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THE SHEAF OF H^p-FUNCTIONS IN PRODUCT DOMAINS

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Let $W=W_1\times W_2\times\cdots\times W_n$ be a bounded polydomain in C^n such that the boundary of each W_i consists of finitely many disjoint Jordan curves. The correspondence that assigns to every relatively open polydomain V in $\bar{W}$ (the closure of W) the Hardy space $\mathscr{H}^p(V\cap W)$, defines a sheaf $\hat{\mathscr{H}}^p_W$ over $\bar{W}$. This sheaf is locally determined in the sense that $\Gamma(\bar{W}, \hat{\mathscr{H}}^p_W)$ is canonically isomorphic to $\mathscr{H}^p(W)$. In this paper we prove, for any $0<p<\infty$ and all integers $q\geq 1$, that the cohomology groups $H^q(\bar{W}, \hat{\mathscr{H}}^p_W)$ are trivial.

I. Introduction. The *Hardy spaces* $\mathscr{H}^p(U^n)$, $0<p<\infty$, for the unit polydisc U^n, consist of all functions F which are holomorphic in U^n and satisfy

$$\sup_{0<r<1}\int_0^{2\pi}\cdots\int_0^{2\pi}|F(re^{i\theta_1},\cdots,re^{i\theta_n})|^p d\theta_1\cdots d\theta_n<+\infty\ .$$

The observation ([9, Exercise 3.4.4(b), p. 52]) that $F\in\mathscr{H}^p(U^n)$ if and only if F is holomorphic and $|F|^p$ has an n-harmonic majorant in U^n, leads to a definition of Hardy spaces for arbitrary product domains; the requirement now being that F be holomorphic and $|F|^p$ have an n-harmonic majorant in the polydomain in question.

The symbol $\mathscr{H}^p$ can thus be regarded as a presheaf on the polydomains in C^n. In this paper we concern ourselves with the sheaf induced by $\mathscr{H}^p$ on the closure of a polydomain, and prove, under certain topological restrictions, that the corresponding cohomology groups are trivial.

Specifically, let $W=W_1\times W_2\times\cdots\times W_n$ be a bounded polydomain in C^n, and suppose each W_i is bounded by finitely many disjoint Jordan curves. The correspondence that assigns to each relatively open product domain V in $\bar{W}$ (the closure of W) the linear space $\mathscr{H}^p(V\cap W)$, defines a sheaf $\hat{\mathscr{H}}^p_W$ over $\bar{W}$. This sheaf is *locally determined*, i.e., $\Gamma(\bar{W}, \hat{\mathscr{H}}^p_W)$ is canonically isomorphic to $\mathscr{H}^p(W)$. Our goal is to prove, for any such W, for $0<p<\infty$, and for all integers $q\geqq 1$, that the cohomology groups $H^q(\bar{W}, \hat{\mathscr{H}}^p_W)$ are trivial.

In [8] A. Nagel proved similar results for a wide class of sheaves of holomorphic functions satisfying boundary conditions in polydomains. Although Nagel's methods can be applied to the sheaves $\hat{\mathscr{H}}^p_W$ when $1<p<\infty$, the cases $0<p\leqq 1$ present difficulties. Instead, as in the earlier papers [12], [13], we follow the approach

of E. L. Stout in [11]. In this respect Theorem 3.3, which is central to our study, is the analogue of Lemma 1.2 of [11].

The crux of our work is Theorem 3.3 (the Decomposition Theorem); the proof, together with the necessary groundwork, appears in § III which is essentially self-contained. The basic definitions are listed in § II. In § IV we consider the Čech cohomology with coefficients in $\hat{\mathscr{H}}^p_W$, and prove our main result, Theorem 4.9.

We mention in closing that although most of our results are proven for the case $n > 1$, they are also verified if $n = 1$ (the modifications in the proofs required for this case are always straightforward).

II. Preliminaries. A *polydomain* in C^n is a cartesian product $W_1 \times W_2 \times \cdots \times W_n$ of n open connected subsets (domains) of C. If each W_i is a bounded domain, bounded by finitely many disjoint Jordan curves (a Jordan domain) we say that W is a *Jordan polydomain.*

Possessing an n-harmonic majorant in a Jordan polydomain is a local property (see also [5]):

THEOREM 2.1. [12, Th. 2.10, p. 301]. *Let W be a Jordan polydomain and let $\{U_\alpha\}$ be a relatively open covering of $\bar{W}$. If s is a positive n-subharmonic function in W with "local" n-harmonic majorants u_α in each intersection $U_\alpha \cap W$, then s has an n-harmonic majorant in W.*

DEFINITION 2.2. Let V be a polydomain, and let $0 < p < \infty$. We define the Hardy space $\mathscr{H}^p(V)$ to be the linear space of all functions F which are holomorphic in V and for which $|F|^p$ has an n-harmonic majorant in V. We establish the convention $\mathscr{H}^p(\Phi) = \{0\}$.

DEFINITION 2.3. Let W be a fixed polydomain in C^n. We define the sheaf $\hat{\mathscr{H}}^p_W$ (the sheaf of germs of $\mathscr{H}^p$-functions on $\bar{W}$) as the sheaf over $\bar{W}$ which is induced by the correspondence between the relatively open polydomains $V \subset \bar{W}$ and the linear spaces $\mathscr{H}^p(V \cap W)$.

If W is a Jordan polydomain, it is a direct consequence of Theorem 2.1 that the linear spaces $\Gamma(\bar{W}, \hat{\mathscr{H}}^p_W)$ and $\mathscr{H}^p(W)$ are canonically isomorphic.

If W and V are Jordan polydomains in C^n, with correspondingly conformally equivalent coordinate domains, the sheaves $\hat{\mathscr{H}}^p_W$ and $\hat{\mathscr{H}}^p_V$ are isomorphic; consequently, the cohomology groups of V and W with coefficients in $\hat{\mathscr{H}}^p_V$ and $\hat{\mathscr{H}}^p_W$, respectively, are isomorphic.

This follows from the invariance of the $\mathscr{H}^p$-spaces under n-conformal transformations, and the well known fact that a conformal equivalence between Jordan domains extends to a homeomorphism between their closures.

III. A decomposition theorem. In what follows, U will be the open unit disc $\{z \in \boldsymbol{C}: |z| < 1\}$ and T its boundary, the unit circle. The cartesian product of n copies of U will be denoted by U^n. Similarly, T^n will be the cartesian product of n copies of T. We will denote the normalized Haar measure on T^n by m_n (by m in the particular case $n = 1$); the corresponding $\mathscr{L}^p$-spaces will be indicated by $\mathscr{L}^p(T^n)$, and the $\mathscr{L}^p$-norm by $\|\ \|_{\mathscr{L}^p(T^n)}$. The extended complex plane will be denoted by S^2.

Let F be a holomorphic function in U^n and let $0 < r < 1$. We denote by F_r the function defined on T^n by the equation

$$F_r(w) = F(rw)\ ;$$

and define, for each $0 < p < \infty$,

$$\|F\|_{\mathscr{H}^p(U^n)} = \lim_{r \to 1} \|F_r\|_{\mathscr{L}^p(T^n)}\ .$$

An alternative characterization of the Hardy space $\mathscr{H}^p(U^n)$ is that it consists of all holomorphic F for which

$$\|F\|_{\mathscr{H}^p(U^n)} < +\infty\ .$$

Moreover, if H is the least n-harmonic majorant of $|F|^p$ in U^n, then

$$\|F\|_{\mathscr{H}^p(U^n)} = H(0)\ ,$$

where we denote the n-tuple $(0, 0, \cdots, 0)$ by 0.

We define $\mathscr{H}^p((S^2 - \bar{U}) \times U^{n-1})$ to be the class of all functions F for which the function F^*, defined for $(x, y) \in U \times U^{n-1}$ by

$$F^*(x, y) = F\left(\frac{1}{x}, y\right)\ ,$$

is in $\mathscr{H}^p(U^n)$. If F and F^* are related as above, we write

$$\|F\|_{\mathscr{H}^p((S^2 - \bar{U}) \times U^{n-1})} = \|F^*\|_{\mathscr{H}^p(U^n)}\ .$$

The space of *test functions* on T will be represented by $\mathscr{C}^\infty(T)$, the space of *distributions* on T by $\mathscr{D}(T)$, and the bilinear pairing between $h \in \mathscr{C}^\infty(T)$ and $f \in \mathscr{D}(T)$ by

$$\langle h(\cdot), f(\cdot) \rangle\ .$$

Let $\boldsymbol{Z}$ be the set of integers. For each $j \in \boldsymbol{Z}$ and $w \in T$, we define

$$e_j(w) = w^j\ .$$

The *Fourier coefficients* of $f \in \mathscr{D}(T)$ are the numbers

$$\hat{f}(j) = \langle e_{-j}(\cdot), f(\cdot) \rangle ,$$

where j ranges over $\boldsymbol{Z}$.

Given $F \in \mathscr{H}^p(U)$, $0 < p < \infty$, there exists a unique $f \in \mathscr{D}(T)$ such that the Fourier coefficients $\hat{f}(j)$, with $j \geqq 0$, are the Taylor coefficients of F, and such that $\hat{f}(j) = 0$ whenever $j < 0$. This can be derived, for example, from [3, Th. 6.4, p. 98]. We refer to f as the *boundary distribution* of F.

Let $w \in T$ and $z \in S^2 - T$. The Cauchy kernel $C(z, w)$ is defined by the equation

$$C(z, w) = \frac{1}{1 - \bar{w}z} .$$

If we fix z and allow w to vary, we obtain a test function which we denote by $C(z, \cdot)$. If $F \in \mathscr{H}^p(U)$ has the boundary distribution f, then, for all $z \in U$,

$$F(z) = \langle C(z, \cdot), f(\cdot) \rangle .$$

On the other hand, if $z \notin \bar{U}$,

$$0 = \langle C(z, \cdot), f(\cdot) \rangle .$$

The first part of the next lemma states that the *Toeplitz operators* induced by the functions in $\mathscr{C}^\infty(T)$ extend or restrict to bounded operators on $\mathscr{H}^p(U)$ for $0 < p < \infty$. This was proven in an earlier paper ([14, Th. 3.2]). A straightforward modification of the proof yields part (2).

LEMMA 3.2. [14, Th. 3.2]. *Let* $h \in \mathscr{C}^\infty(T)$, *let* $F \in \mathscr{H}^p(U)$, $0 < p < \infty$, *and let* f *be the boundary distribution of* F. *Define*

$$\mathscr{T}_h F(z) = \langle h(\cdot) C(z, \cdot), f(\cdot) \rangle .$$

There are constants $B = B(p, h)$ *and* $B^* = B^*(p, h)$, *independent of* F, *such that*

$$\text{(1)} \qquad \|\mathscr{T}_h F\|_{\mathscr{H}^p(U)} \leqq B \|F\|_{\mathscr{H}^p(U)} ,$$

and

$$\text{(2)} \qquad \|\mathscr{T}_h F\|_{\mathscr{H}^p(S^2 - \bar{U})} \leqq B^* \|F\|_{\mathscr{H}^p(U)} .$$

For the next theorem, let L_1 and L_2 be disjoint closed arcs on the unit circle T, and define V_j, for $j = 1, 2$, to be the union of the unit disc U, its exterior $S^2 - \bar{U}$, and the interior (relative to T) of L_j.

THEOREM 3.3. (*Decomposition Theorem*). *Let $n > 1$, and let Y be a Jordan polydomain in C^{n-1}. If $F \in H^p(U \times Y)$, $0 < p < \infty$, there exist holomorphic functions F_1 in $V_1 \times Y$ and F_2 in $V_2 \times Y$ such that*

(1) $F(z) = F_1(z) + F_2(z)$ *if* $z \in U \times Y$,

(2) $0 = F_1(z) + F_2(z)$ *if* $z \in (S^2 - \bar{U}) \times Y$,

and, for $j = 1, 2$,

(3) $F_j \in \mathscr{H}^p(U \times Y)$,

(4) $F_j \in \mathscr{H}^p((S^2 - \bar{U}) \times Y)$,

(5) $F_j \in \mathscr{H}^p(D_j \times Y)$ *for some open set* $D_j \subset C$ *that contains* L_j.

Proof. Choose functions $h_j \in \mathscr{C}^\infty(T)$ such that h_j is identically zero on a neighborhood of L_j in T, and such that $h_1(\xi) + h_2(\xi) = 1$ for all $\xi \in T$. If $(x, y) \in U \times Y$ we write $F^y(x) = F(x, y)$. For each $y \in Y$, the function F^y is in $\mathscr{H}^p(U)$; denote its boundary distribution by f^y and define

$$F_j(x, y) = \mathscr{T}_{h_j} F^y(x) = \langle h_j(\cdot)C(x, \cdot), F^y(\cdot)\rangle .$$

We observe that F_j is separately holomorphic in x and y, and hence holomorphic, at all $z = (x, y)$ such that $y \in Y$ and x is not in the closed support of h_j. In particular, F_j is holomorphic in $V_j \times Y$.

Since $h_1 + h_2 \equiv 1$, we have

$$F_1(x, y) + F_2(x, y) = \langle C(x, \cdot), f^y(\cdot)\rangle .$$

Fix $y \in Y$. The right-hand term above, the Cauchy representation formula for F^y, is 0 if $x \in S^2 - \bar{U}$ and $F^y(x) = F(x, y)$ if $x \in U$. This establishes (1) and (2).

To prove the remainder of the theorem, we assume first that Y is the cartesian product of $n - 1$ simply connected domains.

Without loss of generality set $Y = U^{n-1}$. Let H be the least n-harmonic majorant of $|F|^p$ in U^n, and write $H^y(x) = H(x, y)$ for $(x, y) \in U \times U^{n-1}$. The relations

$$F_j(x, y) = \mathscr{T}_{h_j} F^y(x) ,$$

$$\|\mathscr{T}_{h_j} F^y\|_{\mathscr{H}^p(U)} \leqq B\|F^y\|_{\mathscr{H}^p(U)}$$

(part (1) of Lemma 3.2), and

$$\|F^y\|_{\mathscr{H}^p(U)} \leqq H^y(0) ,$$

imply

$$\int_T |F_j(r\xi, r\eta)|^p dm(\xi) \leqq B^p H(0, r\eta)$$

for all $0 < r < 1$ and $w = (\xi, \eta) \in T \times T^{n-1}$. Integrating the above with respect to η, we get

$$\int_{T^n} |F_j(rw)|^p dm_n(w) \leqq B^p H(0) = B^p \|F\|^p_{\mathscr{H}^p(U^n)} .$$

Hence $F_j \in \mathscr{H}^p(U^n)$.

By part (2) of Lemma 3.2 we have

$$\|\mathscr{T}_{h_j} F^y\|_{\mathscr{H}^p(S^2-\bar{U})} \leqq B^* \|F^y\|_{\mathscr{H}^p(U)} .$$

A similar argument to the one used above then establishes $F_j \in \mathscr{H}^p(S^2 - \bar{U}) \times U^{n-1}$.

Finally, for the case $Y = U^{n-1}$, we prove part (5) of the theorem.

Fix $j = 1, 2$. The function h_j will be identically zero on some open connected subset O_j of T which contains the arc L_j. Let H_U and $H_{S^2-\bar{U}}$ be n-harmonic majorants of $|F_j|^p$ in U^n and $(S^2 - \bar{U}) \times U^{n-1}$ respectively. Considered as functions of the single complex variable x, $H_U(x, 0)$ and $H_{S^2-\bar{U}}(x, 0)$ (where 0 is the zero element in C^{n-1}), are positive harmonic functions (in U, and in $S^2 - \bar{U}$). As is well known, they must have nontangential boundary values at almost all points of T. Choose in each of the two connected components of $O_j - L_j$ a point where both $H_U(x, 0)$ and $H_{S^2-\bar{U}}(x, 0)$ simultaneously have a nontangential boundary value. Call these points ζ' and ζ'', and let C be a circle that intersects T precisely at ζ' and ζ''. Let a be the center and ρ the radius of C, we write $C = a + \rho T$ and let D_j be the disc bounded by $a + \rho T$. The function F_j is holomorphic in a neighborhood of $\bar{D}_j \times U^{n-1}$; we proceed to show that $F_j \in \mathscr{H}^p(D_j \times U^{n-1})$, or equivalently, that the function G, defined by $G(x, y) = F_j(a + \rho x, y)$, is in $\mathscr{H}^p(U^n)$.

Since the circle $a + \rho T$ intersects T nontangentially at ζ' and ζ'', there is a constant K such that

$$H_U(x, 0) \leqq K$$

for $x \in (a + \rho T) \cap U$, and

$$H_{S^2-\bar{U}}(x, 0) \leqq K$$

for $x \in (a + \rho T) \cap (S^2 - \bar{U})$. Hence, for all $0 < r < 1$, we have

$$\text{(3.3.1)} \quad \int_{T^{n-1}} |F_j(a + \rho\xi, rn)|^p dm_{n-1}(\eta) \leqq \int_{T^{n-1}} H_U(a + \rho\xi, r\eta) dm_{n-1}(\eta) = H_U(a + \rho\xi, 0) \leqq K$$

whenever $\xi \in T$ is such that $a + \xi \in U$, and

$$\text{(3.3.2)} \quad \int_{T^{n-1}} |F_j(a + \rho\xi, rn)|^p dm_{n-1}(\eta) \leqq \int_{T^{n-1}} H_{S^2-\bar{U}}(a + \rho\xi, r\eta) dm_{n-1}(\eta) = H_{S^2-\bar{U}}(a + \rho\xi, 0) \leqq K$$

whenever $\xi \in T$ is such that $a + \rho\xi \in S^2 - \bar{U}$.

The inequalities (3.3.1) and (3.3.2) yield, for all $0 < r < 1$,

$$\int_T \int_{T^{n-1}} |F_j(a + \rho\xi, r\eta)|^p dm_{n-1}(\eta) dm(\xi) \leqq K .$$

Recalling the definition $G(x, y) = F_j(a + \rho x, y)$, and writing $w = (\xi, \eta)$, we obtain

$$\begin{aligned}\int_{T^n} |G(rw)|^p dm_n(w) &= \int_{T^{n-1}} \int_T |F_j(a + \rho r\xi, r\eta)|^p dm(\xi) dm_{n-1}(\eta) \\ &\leqq \int_{T^{n-1}} \int_T |F_j(a + \rho\xi, r\eta)|^p dm(\xi) dm_{n-1}(\eta) \leqq K .\end{aligned}$$

It follows that $G \in \mathscr{H}^p(U^n)$, or equivalently that $F_j \in \mathscr{H}^p(D_j \times U^{n-1})$.

We next assume that $Y = Y_1 \times Y_2 \times \cdots \times Y_n$ *is an arbitrary Jordan polydomain in* C^{n-1}.

Decompose each Y_i as a finite union $Y_i = \bigcup_k U_i^{(k)}$, where the sets $U_i^{(k)}$ are simply connected domains in C, and where every boundary point of Y_i has a neighborhood that intersects inside some $U_i^{(k)}$. Let $\mathscr{U}$ be the class of all cartesian products $U_1^{(k_1)} \times U_2^{(k_2)} \times \cdots \times U_{n-1}^{(k_{n-1})}$.

The members of $\mathscr{U}$ are cartesian products of simply connected domains in C; accordingly, as was proven earlier, for each $Q \in \mathscr{U}$ we have $F_j \in \mathscr{H}^p(U \times Q)$, $F_j \in \mathscr{H}^p((S^2 - \bar{U}) \times Q)$, and $F_j \in \mathscr{H}^p(D_j^Q \times Q)$, where D_j^Q is a disc, depending on Q, which contains L_j. From our construction of $\mathscr{U}$ it follows that $\{U \times Q\}_{Q \in \mathscr{U}}$ is a covering of $U \times Y$ that satisfies the requirements of Theorem 2.1; the same is the case for the coverings $\{(S^2 - \bar{U}) \times Q\}_{Q \in \mathscr{U}}$ of $(S^2 - \bar{U}) \times Y$, and $\{D_j \times Q\}_{Q \in \mathscr{U}}$ of $D_j \times Y$, where D_j is the intersection of the (finitely many) discs D_j^Q. If we apply Theorem 2.1 to the n-subharmonic function $|F_j|^p$, we conclude that $F_j \in \mathscr{H}^p(U \times Y)$, $F_j \in \mathscr{H}^p((S^2 - \bar{U}) \times Y)$, and $F_j \in \mathscr{H}^p(D_j \times Y)$. This completes the proof of the theorem.

IV. The Čech cohomology with coefficients in $\mathscr{H}_W^p$. Throughout this section $0 < p < \infty$ will be fixed. We assume $n > 1$. Our goal is to prove, for any Jordan polydomain W in C^n and all integers $q \geqq 1$, that $H^q(\bar{W}, \hat{\mathscr{H}}_W^p) = 0$.

It simplifies matters if we take our coefficients in the presheaf $\mathscr{H}^p$ rather than in its completion, the sheaf $\hat{\mathscr{H}}^p$. We specify below what we mean by the *Čech cohomology theory with coefficients in* $\mathscr{H}^p$.

Let W be a polydomain in C^n. We define a class Ω_W of open coverings of W as follows.

An open covering $\mathscr{U}$ *of* W *belongs to* Ω_W *if and only if*:

(1) *Each member of* $\mathscr{U}$ *is a polydomain.*

(2) *For every point* b *on the boundary of* W *there exists a neighborhood* $N(b)$ *and a set* $U \in \mathscr{U}$ *such that* $N(b) \cap W \subset U$.

Equivalently, $\mathscr{U} \in \Omega_W$ if and only if $\mathscr{U}$ is the restriction to W of a family of polydomains that covers $\bar{W}$.

Let $\mathscr{U} \in \Omega_W$. A *q-simplex* σ of $\mathscr{U}$ is a $q+1$-tuple $(U_0, U_1, \cdots, U_q)$ of members of $\mathscr{U}$; its support $|\sigma|$ is the set $U_0 \cap U_1 \cap \cdots \cap U_q$. We denote by $S_q(\mathscr{U})$ the collection of all q-simplices of $\mathscr{U}$, and by $C^q(\mathscr{U}, \mathscr{H}^p)$ the group of all functions γ (*q-cochains*) that assign to each $\sigma \in S^q(\mathscr{U})$ an element $\gamma(\sigma)$ of $\mathscr{H}^p(|\sigma|)$.

The graded group $C^q(\mathscr{U}, \mathscr{H}^p)$, together with the obvious coboundary operator $\delta: C^q(\mathscr{U}, \mathscr{H}^p) \to C^{p+1}(\mathscr{U}, \mathscr{H}^p)$, constitutes a *cochain complex* with *cocycles* $Z^q(\mathscr{U}, \mathscr{H}^p)$, *coboundaries* $B^q(\mathscr{U}, \mathscr{H}^p)$, and *cohomology group* $H^q(\mathscr{U}, \mathscr{H}^p)$. The relation of refinement induces a partial ordering on Ω_W. The corresponding direct limit groups

$$H^q(W, \mathscr{H}^p) = \lim_{\mathscr{U} \in \Omega_W} H^q(\mathscr{U}, \mathscr{H}^p)$$

are the *cohomology groups of* W *with coefficients in the presheaf* $\mathscr{H}^p$.

As can be easily verified ([10, Cor. 18, p. 329]):

LEMMA 4.1. *The groups* $H^q(\bar{W}, \hat{\mathscr{H}}^p_{\bar{W}})$ *and* $H^q(W, \mathscr{H}^p)$ *are isomorphic for all integers* $q \geqq 0$.

If $V \subset W$ are polydomains, and if $\mathscr{U} \in \Omega_W$, we denote by $\mathscr{U}(V)$ the restriction of $\mathscr{U}$ to V (in particular $\mathscr{U} = \mathscr{U}(W)$). We then have *restriction homomorphisms* $C^q(\mathscr{U}(W), \mathscr{H}^p) \to C^q(\mathscr{U}(V), \mathscr{H}^p)$, which as can be easily verified, commute with the coboundary operators. If $\gamma \in C^q(\mathscr{U}(W), \mathscr{H}^p)$ we denote its restriction to $\mathscr{U}(V)$ by the same symbol γ.

LEMMA 4.2. *Let* W *be a polydomain in* $\boldsymbol{C}^n$, *and let* $W = \{W^{(1)}, W^{(2)}\}$ *be a covering in* Ω_W.

If $\mathscr{U} \in \Omega_W$ *satisfies the conditions*:

(1) *For every simplex* $\sigma \in S^q(\mathscr{U})$, *the support* $|\sigma|$ *is either a Jordan polydomain or the empty set.*

(2) *For every* $\sigma \in S^q(\mathscr{U})$, *the homomorphism*

$$\mathscr{H}^p(|\sigma| \cap W^{(1)}) \oplus \mathscr{H}^p(|\sigma| \cap W^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(|\sigma| \cap W^{(1)} \cap W^{(2)}),$$

defined by $\psi(g^{(1)}, g^{(2)}) = g^{(1)} + g^{(2)}$, *is onto.*

Then there is an exact sequence of groups and homomorphisms

$$\begin{aligned}0 \longrightarrow \cdots \xrightarrow{\mathscr{T}^*} H^q(\mathscr{U}(W), \mathscr{H}^p) \xrightarrow{\psi^*} H^q(\mathscr{U}(W^{(1)}), \mathscr{H}^p) \oplus H^q(\mathscr{U}(W^{(2)}), \mathscr{H}^p) \\ \xrightarrow{\psi^*} H^q(\mathscr{U}(W^{(1)} \cap W^{(2)}), \mathscr{H}^p) \xrightarrow{\mathscr{T}^*} H^{q+1}(\mathscr{U}(W), \mathscr{H}^p) \xrightarrow{\psi^*} \cdots .\end{aligned}$$

(Such a sequence will be called a Mayer-Vietoris sequence.)

Proof. For each $\sigma \in S^q(\mathscr{U})$ define

$$\mathscr{H}^p(|\sigma|) \xrightarrow{\psi} \mathscr{H}^p(|\sigma| \cap W^{(1)}) \oplus \mathscr{H}^p(|\sigma| \cap W^{(2)})$$

by the equation $\phi(g) = (g, -g)$, with suitable restrictions.

By hypothesis $|\sigma|$ is a Jordan polydomain (or the empty set). We can then invoke Theorem 2.1, and conclude that the image of ϕ and the kernel of ψ are the same. Since also ϕ is one-one, we have, for each $\sigma \in S^q(\mathscr{U})$, a short exact sequence

$$\begin{aligned}0 \longrightarrow \mathscr{H}^p(|\sigma|) \xrightarrow{\phi} \mathscr{H}^p(|\sigma| \cap W^{(1)}) \oplus \mathscr{H}^p(|\sigma| \cap W^{(2)}) \\ \xrightarrow{\psi} \mathscr{H}^p(|\sigma| \cap W^{(1)} \cap W^{(2)}) \longrightarrow 0 ,\end{aligned}$$

which in turn induces a short exact sequence of graded groups

$$\begin{aligned}0 \longrightarrow C^q(\mathscr{U}(W), \mathscr{H}^p) \xrightarrow{\phi} C^q(\mathscr{U}(W^{(1)}), \mathscr{H}^p) \oplus C^q(\mathscr{U}(W^{(2)}), \mathscr{H}^p) \\ \xrightarrow{\psi} C^q(\mathscr{U}(W^{(1)} \cap W^{(2)}), \mathscr{H}^p) \longrightarrow 0 ;\end{aligned} \tag{4.2.1}$$

for if V is a polydomain in W, then

$$C^q(\mathscr{U}(V), \mathscr{H}^p) = \underset{\sigma \in S^q(\mathscr{U})}{\Pi} \mathscr{H}^p(|\sigma| \cap V) .$$

Since the homomorphisms ϕ and ψ of (4.2.1) commute with the coboundary operators, the sequence (4.2.1) is a short exact sequence of cochain complexes. As is well known ([4, Th. 3.7, p. 128]) there is then an associated exact cohomology sequence. This completes the proof.

Our next lemma is a direct consequence of Theorem 2.1.

LEMMA 4.3. *If W is a Jordan polydomain, and if $\mathscr{U} \in \Omega_W$, then $H^0(\mathscr{U}, \mathscr{H}^p)$ and $\mathscr{H}^p(W)$ are canonically isomorphic.*

Henceforth, unless otherwise indicated, $W = W_1 \times W_2 \times \cdots \times W_n$ will be a Jordan polydomain.

Towards our goal of establishing $H^q(W, \mathscr{H}^p) = 0$ we consider two cases.

1. *The Simply Connected Case.* We follow the argument of [13]. The proofs are identical (replacing the symbol P by $\mathscr{H}^p$, and using Theorem 3.3 instead of [13, Lemma 3.1, p. 269]). We outline the procedure. Without loss of generality we take W to be a polyrectangle; this will allow a systematic partitioning into smaller polyrectangles.

Let I, I_1, and I_2 be the open intervals $(-1, 1)$, $(-1, \frac{1}{2})$, and $(-\frac{1}{2}, 1)$, respectively. Consider the rectangles $R = I + iI$, $R_1 = I_1 + iI$, $R_2 = I_2 + iI$. For Lemmas 4.4 and 4.5 we write $W = R^n$, $W^{(1)}_{(1)} = R_1 \times R^{n-1}$, $W^{(2)}_{(1)} = R_2 \times R^{n-1}$; and let $\mathscr{U}$ be a finite open covering of W consisting of polyrectangles with edges parallel to the real and imaginary axes of C.

LEMMA 4.4. *If $\sigma \in S^q(\mathscr{U})$ and $g \in \mathscr{H}^p(|\sigma| \cap W^{(1)}_{(1)} \cap W^{(2)}_{(1)})$, there exist $g^{(1)} \in \mathscr{H}^p(|\sigma| \cap W^{(1)}_{(1)})$, $g^{(2)} \in \mathscr{H}^p(|\sigma| \cap W^{(2)}_{(1)})$, such that $g = g^{(1)} + g^{(2)}$.*

LEMMA 4.5. *For all integers $q \geqq 1$, the cohomology groups $H^q(\mathscr{U}, \mathscr{H}^p)$ are trivial.*

THEOREM 4.6. *If W is a simply connected Jordan polydomain in C, then $H^q(W, \mathscr{H}^p) = 0$ for all integers $q \geqq 1$.*

2. *The Multiply Connected Case.* We first observe that Theorem 3.3 remains valid if we substitute the unit disc by a suitable doubly connected domain.

Let $0 < r_1 < r_2$, and $r_2 - r_1/2 < \rho < r_2 + r_1/2$. Write

$$A = \{z \in C: r_1 < |z| < r_2\} ,$$
$$\Omega_1 = \left\{z \in C: \left|z - \frac{r_1 + r_2}{1}\right| < \rho\right\} ,$$
$$\Omega_2 = \left\{z \in C: \left|z + \frac{r_1 + r_2}{2}\right| < \rho\right\} ,$$

and define $B(r_1, r_2; \rho) = A \cup \Omega_1 \cup \Omega_2$. The set $B = B(r_1, r_2; \rho)$ is the union of the annulus A with the symmetric discs Ω_1 and Ω_2. Any such region will be called a *buldged annulus.*

We write

$$C^+ = \{z \in C: \operatorname{Im} z > 0\} ,$$

and set $A^{(1)} = A \cap C^+$, $A^{(2)} = A \cap (-C^+)$, $B^{(1)} = A^{(1)} \cup \Omega_1 \cup \Omega_2$, and $B^{(2)} = A^{(2)} \cup \Omega_1 \cup \Omega_2$.

LEMMA 4.7. *Let Y be a Jordan polydomain in C^{n-1}. If $g \in \mathscr{H}^p((\Omega_1 \cup \Omega_2) \times Y)$, there exist $g^{(1)} \in \mathscr{H}^p(B^{(1)} \times Y)$ and $g^{(2)} \in \mathscr{H}^p(B^{(2)} \times Y)$ such that $g(z) = g^{(1)}(z) + g^{(2)}(z)$ whenever $z \in (\Omega_1 \cup \Omega_2) \times Y$.*

Proof. Let C_1 and C_2 be the boundaries of Ω_1 and Ω_2 respectively. Consider the disjoint closed arcs $L_i^{(j)} = C_i \cap A^{(j)}$, for $i, j = 1, 2$.

It is clear that Theorem 3.3 remains valid if we replace the unit disc U by the disc Ω_1. We apply Theorem 3.3 to $\Omega_1 \times Y$, the restriction of g to $\Omega_1 \times Y$, and the closed arcs $L_1^{(1)}$, $L_{(1)}^{(2)}$, to obtain holomorphic functions $g_1^{(1)}$ and $g_2^{(2)}$, which by Theorem 2.1 are in $\mathscr{H}^p(A^{(1)} \times Y)$ and in $\mathscr{H}^p(A^{(2)} \times Y)$ respectively, such that

$$g(z) = g_1^{(1)}(z) + g_1^{(2)}(z) ,$$

if $z \in \Omega_1 \times Y$, and

$$0 = g_1^{(1)}(z) + g_1^{(2)}(z) ,$$

if $z \notin \bar{\Omega}_1 \times Y$.

Similarly, by applying Theorem 3.3 to $\Omega_2 \times Y$, the restriction of g to $\Omega_2 \times Y$, and the closed arcs $L_2^{(1)}$, $L_2^{(2)}$, we obtain $g_2^{(1)} \in \mathscr{H}^p(A^{(1)} \times Y)$ and $g_2^{(2)} \mathscr{H}^p(A^{(2)} \times Y)$, such that

$$g(z) = g_2^{(1)}(z) + g_2^{(2)}(z) ,$$

if $z \in \Omega_2 \times Y$, and

$$0 = g_2^{(1)}(z) + g_2^{(2)}(z) ,$$

if $z \notin \bar{\Omega}_2 \times Y$.

If we define $g^{(j)} = g_1^{(j)} + g_2^{(j)}$, for $j = 1, 2$, the lemma is verified.

We next prove that the set of buldged annulli is a canonical class for the doubly connected domains in $\boldsymbol{C}$.

LEMMA 4.8. *Let A be a doubly connected domain in $\boldsymbol{C}$. There exists a buldged annulus which is conformally equivalent to A. If A is bounded by two Jorden curves, the conformal equivalence extends to a homeomorphism between the closures.*

Proof. Without loss of generality let A be an annulus centered at the origin. To prove the lemma it suffices to show that there exists a buldged annulus with the same modulus as A.

The modulus $M(D)$ of a doubly connected domain D, we recall, is a conformal invariant which in the special case of an annulus of radii $a < b$ reduces to $1/2\pi \log b/a$. Moreover, two doubly connected regions with the same modulus are necessarily equivalent ([6, Th. 2, p. 208]).

Let $B = B(r_1, r_2; \rho)$ be a buldged annulus contained in A. Since B separates the boundaries of A, we must have ([6, Th. 3, p. 209])

$$M(B) \leqq M(A) . \tag{4.8.1}$$

For each $0 \leqq t < \infty$ define $B_t = B(r_1, r_2 + t; \rho + t/2)$. Given any

$\lambda > 0$ there exists $t > 0$ such that

$$M(B_t) \geqq \lambda\,; \tag{4.8.2}$$

for we can always find an annulus of inner radius r_1 and modulus λ contained in B_t if we choose t sufficiently large.

A direct calculation (using the extremal length characterization of the modulus $M(B_t)$) shows that $M(B_t)$ varies continuously with t. The function $f(t) = M(B_t)$ is therefore continuous on $[0, \infty)$. By (4.9.1) and (4.8.2), we have $f(0) \leqq M(A)$ and $\lim_{t\to+\infty} f(t) = +\infty$, respectively. Consequently, for some t_0 we must have $M(B_{t_0}) = f(t_0) = M(A)$. This proves the first assertion of the lemma.

As is well known ([6, Th. 1, p. 208]), if two conformally equivalent doubly connected domains are bounded by Jordan curves, any conformal equivalence between them extends to a homeomorphism between their closures.

THEOREM 4.9. *If W is a Jordan polydomain in C^n, then $H^q(W, \mathscr{H}^p) = 0$ for all integers $q \geqq 1$.*

Proof. Denote by Z_+^n the set of all n-tuples of positive integers. If μ and y are in Z_+^n, and if $\mu_i \leqq \nu_i$ for all $1 \leqq i \leqq n$, we write $\mu \leqq \nu$. We say that a polydomain $W = W_1 \times W_2 \times \cdots \times W_n$ is μ-connected (for some $\mu \in Z_+^n$) if each W_i has connectively μ_i.

For each $\nu \in Z_+^n$ let $P(\nu)$ be the proposition:
$P(\nu)$: *For all μ-connected polydomains W, $\mu \leqq \nu$, and all integers $q \geqq 1$, the cohomology groups $H^q(W, \mathscr{H}^p)$ are trivial.*

Since a Jordan polydomain is necessarily finitely connected, the theorem will be proven if we verify $P(\nu)$ for all $\nu \in Z_+^n$.

Suppose $P(\nu)$ is true for some $\nu \in Z_+^n$. Fix $1 \leqq k \leqq n$, and denote by ν' the n-tuple define by $\nu_i' = \nu_i$, if $i \neq k$, and $\nu_k' = \nu_k + 1$. We claim that $P(\nu')$ is true. Without loss of generality take $k = 1$.

We first consider the case $\nu_1 = 1$. Let W be an arbitrary ν'-connected Jordan polydomain, and write $W = B \times Y$, where B is a doubly connected Jordan domain in C and where $Y \subset C^{n-1}$. By Lemma 4.8 there is no loss of generality if we let B a buldged annulus. As in Lemma 4.7, decompose $B = B^{(1)} \cup B^{(2)}$, with $B^{(1)} \cap B^{(2)} = \Omega_1 \cup \Omega_2$. Define $W^{(1)} = B^{(1)} \times Y$, and $W^{(2)} = B^{(2)} \times Y$.

We consider the coverings in Ω_W that satisfy the following condition: the support $|\sigma|$ of any simplex σ is a Jordan polydomain (or the empty set) contained in either $W^{(1)}$ or $W^{(2)}$. Such coverings satisfy the hypotheses of Lemma 4.2; and the collection of them constitutes a cofinal subclass of Ω_W. By taking the direct limit of the corresponding Mayer-Vietoris sequences, we obtain the exact sequence

$$\begin{aligned}0 &\longrightarrow \mathscr{H}^p(W) \xrightarrow{\phi} \mathscr{H}^p(W^{(1)}) \oplus \mathscr{H}^p(W^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(W^{(1)} \cap W^{(2)})\\ &\xrightarrow{\mathscr{T}^*} \cdots \xrightarrow{\mathscr{T}^*} H^q(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^q(W^{(1)}, \mathscr{H}^p) \oplus H^q(W^{(2)}, \mathscr{H}^p)\\ &\xrightarrow{\psi^*} H^q(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) \xrightarrow{\mathscr{T}^*} H^{q+1}(W, \mathscr{H}^p) \longrightarrow \cdots .\end{aligned}$$

By Lemma 4.7, the first row above is a short exact sequence; we disregard it, and retain the exact sequence

$$\begin{aligned}0 &\longrightarrow H^1(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^1(W^{(1)}, \mathscr{H}^p) \oplus H^1(W^{(2)}, \mathscr{H}^p)\\ &\xrightarrow{\psi^*} H^1(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) \xrightarrow{\mathscr{T}^*} \cdots \longrightarrow H^{q-1}(W^{(1)} \cap W^{(2)}, \mathscr{H}^p)\\ &\xrightarrow{\mathscr{T}^*} H^q(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^q(W^{(1)}, \mathscr{H}^p) \oplus H^q(W^{(2)}, \mathscr{H}^p) \xrightarrow{\psi^*} .\end{aligned} \tag{4.9.1}$$

Since $W^{(1)}$ and $W^{(2)}$ are Jordan polydomains of connectivity $\leqq \nu$, and since $W^{(1)} \cap W^{(2)}$ is the disjoint union of two Jordan polydomains of connectivity $\leqq \nu$, the inductive hypothesis implies $H^q(W^{(1)}, \mathscr{H}^p) = 0$, $H^q(W^{(2)}, H^q) = 0$, and $H^q(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) = 0$. The exactness of (4.9.1) then establishes $H^q(W, \mathscr{H}^p) = 0$ for all $q \geqq 1$.

We next consider the case $\nu_1 > 1$. As before, let W be an arbitrary ν'-connected polydomain. Write $W = X \times Y$, where $Y \subset C^{n-1}$, and where X is a domain in C of connectivity $k = \nu_1 + 1$ which is bounded by an outer contour C_k and k inner contours $C_0, C_1, \cdots, C_{k-1}$.

Let B be the doubly connected domain bounded by C_0 and C_k, and let $A^{(1)}$ and $A^{(2)}$ be simply connected Jordan domains such that

(1) $A^{(1)} \cup A^{(2)} = B$,

(2) $A^{(1)} \cap A^{(2)}$ is the disjoint union of two simply connected domains,

(3) each contour $C_1, C_2, \cdots, C_{k-1}$ is entirely contained in either $A^{(1)} - A^{(2)}$ or $A^{(2)} - A^{(1)}$.

We define $X^{(1)} = A^{(1)} \cap X$, $X^{(2)} = A^{(2)} \cap X$; and consider the Jordan polydomain $V^{(1)} = A^{(1)} \times Y$, $V^{(2)} = A^{(2)} \times Y$, $V = B \times Y$, $W^{(1)} = X^{(1)} \times Y$, and $W^{(2)} = X^{(2)} \times Y$.

As in the previous case of the theorem, by taking suitable coverings, applying Lemma 4.2, and taking the direct limit of the Mayer-Vietoris sequences that correspond to such coverings, we obtain the exact sequences

$$\begin{aligned}0 &\longrightarrow \mathscr{H}^p(V) \xrightarrow{\phi} \mathscr{H}^p(V^{(1)}) \oplus \mathscr{H}^p(V^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(V^{(1)} \cap V^{(2)})\\ &\xrightarrow{\mathscr{T}^*} H^1(V, \mathscr{H}^p) \longrightarrow \cdots\end{aligned}$$

and

$$\begin{aligned}0 &\longrightarrow \mathscr{H}^p(W) \xrightarrow{\phi} \mathscr{H}^p(W^{(1)}) \oplus \mathscr{H}^p(W^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(W^{(1)} \cap W^{(2)})\\ &\xrightarrow{\mathscr{T}^*} \cdots \xrightarrow{\mathscr{T}^*} H^q(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^q(W^{(1)}, \mathscr{H}^p) \oplus H^q(W^{(2)}, \mathscr{H}^p)\end{aligned} \tag{4.9.2}$$

$$\xrightarrow{\psi^*} H^q(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) \xrightarrow{*} H^{q+1}(W, \mathscr{H}^p) \longrightarrow \cdots .$$

The polydomain V has connectivity μ, with $\mu_1 = 2$, and $\mu_i = \nu_i$ for $i = 2, 3, \cdots, n$. Consequently, as was established earlier, $H^q(V, \mathscr{H}^p) = 0$. In particular

$$0 \longrightarrow \mathscr{H}^p(V) \xrightarrow{\phi} \mathscr{H}^p(V^{(1)}) \oplus \mathscr{H}^p(V^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(V^{(1)} \cap V^{(2)}) \longrightarrow 0$$

is exact. Since $W^{(1)} \cap W^{(2)} = V^{(1)} \cap V^{(2)}$. and since $W^{(1)} \subset V^{(1)}$, $W^{(2)} \subset V^{(2)}$, it follows that

$$0 \longrightarrow \mathscr{H}^p(W) \xrightarrow{\phi} \mathscr{H}^p(W^{(1)}) \oplus \mathscr{H}^p(W^{(2)}) \xrightarrow{\psi} \mathscr{H}^p(W^{(1)} \cap W^{(2)}) \longrightarrow 0$$

is also exact. We can then disregard the first row of (4.9.2) and retain exactness in

$$\begin{aligned} &0 \longrightarrow H^1(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^1(W^{(1)}, \mathscr{H}^p) \oplus H^1(W^{(2)}, \mathscr{H}^p) \\ (4.9.3) \quad &\xrightarrow{\psi^*} H^1(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) \longrightarrow \cdots \longrightarrow H^{q-1}(W^{(1)} \cap W^{(2)}, \mathscr{H}^p) \\ &\xrightarrow{\mathscr{T}^*} H^q(W, \mathscr{H}^p) \xrightarrow{\phi^*} H^q(W^{(1)}, \mathscr{H}^p) \oplus H^q(W^{(2)}, \mathscr{H}^p) \longrightarrow \cdots . \end{aligned}$$

The inductive hypothesis, together with the exactness of (4.9.3), implies $H^q(W, \mathscr{H}^p) = 0$ for all $q \geqq 1$; for $W^{(1)}$ and $W^{(2)}$ are Jordan polydomains of connectivity $\leqq \nu$, and $W^{(1)} \cap W^{(2)}$ is the disjoint union of two Jordan polydomains of connectivity $\leqq \nu$.

We have thus established $P(\nu')$ in all cases. Since, as was proven in Theorem 4.6, $P(\nu)$ is true for $\nu = (1, 1, \cdots, 1)$, by the principal of mathematical induction $P(\nu)$ must also be true for all $\nu \in \boldsymbol{Z}_+^n$. This concludes the proof.

V. Remarks.

1. *The Gleason Problem for* $\mathscr{H}^p(W)$. Let $F \in \mathscr{H}^p(W)$, let $a \in W$, and suppose $F(a) = 0$. The problem asks if there exist $F_1, \cdots, F_n \in \mathscr{H}^p(W)$ such that $F(z) = (z_1 - a_1)F_1(z) + \cdots + (z_n - a_n)F_n(z)$ for all $z \in W$. The method of [7], together with the vanishing of the cohomology of $\mathscr{H}^p$, gives an affirmative answer when W is a Jordan polydomain. A non cohomological treatment of the Gleason problem for various other functions spaces is given in [1].

2. *The extension of* $\mathscr{H}^p$*-functions from hypersurfaces in* W. Let S be the zero set of a bounded holomorphic function in U^n. In [2] Andreotti and Stoll defined a *strictly* $\mathscr{H}^\infty$*-function* to be a function $f: S \to \boldsymbol{C}$ for which there exists a covering $\{U_\alpha\}$ of $\bar{U}^n$, and functions $f_\alpha \in \mathscr{H}^\infty(U_\alpha \cap U^n)$ and $g_{\alpha\beta} \in \mathscr{H}^\infty(U_\alpha \cap U_\beta \cap U^n)$ such that

(i) $f = f_\alpha$ on $S \cap U_\alpha$

(ii) $f_\beta - f_\alpha = hg_{\alpha\beta}$ on $U_\alpha \cap U_\beta \cap U^n$;
and proved, as a direct consequence of the vanishing of $H^1(\bar{U}^n, \mathscr{H}^\infty)$, that any such function has an extension in $\mathscr{H}^\infty(U^n)$.

If W is a Jordan polydomain, S is the zero set of an $\mathscr{H}^\infty$-function in W, and $f: S \to C$ is a strictly $\mathscr{H}^p$-function (defined as above, but requiring now that f_α and $g_{\alpha\beta}$ be in the corresponding $\mathscr{H}^p$-spaces), the vanishing of the cohomology of $\mathscr{H}^p$ establishes the existence of an extension $F \in \mathscr{H}^p(W)$ of f.

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