 1$, $T(D, \lambda, U)$ consists entirely of supercuspidal summands, we can reduce the problem to that of finding ρ.

The method we use is to compare directly the matrices operating on $H_0(\theta, \lambda, \rho)$ with those operating on $H_0^1(U) \oplus H_0^\theta(U)$.

First let $\theta = \varepsilon$ and let s be odd. Then the vectors F_y span a K-invariant subspace of $H_0^1(U) \oplus H_0^\varepsilon(U)$. This is because the matrix given by $M_{xy}^U(g)$ is identical with the unitary matrix $m_{xy}^{\psi}(g)$ for all $g \in K$. This additionally shows that $\psi = \rho$ and proves the theorem in this case.

Now assume $\theta = \pi$ (or $\varepsilon\pi$). It is not hard to see that the matrices given by $m_{xy}^{\tilde{\psi}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are conjugate by a diagonal unitary matrix. Since the matrices for other generators of K are equal and diagonal, the theorem follows.

When $\theta = \varepsilon$ and s is even, we need to say more. When $\bar{y}x \notin \mathfrak{k}$, $M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is smaller by a factor of q than the corresponding $m_{xy}^{\tilde{\psi}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since all representations are unitary we must conclude that $\{F_x : x \in J_\varepsilon(s)\}$ does not span a K-invariant space when s is even.

Let $J_\varepsilon(s) = X \cup Y$ where $H_0^1(U)$ contains $\{F_x : x \in X\}$ and $H_0^\varepsilon(U)$ contains $\{F_y : y \in Y\}$. We can safely assume that $\bar{y}x \in \mathfrak{k}$ if and only if x and y are both in X or both in Y. We need to find a set $\{\tilde{F}_y : y \in Y\}$ such that $\{F_x : x \in X\} \cup \{\tilde{F}_y : y \in Y\}$ spans a K-invariant space. The following lemmas are derived from the constructions of the functions F_x and f_x, $x \in J_\varepsilon(s)$, and the formulae in [4].

LEMMA 7.4. *$H_0^\varepsilon(U)$ is the orthogonal direct sum of eigenspaces W_y of $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathfrak{o}\right\}$. The dimension of each W_y is d^2. For $y \in Y$, $F_y \in W_y$ and W_y is spanned by the left and right translates of F_y by elements in Γ.*

LEMMA 7.5. *Fix $y_0 \in Y$. Then for each $y \in Y$ there exists a $g_y \in K$ such that*

(a) $[T(\varepsilon, \lambda, \psi)(g_y)](f_{y_0}) = f_y$.

(b) $[T(D, \lambda, U)(g_y)](F_{y_0}) = F_y$.

Let W be the irreducible K-space generated by $\{F_x : x \in X\}$. Since $s \neq 1$ there exists x such that $m_{xy_0}^{\psi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0$ and therefore $M_{xy_0}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0$. Hence we may pick $\tilde{F}_{y_0} \in W_{y_0}$ such that $\tilde{F}_{y_0} \in W$ and $\|\tilde{F}_{y_0}\|_2 = 1$. For $y \in Y$, set $\tilde{F}_y = [T(D, \lambda, U)(g_y)](\tilde{F}_{y_0})$. The set $\{F_x, \tilde{F}_y : x \in X, y \in Y\}$ is thus a basis for a K-invariant subspace of $H_0^1(U) \oplus H_0^\varepsilon(U)$. Let $\{\tilde{M}_{xy}^U\}$ be the set of matrix coefficient functions on K with respect to this new basis.

LEMMA 7.6. *There exists a constant $\mu \neq 0$ such that $\tilde{F}_y = \mu F_y + P_y$ where P_y is orthogonal to F_y.*

Proof. For each $y \in Y$, $\langle \tilde{F}_y | F_y \rangle = \langle \tilde{F}_{y_0} | F_{y_0} \rangle$ since $T(D, \lambda, U)(g_y)$ is a unitary transformation. If μ were zero, then all $M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ would be zero for $x \in X$, $y \in Y$. Then $m_{xy}^{\psi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ would also be zero in these cases. This contradicts the irreducibility of $T(\varepsilon, \lambda, \psi)$ for $\psi^2 \neq 1$. (See [2] and [7].) □

LEMMA 7.7. *There exists a K-space isomorphism* $\alpha: F_y \to \tilde{F}_y$ *for each* $y \in Y$.

Proof. By Lemmas 7.3(c) and 7.4, we can find such α for $y = y_0$. Since α commutes with action by K the lemma follows. □

LEMMA 7.8. *For x and y both in X or both in Y we have* $\tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. For x and $y \in X$ it is clear. For x and y in Y, use Lemma 7.7. □

LEMMA 7.9. *Let $x \in X$ and $y \in Y$. Then*

$$\tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mu^{-1} M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Let w be the projection of $\left[T(D, \lambda, U)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right](F_x)$ on W_y. Then $w = \tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\tilde{F}_y = \tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(\mu F_y + P_y)$. Now $\langle w | F_y \rangle = M_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by definition and is equal to $\mu \tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by computation. □

The fact that the functions $\tilde{M}_{xy}^U\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $m_{xy}^{\psi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ form unitary matrices forces $|\mu| = q^{-1}$. It is now clear that these matrices are conjugate by some diagonal unitary matrix. Theorem 7.1 follows. □

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