PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 1, 1981

NOTE ON BOUNDED L^p-SOLUTIONS OF A GENERALIZED LIÉNARD EQUATION

ALLAN J. KROOPNICK

Two theorems are presented giving sufficient conditions for all solutions to $y''+c(t)f(y)y'+a(t)b(y)=0$ to be bounded. Furthermore, two other theorems are given guaranteeing when these solutions are L^p-solutions. Asymptotic stability is then discussed as well as several applications of these results.

In this paper, sufficient conditions will be given so that all solutions to a generalized Liénard equation,

$$y'' + c(t)f(y)y' + a(t)b(y) = 0 \tag{1}$$

will be L^p-solutions ($p \geqq 1$) on $[0, \infty)$. By an L^p-solution we shall mean a solution to (1) such that $\left(\int_0^\infty |y|^p dt\right) < \infty$. This note shall generalize some previous results (see [1] – [3]). We first need the following theorem.

THEOREM I. *Suppose $a(t)$ and $c(t)$ are continuous functions on $[0, \infty)$ and let $b(y)$ and $f(y)$ be continuous on $(-\infty, +\infty)$. Furthermore, suppose for some positive constant a_0, $a(t) \geqq a_0$, $a'(t) \leqq 0$, $c(t) \geqq 0$ for $0 \leqq t < \infty$, and $f(y) > 0$. Finally, if $B(y) = \int_0^y b(u)du \to +\infty$ as $|y| \to \infty$, then every solution of* (1) *exists on $[0, \infty)$ and $|y(t)|$, $|y'(t)|$ are bounded as $t \to \infty$.*

Proof. By standard existence theory (1) has at least one solutions satisfying $y(0) = y_0$, $y'(0) = \dot{y}_0$, and existing on some interval $[0, T)$, $T > 0$. Consider any such solutions of (1) on $[0, T)$. Multiply (1) by y' and integrate from 0 to $t < T$ to obtain,

$$\frac{1}{2}y'(t)^2 + \int_0^t c(s)f(y)y'^2 ds + \int_0^t a(s)b(y)y' ds = \frac{1}{2}\dot{y}_0^2 . \tag{2}$$

Integrating (2) by parts we have

$$\begin{aligned} &\frac{1}{2}y'(t)^2 + \int_0^t c(s)f(y)y'^2 ds + a(t)B(y(t)) \\ &\qquad - \int_0^t a'(s)B(y(s))ds = \frac{1}{2}\dot{y}_0^2 + a(0)B(y_0) \quad (0 \leqq t < T) . \end{aligned} \tag{3}$$

For $|y|$, $|y'|$ large all terms on the LHS of (3) are positive except, perhaps, for $-\int_0^t a'(s)B(y)ds$. We show that this term is bounded from below. By our hypotheses, $B(y) > -d^2$ for some real constant d and $-\int_0^t a'(s)B(y(s))ds > d^2\int_0^t a'(s)ds > -a(0)d^2$. Since

$$\int_0^t c(s)f(y(s))(y'(s))^2 ds \geqq 0$$

and since $a(t)B(y(t)) > -a(0)d^2$, (3) becomes

$$(3') \qquad \begin{aligned} -2a(0)d^2 &< \frac{1}{2}y'^2(t) + a(t)B(y(t)) - \frac{1}{2}\int_0^t a'(s)B(y(s))ds \\ &\leqq \frac{1}{2}y_0^2 + a(0)B(y_0) \quad (0 \leqq t < T)\,. \end{aligned}$$

By assumption $B(y) \to +\infty$ as $|y| \to \infty$, and the left and right sides of the inequality (3′) are a priori lower and upper bounds, independent of T. Therefore, a standard argument completes the proof of global existence of solutions y such that $|y(t)|$ and $|y'(t)|$ are bounded on $[0, \infty)$.

REMARKS (1). If one also assumes $yb(y) \geqq 0$, then all terms on the LHS of (3) are positive making the proof considerably simpler. (2) If the assumptions concerning a, c in Theorem I hold on an interval $[t_0, \infty)$ for some $t_0 \neq 0$, the conclusions are valid on the interval $[t_0, \infty)$. The same remark applies to Theorems II, III, and IV below.

We now proceed to our main theorems.

THEOREM II. *Let the hypotheses of Theorem I hold. In addition, let there exist constants* $c_0 > 0$, $f_0 > 0$, $M > 0$, $p \geqq 1$ *such that* $c(t) \geqq c_0$, $c'(t) \leqq 0$ $(0 \leqq t < \infty)$, *and* $f(y) \geqq f_0$, $yb(y) \geqq M|y|^p$ $(y \in \boldsymbol{R})$. *Then* $\int_0^\infty |y(t)|^p dt < \infty$.

Proof. From (3) (which now holds on $0 < t < \infty$) we see that $c_0 f_0 \int_0^\infty y'(t)^2 dt \leqq (1/2)y_0^2 + a(0)B(y_0)$, so y' is square integrable. Multiply (1) by y and integrate by parts from 0 to t obtaining,

$$(4) \quad y(t)y'(t) - \int_0^t y'(s)^2 ds + \int_0^t c(s)f(y)yy'ds + \int_0^t a(s)b(y)yds = y_0\dot{y}_0\,.$$

Letting $F(y) = \int_0^y uf(u)du$ we have upon another integration by parts,

$$y(t)y'(t) - \int_0^t y'(s)^2 ds + c(t)F(y(t)) - \int_0^t c'(s)F(y)ds + \int_0^t a(s)b(y)yds = y_0\dot{y}_0 + c(0)F(y_0) . \tag{5}$$

As $t \to \infty$ all terms on the LHS of (5) stay finite. Specifically, (5) shows that

$$0 \leqq a_0 M \int_0^t |y(s)|^p ds \leqq K_1 \quad (0 \leqq t < \infty) \tag{6}$$

where $K_1 = |y_0||\dot{y}_0| + K^2 + 3c(0) \sup_{-K \leqq y \leqq K} F(y) + \int_0^\infty y'(s)^2 ds$ and K is the bound of $|y|$ and $|y'|$ on $[0, \infty)$. Since K_1 is independent of t, the result follows.

REMARK. Theorem II is still true under the assumptions $c_0 \leqq c(t) \leqq c_1 (c_0, c_1 > 0)$ and $c' \in L^1[0, \infty)$ because

$$\left| \int_0^t c'(s)F(y)ds \right| \leqq \sup_{-K \leqq y \leqq K} F(y) \int_0^\infty |c'(s)| ds . \tag{7}$$

This implies all terms on the LHS of (5) are bounded so that inequality (6) still holds although K_1 will be different.

The last two theorems use the well-known Gronwall-Bellman inequality.

THEOREM III. *The hypotheses are the same as Theorem I except that* $a'(t) \geqq 0 (0 \leqq t < \infty)$. *Furthermore, if* $yb(y) \geqq 0 (y \in \boldsymbol{R})$, *then all solutions to* (1) *are bounded as* $t \to \infty$.

Proof. From (3) we obtain the following inequality,

$$0 \leqq a(t)B(y(t)) \leqq K + \int_0^t a'(s)B(y(s))ds \tag{8}$$

where $K = a_0 B(y_0) + (1/2)\dot{y}_0^2$. (8) may be rewritten as

$$0 \leqq a(t)B(y(t)) \leqq K + \int_0^t \frac{a'(s)}{a(s)} a(s)B(y(s))ds . \tag{9}$$

Applying the Gronwall-Bellman inequality to (9) yields,

$$a(t)B(y(t)) \leqq K\left(\exp \int_0^t \frac{a'(s)}{a(s)} ds\right) = K\frac{a(t)}{a(0)} \tag{10}$$

or

$$B(y(t)) \leqq \frac{K}{a_0} . \tag{11}$$

Since $B(y) \to \infty$ as $|y| \to \infty$, $y(t)$ must stay finite on $0 \leqq t < \infty$.

REMARK. If one does not assume $a'(t) \leqq 0$ (or $a'(t) \geqq 0$) for $0 \leqq t < \infty$, but only that $0 < a_0 \leqq a(t) < \infty$, and $a' \in L^1[0, \infty)$, one can still obtain the conclusion that $y(t)$ remains bounded, for from (8) one has

$$a(t)B(y(t)) \leqq K + \frac{1}{a_0}\int_0^t |a'(s)| a(s)B(y(s))ds \, . \tag{12}$$

Thus by the Gronwall-Bellman inequality

$$a(t)B(y(t)) \leqq K\Big(\exp\Big(\frac{1}{a_0}\int_0^\infty |a'(s)|ds\Big)\Big) < \infty \, . \tag{13}$$

By strengthening the hypotheses of Theorem III slightly, we are able to show that all solutions are in $L^p[0, \infty)$. This is the substance of our final theorem.

THEOREM IV. *The hypotheses are the same as Theorem III. In addition, let there exist positive constants c_0 and f_0 such that $c(t) \geqq c_0$ and $f(y) \geqq f_0$, $c'(t) \leqq 0$, $yb(y) \geqq M|y|^p$ (for some positive constants M and $p \geqq 1$), and if $a(t) \leqq A_0$ where A_0 is a fixed positive constant, then $\int_0^\infty |y|^p dt < \infty$.*

Proof. From (3) we see that if $a(t)$ is bounded from above, then $\int_0^T a'(s)B(y)ds$ is bounded and, therefore, y' is bounded since all other terms in (3) are bounded. The remainder of the proof is identical to Theorem II and the remark after theorem is still true.

REMARKS. Had we only required $yb(y) \geqq 0$, then Theorems II and IV would yield $\int_0^\infty y(t)b(y(t))dt < \infty$.

Actually, more may be proven. Under the hypotheses of Theorem II, the solutions to (1) are asymptotically stable (cf. Lemma 2.2 and Remarks 2.1 and 2.2 of [2]). In order to see this, write (1) as the following two dimensional system,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -c(t)f(y_1)y_2 - a(t)b(y_1) \, . \end{aligned} \tag{14}$$

Now consider the following Liapunov function $V(t, y_1, y_2) = (1/2)y_2^2 + a(t)B(y_1)$. Differentiating we have $dV/dt = y_2y_2' + a'(t)B(y_1) + a(t)b(y_1)y_1' = a'(t)B(y_1) - c(t)f(y_1)y_2^2 < 0$ for $(y_1, y_2) \neq (0, 0)$ implying the solutions are asymptotically stable.

EXAMPLE 1. Consider the differential equation

$$y'' + \frac{1}{t^n} y = 0 \quad (0 < t_0 \leqq t < \infty,\ n \text{ a positive integer}) . \tag{15}$$

The substitution $y = x(t)\exp(t)$ transforms (15) into

$$x'' + 2x' + \left(1 + \frac{1}{t^n}\right)x = 0 \quad (0 < t_0 \leqq t < \infty) . \tag{16}$$

Therefore, applying Theorem II and Remark 2 following Theorem I, any solution to (15) is the product of the exponential function and a function in $L^2[a, \infty)$ $(a > 0)$ since $xb(x) = x^2$.

EXAMPLE 2. It is well-known that the solutions to the homogeneous Duffing equation

$$y'' + cy' + ay(1 + dy^2) = 0 \quad (a, c, d > 0) \tag{17}$$

are asymptotically stable. By Theorem II the solutions are in both $L^2[0, \infty)$ and $L^4[0, \infty)$ since $yb(y) = y^2(1 + dy^2) \geqq y^2$ and $yb(y) \geqq dy^4$.

EXAMPLE 3. Emden's equation

$$y'' + \frac{2}{x} y' + y^n = 0 \quad (n \text{ a positive integer}) \tag{18}$$

has important applications in astrophysics. When n is an odd positive integer, all solutions to (18) are bounded on $0 < x_0 \leqq x < \infty$ using Theorem I and Remark 2 following Theorem I.

The author would like to thank the referee for several helpful suggestions concerning this paper.

REFERENCES

1. A. Kroopnick, *L^2-solutions to $y''+c(t)y'+a(t)b(y)=0$*, Proc. Amer. Math. Soc., **39** (1973), 217-218.

2. J. J. Levin and J. A. Nohel, *Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics*, Archive Rat. Mech. Anal., **5** (1960), 194-211.

3. A. Strauss, *Liapunov functions and L^p-solutions of differential equations*, Trans. Amer. Math. Soc., **119** (1956), 37-50.

Received April 10, 1978 and in revised form August 10, 1980.

SOCIAL SECURITY ADMINISTRATION
OFFICE OF RESEARCH AND STATISTICS
MEADOWS EAST BUILDING
6401 SECURITY BLVD
BALTIMORE, MD 21235