PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 1, 1981

CONCERNING THE MINIMUM OF PERMANENTS ON DOUBLY STOCHASTIC CIRCULANTS

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Let P_n be the permutation matrix such that $(P_n)_{ij} = 1$ if $j = i + 1 \pmod{n}$. Minc [2] proved that the minimum of the permanent on the collection of $n \times n$ doubly stochastic circulants $\alpha I_n + \beta P_n + \gamma P_n^2$ is in $(1/2^n, 1/2^{n-1}]$, and if $n \geqq 5$ then the minimum is not achieved at $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$. This paper proves that if $n \geqq 3$ then the minimum of such permanents is less than $1/2^{n-1}$, and if $n \in \{3, 4\}$ then this minimum is uniquely achieved at $(1/3)I_n + (1/3)P_n + (1/3)P_n^2$.

Introduction. Let n be a positive integer, let I_n denote the $n \times n$ identity matrix, and let P_n denote the full cycle permutation matrix such that $(P_n)_{ij} = 1$ if $j = i + 1 \pmod{n}$. Minc [2] studied the permanent of circulants $\alpha I_n + \beta P_n + \gamma P_n^2$ and proved the following three theorems:

THEOREM 1. *If $n \geqq 3$ then*

$$\operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) = \left(\frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2}\right)^n + \left(\frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2}\right)^n + \alpha^n + \gamma^n .$$

THEOREM 2. *If α, β, γ are nonnegative then*

$$\frac{1}{2^n} < \min_{\alpha+\beta+\gamma=1} \operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) \leqq \frac{1}{2^{n-1}} .$$

THEOREM 3. *If α, β, γ are nonnegative, $n \geqq 5$, then*

$$\min_{\alpha+\beta+\gamma=1} \operatorname{per}(\alpha I_n + \beta P_n + \gamma P_n^2) < \operatorname{per}\left(\frac{1}{3}I_n + \frac{1}{3}P_n + \frac{1}{3}P_n^2\right) .$$

MAIN RESULTS. Let $S = \{(\alpha, \gamma) \mid 0 \leqq \alpha, 0 \leqq \gamma, \alpha + \gamma \leqq 1\}$, and let f_n denote the function on S such that

$$f_n(\alpha, \gamma) = \operatorname{per}(\alpha I_n + (1 - \alpha - \gamma)P_n + \gamma P_n^2) .$$

THEOREM 4. *If $n \geqq 3$ then f_n is not minimum on the boundary of S.*

LEMMA TO THEOREM 4. *The minimum of f_n on the boundary of*

S is $1/2^{n-1}$. If n is even this minimum is achieved only on $\{(1/2, 0), (0, 1/2)\}$, and if $n > 1$ and n is odd this minimum is achieved only on $\{(1/2, 0), (1/2, 1/2), (0, 1/2)\}$.

Proof. The lemma is clearly true is case $n \in \{1, 2\}$. Suppose $n \geqq 3$. Since

$$f_n(1/2, 0) = f_n(0, 1/2) = \frac{1}{2^{n-1}} < 1 = f_n(1, 0) = f_n(0, 0)f_n(0, 1)\ ,$$

then it is sufficient to consider only points belonging to the interior of the boundary of S. The only real number α satisfying $D_1 f_n(\alpha, 0) = 0$ is $1/2$. Therefore, since $f_n(\alpha, \gamma) = f_n(\gamma, \alpha)$, then the minimum of f_n on $\{(\alpha, \gamma) \mid \alpha\gamma = 0\}$ is $1/2^{n-1}$. Let $g(\alpha) = f_n(\alpha, 1 - \alpha)$. If n is even, put $k = n/2$ and observe that $g(\alpha) = (\alpha^k + (1 - \alpha)^k)^2$. If n is odd then $g(\alpha) = \alpha^n + (1 - \alpha)^n$. In either case, $1/2$ is the only real number α such that $g'(\alpha) = 0$. If n is even then $f_n(1/2, 1/2) = 1/2^{n-2} > 1/2^{n-1}$, and if n is odd then $f_n(1/2, 1/2) = 1/2^{n-1}$.

Proof of Theorem 4. By the lemma it is sufficient to show there is a point q of S so that $f_n(q) < f_n(1/2, 0)$. Observe that $D_1 f_n(\alpha, \gamma)$ is

$$\begin{aligned}
&\frac{n}{2}\left(\frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-1}\left(-1+\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\right)\\
&\quad+\frac{n}{2}\left(\frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-1}\left(-1-\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\right)\\
&\quad+n\alpha^{n-1}\ .
\end{aligned}$$

Thus $D_1 f_n(1/2, 0) = 0$ and therefore, since $D_1 f_n(\alpha, \gamma) = D_2 f_n(\gamma, \alpha)$, then $(1/2, 0)$ is a critical point for f_n. Now observe that $D_{1,1}(\alpha, \gamma)$ is

$$\begin{aligned}
&\frac{n}{2}\left[\frac{(n-1)}{2}\left(\frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-2}\left(-1+\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\right)^2\right.\\
&\left.+\left(\frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-1}\left(\frac{(1-\alpha-\gamma)^2+4\alpha\gamma-(-1+\alpha+3\gamma)^2}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}}\right)\right]\\
&+\frac{n}{2}\left[\frac{(n-1)}{2}\left(\frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-2}\left(-1-\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\right)^2\right.\\
&\left.+\left(\frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\right)^{n-1}\left(\frac{-(1-\alpha-\gamma)^2+4\alpha\gamma+(-1+\alpha+3\gamma)^2}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}}\right)\right]\\
&+n(n-1)\alpha^{n-2}\ .
\end{aligned}$$

Thus $D_{1,1}f_n(1/2, 0) = n(n-1)/2^{n-3}$, and since $D_{2,2}f_n(\alpha, \gamma) = D_{1,1}(\gamma, \alpha)$ then $D_{2,2}f_n(1/2, 0) = 0$. Finally, observe that $D_{1,2}f_n(\alpha, \gamma)$ is

$$\frac{n}{2}\Big[\frac{(n-1)}{2}\Big(\frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\Big)^{n-2}\Big(-1+\frac{-1+3\alpha+\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\Big)$$

$$\times\Big(-1+\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\Big)+\Big(\frac{1-\alpha-\gamma+\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\Big)^{n-1}$$

$$\times\Big(\frac{3((1-\alpha-\gamma)^2+4\alpha\gamma)-(-1+\alpha+3\gamma)(-1+3\alpha+\gamma)}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}}\Big)\Big]$$

$$+\frac{n}{2}\Big[\frac{(n-1)}{2}\Big(\frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\Big)^{n-2}\Big(-1-\frac{-1+3\alpha+\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\Big)$$

$$\times\Big(-1-\frac{-1+\alpha+3\gamma}{\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}\Big)+\Big(\frac{1-\alpha-\gamma-\sqrt{(1-\alpha-\gamma)^2+4\alpha\gamma}}{2}\Big)^{n-1}$$

$$\times\Big(\frac{-3((1-\alpha-\gamma)^2+4\alpha\gamma)+(-1+\alpha+3\gamma)(-1+3\alpha+\gamma)}{((1-\alpha-\gamma)^2+4\alpha\gamma)^{3/2}}\Big)\Big].$$

Thus $D_{1,2}f_n(1/2, 0) = n/2^{n-3} = D_{2,1}f_n(1/2, 0)$.

Let H denote the Hessian matrix for f_n at $(1/2, 0)$. H has characteristic values

$$\lambda_1 = \frac{n}{2^{n-2}}(n-1+\sqrt{(n-1)^2+4})$$

and

$$\lambda_2 = \frac{n}{2^{n-2}}(n-1-\sqrt{(n-1)^2+4}).$$

Since $\lambda_2 < 0 < \lambda_1$ then $(1/2, 0)$ is a saddle point for f_n. Let $x = (\lambda_2, 1)$ and put $|x| = \sqrt{\lambda_2^2+1}$. By Taylor's theorem there is a positive number δ so that if $|x| < \delta$ then there is a number $R(x)$ so that $f_n((1/2, 0) + x)$ is

$$\frac{1}{0!}f_n(1/2, 0) + \frac{1}{1!}\sum_{k=1}^{2}(x)_k D_k f_n(1/2, 0) + \frac{1}{2!}\sum_{i,j=1}^{2}(x)_i(x)_j D_{i,j}f_n(1/2, 0) + R(x)$$

and therefore, since $(1/2, 0)$ is a critical point for f_n, and since $Hx^T = \lambda_2 x^T$, then

$$f_n((1/2, 0) + x) = f_n(1/2, 0) + \lambda_2|x|^2 + R(x).$$

Since $\lambda_2 < 0$ then there is a positive number $\omega < \delta$ such that if $|x| < \omega$ then $\lambda_2|x|^2 + R(x) < 0$, and therefore $f_n((1/2, 0) + x) < f_n(1/2, 0)$. Let $q = (1/2, 0) + \omega|x|^{-1}x$, observe that $q \in S$ and that $f_n(q) < f_n(1/2, 0)$.

THEOREM 5. *If* $n \in \{3, 4\}$ *then* f_n *is minimum, uniquely, at* $(1/3, 1/3)$.

Proof. In [1] Marcus and Newman proved the van der Waerden conjecture true in case $n = 3$, and hence this theorem is also true in this case. Let (α, γ) be a point of S at which f_4 is minimum. Observe that $f_4(\alpha, \gamma)$ is

$$2\alpha^4 - 4\alpha^3 + 6\alpha^2 - 4\alpha + 2\gamma^4 + 6\gamma^2 - 4\gamma - 20\gamma^2 + 8\alpha\gamma^3 + 16\alpha^2\gamma^2 + 8\alpha^3\gamma - 20\alpha^2\gamma + 16\alpha\gamma + 1 ,$$

that $D_1 f_4(\alpha, \gamma)$ is

$$8\alpha^3 - 12\alpha^2 + 12\alpha - 4 - 20\gamma^2 + 8\gamma^3 + 32\alpha\gamma^2 + 24\alpha^2\gamma - 40\alpha\gamma + 16\gamma ,$$

and that $D_2 f_4(\alpha, \gamma)$ is

$$8\gamma^3 - 12\gamma^2 + 12\gamma - 4 - 40\alpha\gamma + 24\alpha\gamma^2 + 32\alpha^2\gamma + 8\alpha^3 - 20\alpha^2 + 16\alpha .$$

By Theorem 4, (α, γ) is not on the boundary of S and so $D_1 f_4(\alpha, \gamma) = 0 = D_2 f_4(\alpha, \gamma)$. Thus $D_1 f_4(\alpha, \gamma) - D_2 f_4(\alpha, \gamma) = 0$ and therefore

$$(1) \qquad (\alpha - \gamma)(2(\alpha + \gamma) - 1 - 2\alpha\gamma) = 0 .$$

Since $D_1 f_4(\alpha, \alpha) = (\alpha - 1/3)(18\alpha^2 - 12\alpha + 3)$ then the only critical point on the diagonal of S is $(1/3, 1/3)$. Suppose

$$(2) \qquad f_4(\alpha, \gamma) < f_4\left(\frac{1}{3}, \frac{1}{3}\right)$$

and observe from (1) that

$$(3) \qquad 2(\alpha + \gamma) - 1 - 2\alpha\gamma = 0 .$$

Let $\beta = 1 - \alpha - \gamma$. It follows from (3) that $\beta^2 = \alpha^2 + \gamma^2$ and from (2) and (3) that

$$f_4(\alpha, \gamma) = \beta^4 + 2\beta^2(2\alpha\gamma) + (\alpha^2 + \gamma^2)^2 = 2\beta^2(1 - \beta)^2 < \frac{1}{9} .$$

Hence $\beta(1 - \beta) < 1/3\sqrt{2}$ and therefore

$$(4) \qquad \text{either} \quad \beta < \frac{1 - \sqrt{1 - \dfrac{2\sqrt{2}}{3}}}{2} \quad \text{or} \quad \beta > \frac{1 + \sqrt{1 - \dfrac{2\sqrt{2}}{3}}}{2} .$$

It also follows from (3) that $2\gamma^2 - 2(1 - \beta)\gamma + 1 - 2\beta = 0$ and therefore, since γ is a real number, then

$$(5) \qquad \beta \geqq \sqrt{2} - 1 .$$

Finally, (3) implies that $1 - 2\beta - 2\alpha\gamma = 0$, and therefore since $\alpha\gamma \geqq 0$, then

$$(6) \qquad \beta \leqq 1/2 .$$

Inequalities (4), (5) and (6) constitute a contradiction.

I would like to thank Professor Sinkhorn for his assistance and suggestions in the preparation of this paper.

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Received July 3, 1973.

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