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JETS WITH REGULAR ZEROS

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If a mapgerm $f: \boldsymbol{R}^n, 0 \to \boldsymbol{R}^p, 0$ is a submersion ($rkf = p$), then its zero set is regular (the germ of a manifold) by the Implicit Function Theorem. Of course, there are also critical maps ($rkf < p$) whose zero sets are manifolds. Submersions have the added feature that one can discern that the zero set is regular from the first derivative of f at 0. Are there other instances in which one can tell purely from the derivatives of f at 0 that the zero set is regular? In this paper we show that there are, and go part way toward the eventual goal of describing them all.

The k-jet $j^k f(x)$ is $(f(x), Df(x), \cdots, D^k f(x))$ if $k < \infty$ or $(f(x), Df(x), \cdots)$ if $k = \infty$. A k-jet z is said to have regular zeros if every representative f (a germ such that $j^k f(x) = z$) has regular zero set. Suppose f has regular zero set V_f. In §2 we show that $j^\infty f$ has regular zeros iff f is ∞-$\mathscr{K}$-determined. In this case $\dim V_f = 0$ or $n - p$. If this dimension is 0, then f is ∞-$\mathscr{C}$-determined. If $p = 1$ and $\dim V_f = n - 1$, then $f = g \cdot h$ where g is a submersion and h is ∞-$\mathscr{C}$- determined. (If f is analytic, then it is ∞-$\mathscr{K}$-determined at x iff it is a submersion at each point of $V_f - \{x\}$ and is ∞-$\mathscr{C}$-determined at x iff $V_f = \{x\}$.) In §3 we show (again assuming V_f regular) that $j^k f$ has regular zeros for some finite k iff f is ∞-$\mathscr{K}$-determined and either $\dim V_f = 0$ or $p = n - 1$. In this section we especially consider finitely $\mathscr{K}$-determined mapgerms. (If f is analytic, then it is finitely $\mathscr{K}$-determined at x iff it is a submersion at each of its complex zeros except possibly x.) Among the examples given are $x(x^2 + y^2)$, $x(x^2 + y^2)^2$ and $x(x^2 + y^2 + z^2)$; the first example is finitely determined and its 3-jet has regular zeros, the second is ∞- but not finitely determined and its 5-jet has regular zeros, and the third is ∞-determined and its ∞-jet but no finite jet has regular zeros.

For notational simplicity, we restrict our study of regular zeros to jets of germs at 0. Let $E_{n,p}$ denote the germs at 0 of C^∞ maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}^p$, $m_{n,p}$ those which are 0 at 0, $E_n = E_{n,1}$ and $m_n = m_{n,1}$. Let $\mathscr{K}$ be the set of pairs (R, A), where $R \in m_{n,n}$ is invertible and A is a $p \times p$ matrix with entries in E_n such that $A(0)$ is invertible. Define a group structure on $\mathscr{K}$ by $(R', A') \cdot (R, A) = (R' \circ R, (A' \circ R)A)$ and a left action of $\mathscr{K}$ on $m_{n,p}$ by $(R, A) \cdot f = (Af) \circ R^{-1}$. Note that while this definition of $\mathscr{K}$ differs from that of Mather (see §2 of [6]), the $\mathscr{K}$ orbits are identical under both definitions. $\mathscr{R}$ and $\mathscr{C}$ are the subgroups in which A or R is the identity, respectively.

Two germs f and g are $\mathscr{K}$-, $\mathscr{R}$- or $\mathscr{C}$-equivalent if they lie in the same $\mathscr{K}$, $\mathscr{R}$ or $\mathscr{C}$ orbits, respectively. Note that if f and g are $\mathscr{K}$-equivalent, then their zero sets are diffeomorphic. A jet z is $\mathscr{K}$-, $\mathscr{R}$- or $\mathscr{C}$-sufficient if all representatives are $\mathscr{K}$-, $\mathscr{R}$- or $\mathscr{C}$-equivalent, respectively. A mapgerm f is k-$\mathscr{K}$-determined if its k-jet is $\mathscr{K}$-sufficient and is finitely $\mathscr{K}$-determined if the above k is finite (etc. for $\mathscr{R}$ or $\mathscr{C}$).

Next we review from [6], [7] and [10] some facts about $\mathscr{K}$-determined mapgerms. Let fE_n denote the ideal generated by $f_1, \cdots, f_p$. Let Jf denote the ideal generated by the $p \times p$ subdeterminants of the Jacobian matrix of f. Let δf be the ideal generated by fE_n and Jf. Then f is finitely $\mathscr{K}$-, $\mathscr{R}$- or $\mathscr{C}$-determined iff (respectively) δf, Jf or fE_n contains m_n^l for some finite l, and is ∞-determined iff the ideal contains m_n^∞. It follows that an analytic f is finitely $\mathscr{K}$-determined iff its complexification F is a submersion on $V_F - \{0\}$. In V. 4.3 of [8] it is shown that an ideal gE_n contains m_n^∞ iff g satisfies a Lojasiewicz inequality $|g(x)| \geqq c|x|^r$, $c > 0$, $r \geqq 0$ (r is called the order of the inequality). Let S denote the set of 1-jets of mapgerms with value 0 and rank less than p. It is easy to see (and is shown in [10]) that f is infinitely $\mathscr{K}$-determined iff $\operatorname{dist}(j^1f, S)$ satisfies a Lojasiewicz inequality. If x_i converges to x in $\boldsymbol{R}^n$, a sequence a_i is flat along x_i if, for each $r > 0$, there corresponds an N such that $i \geqq N$ implies $|a_i| \leqq |x_i - x|^r$. Thus f is not ∞-$\mathscr{K}$-determined iff there is a sequence x_i converging to 0 along which $\operatorname{dist}(j^1f, S)$ is flat.

2. Infinite jets with regular zeros.

A $\mathscr{K}$-sufficient jet z has regular zeros iff any one representative of z has regular zero set. Our first theorem shows we can restrict our attention to $\mathscr{K}$-sufficient jets.

THEOREM 1. *If an ∞-jet is not $\mathscr{K}$-sufficient, then some representative of z has a singular zero set.*

The following is a special case of Lemma 3.3 of [10].

LEMMA 2. *If a sequence of k-jets z_i is flat along x_i, then there is an $f \in m_n^\infty$ such that $j^kf(x_i) = z_i$ for infinitely many i.*

Proof of Theorem 1. Suppose z is not $\mathscr{K}$-sufficient. Let f be a representative of z. There is a sequence x_i converging to 0 and a sequence $z_i \in S$ such that $\operatorname{dist}(j^1f(x_i), z_i)$ is flat along x_i.

Assume $n \geqq p$. Let π denote the projection of J^2 onto J^1. Since the fold germs with value 0 (those germs which are $\mathscr{K}$-

equivalent to $(x_1, \cdots, x_{p-1}, x_p^2 \pm \cdots \pm x_n^2)$ — see [4]) are dense in $\pi^{-1}(S)$, we can find a sequence of fold jets q_i with value 0 such that $j^2f(x_i) - q_i$ is flat along x_i. Then by Lemma 2 there is a representative g of z such that $j^2g(x_i) = q_i$ for infinitely many i. The real zero set of g in a small neighborhood of x_i is either an isolated point or is singular. If $n < p$, we choose g instead so that g is an immersion at each x_i with $g(x_i) = 0$. In either case, the zero set of g is not a manifold in a neighborhood of 0. □

Since an ∞-$\mathscr{K}$-determined f is nonsingular on V_f except possibly at 0, V_f is of dimension either 0 or $n - p$. In [10] it is proved that f is ∞-$\mathscr{K}$-determined with isolated zero set iff f is ∞-$\mathscr{C}$-determined.

PROPOSITION 3. *An ∞-jet z has regular zeros of dimension $n-1$ iff each (equivalently, any one) representative is $\mathscr{R}$-equivalent to a function $f(x_1, \cdots, x_n) = x_1 g(x_1, \cdots, x_n)$, where g is ∞-$\mathscr{C}$-determined.*

Proof. Suppose z is an infinite jet with regular zeros of dimension $n - 1$. Then every representative is $\mathscr{R}$-equivalent to a function $f(x_1, \cdots, x_n) = x_1 g(x_1, \cdots, x_n)$. Since f is ∞-$\mathscr{K}$-determined,

$$(f, \partial f/\partial x_1, \cdots, \partial f/\partial x_n) = (x_1 g, x_1 \partial g/\partial x_1 + g, x_1 \partial g/\partial x_2, \cdots, x_1 \partial g/\partial x_n)$$

satisfies a Lojasiewicz inequality of some order r. Let $U = \{|g| \geqq |x|^{r+1}\}$. On U^c, $(x_1 g, x_1 \partial g/\partial x_1, \cdots, x_1 \partial g/\partial x_n)$ satisfies a Lojasiewicz inequality of order r. Thus, on U^c, $(g, \partial g/\partial x_1, \cdots, \partial g/\partial x_n)$ satisfies a Lojasiewicz inequality of order $r - 1$. On U, g and hence $(g, \partial g/\partial x_1, \cdots, \partial g/\partial x_n)$ satisfies a Lojasiewicz inequality of order $r + 1$. Thus g is ∞-$\mathscr{K}$-determined. Clearly $g \neq 0$ when $x_1 \neq 0$. If $x_1 = 0$ and $g = 0$, then f is critical; since f is only critical at 0 along V_f, g is 0 only at 0. Thus g is ∞-$\mathscr{C}$-determined.

Now suppose we have a g which is ∞-$\mathscr{C}$-determined. Let $f = x_1 g$. Then the zero set of f is $x_1 = 0$. Since g satisfies a Lojasiewicz inequality of some order r, f satisfies one of order $r + l$ on $V = \{|x_1| > |x|^l\}$. If $l > r$, then $|\partial f/\partial x_1| \geqq ||g| - |x_1 \partial g/\partial x_1||$ implies that $\partial f/\partial x_1$ satisfies a Lojasiewicz inequality or order r on V^c. Thus $(f, \partial f/\partial x_1)$ and hence $j^1 f$ satisfies a Lojasiewicz inequality. Thus f is ∞-$\mathscr{K}$-determined. □

EXAMPLE 4. Let A be a $p \times p$ matrix whose entries are analytic functions in n variables, $n \geqq p$. Suppose $\det A = 0$ only at 0. Then $f(x) = (x_1, \cdots, x_p) \cdot A$ is ∞-$\mathscr{K}$-determined with zero set $x_1 = \cdots =$

$x_p = 0$. This gives examples of all ranks. For instance, let A be the diagonal matrix with r entries 1 and the rest $x_1^2 + \cdots + x_n^2$. Then f has rank r.

It is difficult to make any systematic list of examples except in the case of finitely determined mapgerms, to which we turn our attention.

3. Finite jets with regular zeros. It is much easier to carry out computations for finite jets with regular zeros. Finite $\mathscr{K}$-sufficient jets are of this type, and considerable study has already been made of these jets. We will see, however, that finite jets with regular zeros form a somewhat limited class of examples, in that the zero sets must have dimension 0 or 1.

PROPOSITION 5. *Suppose* $f: \boldsymbol{C}^n, 0 \to \boldsymbol{C}^p, 0$, $p < n$, *is holomorphic and finitely determined. Then* $R_f = O_n/(f_1, \cdots, f_p)O_n$ (O_n *is the ring of holomorphic germs*) *is reduced. If* V_f *is nonsingular, then* f *is a submersion. If* $p < n - 1$, *then* R_f *is normal.*

Proof. f finitely determined implies that f is a submersion at each nonzero point of its zero set V_f. In particular V_f is of dimension $n - p$. As shown on page 141 of [5], this implies that $f_1, \cdots, f_p$ is an O_n-sequence (i.e., for each i, f_{i+1} is not a zero-divisor of $O_n/(f_1, \cdots, f_i)O_n$). Since $x_1, \cdots, x_n$ is a maximal O_n-sequence, and all maximal O_n-sequences are of the same length (Theorem 18 of [5]) and $p < n$, $f_1, \cdots, f_p$ is not a maximal O_n-sequence. Thus there is a $\delta \in m_n$ which is not a zero-divisor of R_f. It follows easily from Proposition II. 3.6 of [8] that R_f is reduced. By the Nullstellensatz, $I(V_f) = (f_1, \cdots, f_p)O_n$. If V_f is nonsingular, then $I(V_f) = (g_1, \cdots, g_p)O_n$ where $g = (g_1, \cdots, g_p)$ is a submersion. By Proposition 2.3 of [6], there is an invertible matrix A with entries in O_n such that $f = Ag$. It follows that f is a submersion.

A variety whose ideal is generated by an O_n-sequence is called a complete intersection. By Corollary 1 to Theorem 15 of [5], a complete intersection is normal iff its singularities are of codimension at least two. By the first paragraph, if f is finitely determined, then V_f is a complete intersection. Since the singularities of V_f are of codimension $n - p$, the theorem is proved. $\square$

This theorem shows that there is no nontrivial theory of finite jets with regular zeros in the complex analytic category. (There is still an unanswered question: must the complex zero set of a critical ∞-$\mathscr{K}$-determined analytic mapgerm be singular?)

PROPOSITION 6. *If the zero set V of a finitely $\mathscr{K}$-determined germ $f \in m_{n,p}$ is a positive-dimensional manifold, then either f is a submersion or $p = n - 1$ (and V is a curve).*

Proof. If $p \geqq n$, then $(f_1, \cdots, f_p)E_n \supset m_n^k$ for some k, so the zero set is zero-dimensional. Suppose $p \leqq n - 2$. We may as well assume f is analytic. The complex zero set V_C is normal and hence (by definition) irreducible, and is of dimension $n - p$. If V is a positive-dimensional manifold, necessarily of dimension $n - p$, its complexification is a manifold M of complex dimension $n - p$, and is contained in V_C. Since V_C is irreducible, it must equal M. Thus f must be a submersion. □

PROPOSITION 7. *A finitely determined germ in m_2 has zero set a 1-manifold iff it is $\mathscr{R}$-equivalent to $f(x, y) = xg(x, y)$, where g is finitely determined and vanishes only at 0.*

Proof. A function in m_2 has zero set a 1-manifold iff it is $\mathscr{R}$-equivalent to some $f(x, y) = xg(x, y)$, with $g \neq 0$ if $x \neq 0$. We may as well assume g is analytic. Note that $\operatorname{grad} f = (x\partial g/\partial x + g, x\partial g/\partial y)$. Working now in C^2: if $x \neq 0$, then $f = 0$ iff $g = 0$ and, along this zero set $\operatorname{grad} f \neq 0$ iff $\operatorname{grad} g \neq 0$; if $x = 0$, then $g = 0$ iff f is critical. □

If $F \in m_{n+r,p+r}$ is of rank r, it is $\mathscr{K}$-equivalent to a germ $(f(x_1, \cdots, x_n), x_{n+1}, \cdots, x_{n+r})$ by [7]. The zero set of F is a manifold iff that of f is. F is finitely $\mathscr{K}$-determined iff f is. The corank of F is p. Thus Proposition 7 yields a characterization of all corank 1, finitely $\mathscr{K}$-determined mapgerms with zero set a 1-manifold.

EXAMPLE 8. The finitely $\mathscr{K}$-determined real valued germs having simple singularities (see [1]) and zero set a 1-manifold are $\mathscr{R}$-equivalent to one of the germs $f(x, y) = x(x^k + y^2)$, with $k = 2, 4, 6, \cdots$. Damon in [3] studies the topological type of V_f for finitely $\mathscr{K}$-determined germs f having stable unfoldings in the nice range of dimensions, with $n > p$. These are all simple and of corank 1 or 2. He shows that, of the corank 2 germs, only those with normal form $(xy, z^2 + x^2 \pm y^k)$, k odd $\geqq 3$, and $(xy, z^2 + x^\kappa + y^l)$, k even $\geqq 4$, l odd $\geqq 3$, have V_f a topological 1-manifold. But V_f for these germs is not a C^∞ 1-manifold. Thus the only stable maps in the nice range of dimensions whose zero sets are positive-dimensional manifolds are those represented by normal forms $(x(x^2 + y^2) + sy + tx + ux^2, s, t, u)$, which is known as the hyperbolic umbilic, and $(x(x^4 + y^2) + sy + tx + ux^2 + vx^3 + wx^4, s, t, u, v, w)$.

EXAMPLE 9. There are finitely $\mathscr{K}$-determined germs of arbitrary corank having zero set a 1-manifold, as can be seen from the following example. Let $F = (x_1 f_1(x_1, z), \cdots, x_n f_n(x_n, z))$. Assume each f_i is finitely $\mathscr{K}$-determined and vanishes only at 0 in $\boldsymbol{R}^2$. Then F is finitely $\mathscr{K}$-determined and vanishes only along the z-axis in $\boldsymbol{R}^{n+1}$. As a particular example we have $(x(x^2 + z^2), y(y^2 + z^2))$.

EXAMPLE 10. Let $f(x, y, z) = (xy - (y^2 + z^2)z, xz + (y^2 + z^2)y)$. Unlike the above rank 0 examples, this map has quadratic terms. It is finitely $\mathscr{K}$-determined and only vanishes in $\boldsymbol{R}^3$ along the x-axis.

Suppose z is an ∞-jet with regular zeros. Recall that every representative f of z is ∞-$\mathscr{K}$-determined, that V_f has dimension $n - p$ or 0, and that V_f has dimension 0 iff f is ∞-$\mathscr{C}$-determined. Thus every representative f of z has V_f of the same dimension. Say that a finite jet with regular zeros has strictly regular zeros if the zero sets of its representatives are all of the same dimension. I conjecture that every jet with regular zeros has strictly regular zeros (I can prove this for $p = 1$).

THEOREM 11. *Suppose f is a critical ∞-$\mathscr{K}$-determined germ in $m_{n,p}$ with nonsingular zero set. Then some finite jet of f has strictly regular zeros iff V_f has dimension* 0 *or* 1.

Proof. Suppose $z = j^r f(0)$ has strictly regular zeros. By Proposition VII. 6.2 of [8], some representative g of z is finitely $\mathscr{K}$-determined. By Proposition 6, V_g has dimension 0 or 1. Thus V_f also has dimension 0 or 1.

Now we prove the converse. Suppose $\dim V_f = 0$. Then f is ∞-$\mathscr{C}$-determined, hence satisfies a Lojasiewicz inequality of order r, for some positive integer r. Then every representative of $j^r f(0)$ also satisfies a Lojasiewicz inequality of order r. Thus $j^r f$ has strictly regular zeros.

Now assume that $\dim V_f = 1$. Necessarily $p = n - 1$. We may assume, without loss of generality, that V_f is the line $x = 0$, where $x = (x_1, \cdots, x_p)$. Then $f(x, y) = xg(x, y)$, where g is $p \times p$ matrix valued with entries in E_n. Any representative F of $j^r f(0)$ can be expressed as $F(x, y) = xG(x, y) + h(y)$, where the entries of $G - g$ are in m_n^r and the components of h are in m_1^{r+1}.

We are going to apply Tougeron's Implicit Function Theorem (see [9]). The normal derivative of f along its zero set is the $n \times n$ Jacobian matrix $(\partial f/\partial x)(0, y) = g(0, y)$. Since f is ∞-$\mathscr{K}$-determined, there is some positive integer d such that $\det g(0, y) = (\text{unit})y^d$ (note if $d = 0$, f would be a submersion). $(\partial F/\partial x)(0, y) = G(0, y)$ and, if $r > d$, then $\delta(y) = \det G(0, y) = (\text{unit})y^d$. If $r \geqq 2d$, then $F(0, y) =$

$h(y) \in \bigoplus_p \delta^2 m_1$ so, by Tougeron's theorem, there exists $x(y) \in \bigoplus_p m_1^{d+1}$ such that $F(x(y), y) = 0$.

In [10] it is shown that each ∞-$\mathscr{K}$-determined germ f is finitely v-determined, i.e., for some finite k each representative of $j^k f(0)$ has its zero set homeomorphic to V_f. Thus if $r \geqq k$, the zero set of the above F is a topological 1-manifold. Since V_F contains the C^∞ 1-manifold $(x(y), y)$, it is itself a C^∞ 1-manifold. Thus if $r \geqq 2d$ and $r \geqq k$, then $j^r f(0)$ has regular zeros. □

We have not in general computed the smallest r for which $j^r f(0)$ has regular zeros in the above proof. However, consider the following example.

EXAMPLE 12. Let $f(x, y) = x(x^2 + y^2)^2$; f is ∞-$\mathscr{K}$-determined but not finitely $\mathscr{K}$-determined. We will show that $j^5 f(0)$ has regular zeros. Any representative of $j^5 f(0)$ is of the form $F(x, y) = x((x^2 + y^2)^2 + h(x, y)) + y^5 k(y)$ where $k \in m_1$ and $h \in m_2^5$. We search for a solution of $F = 0$ of the form $x = yw(y)$, $w \in m_1$. By cancelling y^5 from both sides of the equation $F(yw, y) = 0$, we see the desired solution exists iff there is a solution $w(y)$ of $l(w, y) = w((w^2 + 1)^2 + y^s h(w, 1)) + k(y)$, where $s \geqq 1$. Since at $(0, 0)$ $f = 0$ and $\partial f/\partial w = 1$, the solution $w(y) \in m_1$ exists.

By the Kuiper-Kuo Theorem (see [2]), $j^5(x(x^2 + y^2)^2)$ is C^0 sufficient. Hence the zero set of F is a topological 1-manifold and, since it contains $(yw(y), y)$, a C^∞ 1-manifold. So $j^5(x(x^2 + y^2)^2)$ has regular zeros as claimed.

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