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# DUAL MAPS OF JORDAN HOMOMORPHISMS AND *-HOMOMORPHISMS BETWEEN $C^*$-ALGEBRAS

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**A geometric characterization of the dual maps of Jordan homomorphisms and *-homomorphisms between $C^*$-algebras is given.**

**Introduction.** In [2] the authors gave a geometric characterization of state spaces of (unital) $C^*$-algebras among compact convex sets. They defined the notion of an orientation of the state space, and showed that the state space as a compact convex set with orientation completely determines the $C^*$-algebra up to *-isomorphism. Our purpose here is to show that this correspondence is categorical by giving a geometric description of the dual maps on the state space induced by unital *-homomorphisms. Along the way we will also characterize dual maps of unital Jordan homomorphisms between $C^*$-algebras, and in fact in the larger category of $JB$-algebras: the normed Jordan algebras introduced in [3]. Finally we remark that the first result on this topic was Kadison's [6]: the dual maps of Jordan isomorphisms are precisely the affine homeomorphisms of the state spaces.

*Characterization of Jordan homomorphisms.* Throughout this paper $A$ will be a $C^*$-algebra with state space $K$. (All $C^*$-algebras mentioned are assumed to be unital.) Assume that $A \subseteq B(H)$ is given in its universal representation, and thus its weak closure can be identified with its bidual $A^{**}$, and $K$ can be identified with normal state space of $A^{**}$ [4, §12].

We will view elements of $A$ and $A^{**}$ as affine functions on $K$. In fact, the self-adjoint parts of $A$ and $A^{**}$ are respectively isometrically order isomorphic to the spaces $A(K)$ and $A^b(K)$ of $w^*$-continuous (respectively, bounded) affine functions on $K$ [6]. If $B$ is also a $C^*$-algebra and $\phi: A \to B$ is a unital positive map then the dual map $\phi^*$ is an affine map from the state space $K_B$ of $B$ into $K = K_A$, and is weak *-continuous; $\phi \to \phi^*$ is a $1-1$ correspondence of unital positive maps and $w^*$-continuous affine maps. Our purpose in this section is to characterize those affine maps from $K_B$ into $K_A$ which correspond to Jordan homomorphisms of $A$ into $B$. (In the case that the $C^*$-subalgebra generated by $\phi(A)$ is all of $B$, another characterization of the dual map has been given by Størmer [10].)

Recall that a convex subset $F$ of $K$ is a *face* of $K$ if $\lambda\sigma + (1-\lambda)\tau \in F$ for $\sigma, \tau \in K$ and $\lambda \in (0, 1)$ implies $\sigma$ and $\tau$ are in $F$. If

$a \in A^{**}$ is positive, then $a^{-1}(0)$ is a face of $K$; such faces are said to be (norm)-*exposed*. In [5] and [7] it is shown that every norm closed face of $K$ is exposed.

Exposed faces of $K$ are in $1-1$ correspondence with projections in $A^{**}$, with the face corresponding to a projection $p$ being $p^{-1}(1)$. Given an exposed face $F$, the corresponding projection $p$ can be recovered as the affine function.

$$(1) \qquad p = \inf\{a \in A^b(K) \mid 0 \leqq a \leqq 1, a = 1 \quad \text{on} \quad F\} .$$

We will write $F^{\#}$ for the face corresponding to $1-p$, i.e., $F^{\#} = (1-p)^{-1}(1) = p^{-1}(0)$. The face $F^{\#}$ is called the *quasicomplement* of $F$ and will play a key role in characterizing dual maps. (For details on other geometric properties of these faces, which lead to the notion of a "projective face", see [1, §§1-3].) Note that when we give $A$ its universal representation all states are vector states; the states in $F$ and $F^{\#}$ are then the vector states $w_\xi$ with $\xi \in pH$ (respectively $\xi \in (1-p)H$).

The key to the role played by $F$ and $F^{\#}$ is their relationship to orthogonality. Recall that each $a = a^* \in A^{**}$ admits a units a unique orthogonal decomposition, $a = a^+ - a^-$ with $0 \leqq a^+$, $0 \leqq a^-$ and $a^+a^- = 0$. To express this in geometric terms, note that $a, b \in (A^{**})^+$, are orthogonal (i.e., $ab = 0$) iff the kernel of $a$ contains (range $b)^- =$ (kernel $b)^\perp$. In terms of the state space:

(2) $a, b, \in (A^{**})^+$ are orthogonal iff there exists an exposed face $F$ with $a = 0$ on $F$, $b = 0$ on $F^{\#}$.

We are now ready for our first result. The natural context is the category of $JB$-algebras: the normed Jordan algebras investigated in [3] which include self-adjoint parts of $C^*$-algebras as a special case.

PROPOSITION 1. *Let $A_1$ and $A_2$ be JB-algebras with state spaces $K_1$ and $K_2$. A $w^*$-continuous affine map $\psi: K_2 \to K_1$ is the dual of a unital Jordan homomorphism from $A_1$ into $A_2$ iff $\psi^{-1}$ preserves quasicomplements,* i.e., *$\psi^{-1}(F^{\#}) = \psi^{-1}(F)^{\#}$ for every exposed face $F$ of $K_1$.*

*Proof.* We will prove the proposition for the case when $A_1$ and $A_2$ are the self-adjoint part of $C^*$-algebras and then indicate the changes needed for $JB$-algebras.

Assume first that $\phi: A_1 \to A_2$ is a unital Jordan homomorphism such that $\phi^* = \psi$, and let $F$ be an exposed face in $K_1$, say $F = p^{-1}(1)$ for $p^2 = p \in A_1^{**}$. Then

$$\psi^{-1}(F) = \psi^{-1}(p^{-1}(1)) = (\phi^{**}(p))^{-1}(1) ,$$

while

$$\psi^{-1}(F^{\#}) = \psi^{-1}(p^{-1}(0)) = (\phi^{**}(p))^{-1}(0) .$$

Since $\phi^{**}: A_1^{**} \to A_2^{**}$ is a Jordan homomorphism, then $\phi^{**}(p)$ is an idempotent, so we have shown that $\psi^{-1}$ preserves quasicomplements.

Conversely, suppose $\psi^{-1}$ preserves quasicomplements. We first show that $\psi^{-1}$ sends exposed faces to exposed faces. If $p^2 = p \in A_1^{**}$ and $F = p^{-1}(0)$, then

$$\psi^{-1}(F) = \psi^{-1}(p^{-1}(0)) = (p \circ \psi)^{-1}(0) .$$

Since $p \circ \psi (A_2^{**})^+$, then $\psi^{-1}(F)$ is a norm exposed face of $K_2$.

Next we show that $\psi$ preserves orthogonality of elements of $A_1^+$. Suppose $a, b \in A_1^+$ and $a \perp b$. Let $F$ be a norm exposed face of $K_1$ such that $a = 0$ on $F$ and $b = 0$ on $F^{\#}$. Now $\phi(a)$ and $\phi(b)$ are positive elements of $A_2$ which are zero on $\psi^{-1}(F)$ and $\psi^{-1}(F^{\#}) = \psi^{-1}(F)^{\#}$ respectively, and so $\phi(a) \perp \phi(b)$.

Now suppose $a$ is any element of $A_1$, with orthogonal decomposition $a = a^+ - a^-$. By virtue of uniqueness of the orthogonal decomposition we conclude that $\phi(a^+) - \phi(a^-)$ is the orthogonal decomposition of $\phi(a)$ in $A_2$; in particular $\phi(a^+) = \phi(a)^+$.

Since $\phi$ is positive and unital, then $\|\phi\| \leqq 1$. Now the set of all $f \in C(\sigma(a))$ such that $\phi(f(a)) = f(\phi(a))$ is seen to be a norm closed vector sublattice of $C(\sigma(a))$; by the Stone-Weierstrass theorem it equals $C(\sigma(a))$. In particular $\phi$ will preserve squares and then also Jordan products. Thus $\phi$ is a Jordan homomorphism. Finally, we consider the more general $JB$-algebra context. We can *define* orthogonality by the property in (2). The proof above then applies without change; the necessary background on the bidual, functional calculus, facial structure and orthogonal decomposition can be found in [8], [3, §2], and [1, §12]. □

As an illustration, let $A_1$ be the $2 \times 2$ real symmetric matrices and $A_2$ the $2 \times 2$ hermitian matrices. The corresponding state spaces are affinely isomorphic to the unit balls of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$ respectively. (See the last section of this paper.) In each case the nontrivial pairs of quasicomplementary faces are just the pairs of antipodal boundary points.

Now suppose $\phi: A_1 \to A_2$ is a unital order isomorphism of $A_1$ into $A_2$, i.e., $a \geqq 0$ iff $\phi(a) \geqq 0$. Now $\phi^*: K_2 \to K_1$ will be surjective, and one readily verifies that $(\phi^*)^{-1}$ must preserve quasicomplements. It follows that every unital order isomorphism from $A_1$ into $A_2$ is a Jordan isomorphism. (This is not true in general.)

*Characterization of *-homomorphisms.* We first recall the notion of orientation defined in [2]. Let $B$ be a 3-ball (i.e., a convex set affinely isomorphic to the closed unit ball of $E^3$ of $\boldsymbol{R}^3$). If $\psi_1$ and $\psi_2$ are affine maps of $E^3$ onto $B$, we say that $\psi_1$ and $\psi_2$ are equivalent if the orthogonal transformation $\psi_1^{-1}\circ\psi_2$ has determinant $+1$. An *orientation* of $B$ is then an equivalence class of affine maps from $E^3$ onto $B$.

Recall that the state space $S(M_2(\boldsymbol{C}))$ of the $2\times 2$ complex matrices is a 3-ball; in fact if we identify $S(M_2(\boldsymbol{C}))$ with the positive matrices of unit trace, then an affine isomorphism $\tau\colon E^3 \to S(M_2(\boldsymbol{C}))$ is given by

$$\tau(a, b, c) = \begin{pmatrix} \frac{1}{2}(1+a) & \frac{1}{2}(b+ic) \\ \frac{1}{2}(b-ic) & \frac{1}{2}(1-a) \end{pmatrix}. \tag{3}$$

We will refer to the associated orientation as the standard orientation for $S(M_2(\boldsymbol{C}))$.

If $B_1$ and $B_2$ are 3-balls with orientations given by $\psi_i\colon E^3 \to B_i$ for $i = 1, 2$, we say an affine map $\gamma$ of $B_1$ onto $B_2$ *preserves orientation* if $\gamma\circ\psi_1$ is equivalent to $\psi_2$; else we say $\gamma$ reverses orientation. It is not difficult to verify that the dual map of any *-automorphism of $M_2(\boldsymbol{C})$ will preserve orientation, while for a *-anti-homomorphism orientation is reversed [2, Lemma 6.1].

Now let $A$ be a $C^*$-algebra with state space $K$. If $\rho$ and $\sigma$ are unitarily equivalent pure states then the smallest face containing $\rho$ and $\sigma$ is a 3-ball, which we denote $B(\rho, \sigma)$. (If $\rho$ and $\sigma$ are inequivalent, the face they generate is the line segment $[\rho, \sigma]$. See [2, Lemma 3.4] for details.) In the future when we refer to a 3-ball of $K$ we will mean a facial 3-ball, i.e., one of the form $B(\rho, \sigma)$.

Let $A(E^3, K)$ denote the set of affine maps from $E^3$ onto 3-balls of $K$, with the topology of pointwise convergence. We let the the orthogonal group $O(3)$ of affine automorphisms of $E^3$ act on $A(E^3, K)$ by composition. Then $A(E^3, K)/SO(3) \to A(E^3, K)/O(3)$ is a locally trivial $\boldsymbol{Z}/2$ bundle cf. [2, Lemma 7.1]. Note that a cross section of this bundle is just a choice of one of the two possible orientations for each 3-ball in $K$. We then define a (global) *orientation* of $K$ to be a continuous cross section of this bundle.

The state space of every $C^*$-algebra is orientable. (Indeed, the fact that face $\{\rho, \sigma\}$ is always of dimension 1 or 3, together with orientability, characterize state spaces of $C^*$-algebras among state space of $JB$-algebras; this is the main result of [2].) To define the standard orientation of $K$, we define the orientation on each 3-ball $B$ in $K$. If $p \in A^{**}$ is the projection corresponding to $B$ (i.e., $p^{-1}(1) = B$), then $pA^{**}p$ is *-isomorphic to $M_2(\boldsymbol{C})$. If $\Phi\colon pA^{**}p \to M_2(\boldsymbol{C})$ is a

$*$-isomorphism, then we define the orientation of $B$ to be that carried over from $S(M_2(C))$ by $\phi^*$. More precisely, let $U_p: A^{**} \to A^{**}$ be the map $a \to p\ a\ p$ and let $\tau: E^3 \to S(M_2(C))$ be the map defined by equation (3); then the orientation of $B$ is given by the map $U_p^* \circ \phi^* \circ \tau: E^3 \to B$. If this orientation is chosen for each 3-ball, then it is shown in [2, Thm. 7.3] that this cross section is continuous, i.e., is a global orientation.

If $\psi: K_2 \to K_1$ is an affine map between state spaces of $C^*$-algebras, we say $\psi$ *preserves orientation* if $\psi$ preserves orientation for each 3-ball of $K_2$ whose image in $K_1$ is a 3-ball of $K_1$. In general $\psi$ will not map 3-balls to 3-balls, even if $\psi$ is the dual of a $*$-homomorphism, but the following lemma shows this happens often enough for our purposes.

The following observation will be useful in the proof. If $\pi: A \to B(H)$ is an irreducible representation, then $\pi^*$ maps the normal state space $N(B(H))$ bijectively onto a face of $K_1$ which we will denote by $F_\pi$. To see that $F_\pi$ is a face, note that $\pi^* N(B(H))$ is just the annihilator in $K$ of the ideal $\ker \tilde{\pi}$, where $\tilde{\pi}: A^{**} \to B(H)$ is the $\sigma$-weakly continuous extension of $\pi$. (In fact $F_\pi$ will be a minimal split face of $K_1$ containing the pure states whose GNS representations are unitarily equivalent to $\pi$, cf. [2, Prop. 2.2], but we will not need this.) Since $\tilde{\pi}$ is surjective, $\pi^*$ will be $1-1$.

LEMMA 2. *Let $A_1$ and $A_2$ be $C^*$-algebras with state spaces $K_1$ and $K_2$, and $\phi: A_1 \to A_2$ a $*$-preserving unital Jordan homomorphism. Then each 3-ball of $K_1$ which lies in $\phi^*(K_2)$ is the image of a 3-ball in $K_2$.*

*Proof.* Let $B = B(\rho, \sigma) \subseteq \phi^*(K_2)$ be a 3-ball of $K_1$. Then $(\phi^*)^{-1}(\rho)$ is a nonempty $w^*$-closed face of $K_2$, so contains a pure state $\tilde{\rho}$. Let $(\pi, H, \xi)$ be the corresponding GNS representation of $A_2$, and let $q$ be the projection on $((\pi \circ \phi)(A_1)\xi)^-$. Identify $qB(H)q$ and $B(qH)$; define $\gamma: A_1 \to (B(qH))$ by

$$\gamma(a) = p(\pi \circ \phi)(a)p\ .$$

Then $\gamma$ is an irreducible representation of $A_1$, and so $\gamma^*$ maps the normal state space $N(B(qH))$ bijectively onto the face $F_\gamma$ of $K_1$. Since $\rho$ and $\sigma$ belong to a 3-ball, they are unitarily equivalent; thus $\sigma = w_\eta \circ \gamma$ for some vector state $w_\eta$ on $B(qH)$. It follows that $B \subseteq F_\gamma$, and thus there is a 3-ball $B^1$ in $q^{-1}(1) \cong (B(qH))$ which is mapped onto $B$ by $(\pi \circ \phi)^*$. Finally, $\pi^*$ maps $N(B(H))$ bijectively onto the face $F_\pi$ of $K_2$, and therefore $\pi^*(B^1)$ is the desired 3-ball of $K_2$. □

PROPOSITION 3. *Let $A_1$ and $A_2$ be $C^*$-algebras with state spaces*

*$K_1$ and $K_2$. A *-preserving unital Jordan homomorphism $\phi: A_1 \to A_2$ is a *-homomorphism iff $\phi^*$ preserves orientation.*

*Proof.* Assume $\phi$ is a *-homomorphism, and let $B_1$ and $B_2$ be 3-balls such that $\phi^*(B_2) = B_1$. Let $p \in A_1^{**}$ be the projection corresponding to $B_1$, i.e. $B = p^{-1}(1)$, and denote by $\phi: A_1^{**} \to A_2^{**}$ the $\sigma$-weakly continuous extension of $\phi$. Now since $pA_1^{**}p$ and $\tilde{\phi}(p)A_2^{**}\tilde{\phi}(p)$ are both isomorphic to $M_2(C)$, it follows that $\tilde{\phi}: pA_1^{**}p \to \tilde{\phi}(p)A_2^{**}\tilde{\phi}(p)$ is a *-isomorphism. From the definition of the standard orientations of $K_1$ and $K_2$, it follows that $\phi^*: B_2 = (\tilde{\phi}(p))^{-1}(1) \to B_1$ preserves orientation. (We note for use below that if $\phi$ were a *-anti-homomorphism, the argument above shows that $\phi^*: B_2 \to B_1$ would reverse orientation.)

Conversely, assume now that $\phi^*: K_2 \to K_1$ preserves orientation. Let $C$ be the $C^*$-subalgebra of $A_2$ generated by $\phi(A_1)$; clearly it suffices to show $\phi: A_1 \to C$ is a *-homomorphism.

We will first show that $\phi^*: K_C \to K_1$ is orientation preserving (where $K_C$ is the state space of $C$). Let $B_C$ and $B_1$ be 3-balls in $K_C$ and $K_1$ with $\phi^*(B_C) = B_1$. By Lemma 2 we can choose a 3-ball $B_2$ in $K_2$ such that the restriction map sends $B_2$ onto $B_C$. By the first paragraph of this proof the restriction map preserves orientation; by assumption so does $\phi^*: B_2 \to B_1$. It follows that $\phi^*: B_C \to B_1$ preserve orientation.

Now let $\pi: C \to B(H)$ be any irreducible *-representation of $C$. Since $\phi(A_1)$ generates $C$, then $\pi \circ \phi: A_1 \to B(H)$ will be an irreducible Jordan homomorphism. By [9, Cor. 3.4] $\pi \circ \phi$ is either a *-homomorphism or *-anti-homomorphism. Let $B$ be any 3-ball in $K_1$ contained in the image of the state space of $B(H)$ under $(\pi \circ \phi)^*$. (By the remarks preceding Lemma 2 such a 3-ball will exist unless $\dim H = 1$.) Now by Lemma 2 there is a 3-ball $B^1$ in $K$ with $\phi^*(B^1) = B$ and a 3-ball $B^2$ in the state space of $B(H)$ with $\pi^*(B^2) = B^1$. Since $\pi^*$ and $\phi^*$ preserve orientation, then $(\pi \circ \phi)^*: B^2 \to B$ does also. By the remarks in the first paragraph of this proof, this rules out the case where $\pi \circ \phi$ is an anti-homomorphism unless $\dim H = 1$, and so in all cases $\pi \circ \phi$ is a *-homomorphism. Since $\pi$ was an arbitrary irreducible representation of $C$, it follows that $\phi$ is a *-homomorphism. □

PROPOSITION 4. *Let $A$ and $B$ be $C^*$-algebras and $\psi$ a $w^*$-continuous affine map from the state space of $B$ into the state space of $A$. Then $\psi$ is the dual of a unital * homomorphism from $A$ into $B$ iff $\psi^{-1}$ preserves quasicomplements and $\psi$ preserves orientation.*

*Proof.* Immediate from Propositions 1 and 3. □

## REFERENCES

1. E, M. Alfsen and F. Shultz, *Non-commutative spectral theory for affine function spaces on convex sets,* Mem. Amer. Math. Soc., **172** (1976).
2. E. M. Alfsen, H. Hanche-Olsen, and F. Shultz, *State spaces of C*-algebras,* Acta Math., **144** (1980), 267-305.
3. E. M. Alfsen, F. Shultz, and E. Størmer, *A Gelfand-Neumark theorem for Jordan algebras,* Advances in Math., **28** (1978), 11-56.
4. J. Dixmier, *Les C*-algêbres et leurs reprêsentations,* Gauthier-Villars, Paris, 1964.
5. E. G. Effros, *Order ideals in a C*-algebra and its dual,* Duke Math. J., **30** (1963), 391-412.
6. R. V. Kadison, *Transformations of states in operator theory and dynamics,* Topology, **3** (1965), 177-198.
7. R. T. Prosser, *On the ideal structure of operator algebras,* Mem. Amer. Math. Soc., **45** (1963).
8. F. Shultz, *On normal Jordan algebras which are Banach dual spaces,* J. Functional Analysis, **31** (1979), 360-376.
9. E. Størmer, *On the Jordan structure of C*-algebras,* Trans. Amer. Math. Soc., **120** (1965), 438-447.
10. ———, *On partially ordered vector spaces and their duals, with applications to simplexes and C*-algebras,* Proc. London Math. Soc., **18** (1968), 245-265.

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