PACIFIC JOURNAL OF MATHEMATICS
Vol. 95, No. 1, 1981

BOUNDARY POINTS OF JOINT NUMERICAL RANGES

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In this paper it is shown that the conical points of the joint numerical range belong to the joint spectrum. Moreover, we discuss the bare points and extreme points of the joint numerical ranges for the n-tuples of commuting normal operators and Toeplitz operators.

Introduction. The notion of the joint numerical range was first investigated by Halmos ([6], Prob. 166). Dash [4] tried to find how much of the knowledge about the numerical range in the single operator case carried over to the analogous situation in the case of an n-tuple of operators. Our purpose is to discuss the same subject as his. Dash [4] studied particularly about the convexity of the numerical range known as the Toeplitz-Hausdorff theorem. Here we shall, however, bring the boundary point of the numerical range into focus. In the case of a single operator, many authors have asserted the results referring to the relation between the numerical range and spectrum. Concerning these, Dash [4], Juneja [8], Abramov [1], Buoni and Wadhwa [3] have investigated the relation between the joint spectrum and joint numerical range. Abramov [1] has shown that the conical point of the closure of the joint numerical range of $A = (A_1, \cdots, A_n)$ belongs to the joint approximate point spectrum of A in the case of the family A consisting of self-adjoint operators. In §1, our result shall be given more clearly than Abramov's one even to the family of arbitrary operators, by means of Hildebrandt's technique [7]. In §2, we shall introduce a class of operator-families called *joint normaloid*. And, in §3, we shall discuss the bare points and extreme points of joint numerical ranges for the operator-families belonging to the joint normaloid.

Notation and definition. Throughout this paper, H will be a complex Hilbert space with the scalar product $(,)$ and the norm $\|\cdot\|$, and all operators on H will be assumed to be linear and bounded. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators on H. The *joint numerical range* of A is the subset $W(A)$ of the n-dimensional unitary space C^n such that

$$W(A) = \{((A_1x, x), \cdots, (A_nx, x)) : x \in H, \|x\| = 1\} .$$

In the case of $n = 1$, it is the usual numerical range of an operator.

We shall say that a point $z = (z_1, \cdots, z_n)$ of C^n is in the *joint approximate point spectrum* $\sigma_\pi(A)$ of A if there exists a sequence $\{x_i\}$ of unit vectors in H such that

$$\|(z_k - A_k)x_i\| \longrightarrow 0(i \longrightarrow \infty)\,, \quad k = 1, \cdots, n\,.$$

A point $z = (z_1, \cdots, z_n)$ will be called a *joint eigenvalue of* A if there exists a nonzero eigenvector x such that

$$A_k x = z_k x\,, \quad k = 1, \cdots, n\,.$$

And a point $z = (z_1, \cdots, z_n)$ will be said to be in the *joint residual spectrum* $\sigma_r(A)$ of A if there exists a nonzero vector x such that

$$A_k^* x = \overline{z_k} x\,, \quad k = 1, \cdots, n\,,$$

where $\overline{z_k}$ denotes the complex conjugate of z_k. (Consult [5].)

Moreover, let $A = (A_1, \cdots, A_n)$ be an n-tuple of *mutually commuting* operators. And let A'' be the double commutant of A. Then we shall say that a point $z = (z_1, \cdots, z_n)$ of C^n is in the *joint spectrum* $\sigma(A)$ of A relative to A'' if

$$\sum_{k=1}^{n} B_k(A_k - z_k) \neq I\,,$$

for all $B_1, B_2, \cdots, B_n$ in A'', where I denotes the identity operator. (Consult [5].)

1. Conical points.

DEFINITION 1. Let a closed subset K of C^n be called a closed convex cone with vertex $(0, \cdots, 0)$ whenever K satisfies the following properties:

$$(1) \qquad K + K \subset K\,,$$

$$(2) \qquad \alpha K \subset K \quad \text{for all} \quad \alpha \geqq 0\,,$$

$$(3) \qquad K \cap (-K) = \{(0, \cdots, 0)\}\,.$$

If, for $F \subset C^n$ and $z = (z_1, \cdots, z_n) \in F$, there exists a closed convex cone K with vertex $(0, \cdots, 0)$ such that $F \subset K - z$, then we shall call the point z a *conical point of* F.

THEOREM 1. *Let* $A = (A_1, \cdots, A_n)$ *be an* n*-tuple of arbitrary operators. If* $z = (z_1, \cdots, z_n)$ *is a conical point of* $\overline{W(A)}$ *(throughout we shall use the bar symbol for closure), then* z *belongs to the joint approximate point spectrum* $\sigma_\pi(A)$ *of* A. *If, moreover,* z *is in* $W(A)$, *then* z *is a joint eigenvalue of* A.

Proof. We may assume without loss of generality that the conical point z of $\overline{W(A)}$ is $(0, \cdots, 0)$. Then we can choose n linearly independent vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ in $\boldsymbol{C}^n$ and n constants $\theta_1, \cdots, \theta_n$ such that $0 \leqq \theta_k < \pi$, $k = 1, \cdots, n$ and

$$\overline{W(A)} \subset \{\alpha_1 \boldsymbol{a}_1 + \cdots + \alpha_n \boldsymbol{a}_n : 0 \leqq \arg \alpha_k \leqq \theta_k,\ k = 1, \cdots, n\} .$$

Let a set $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ of vectors in $\boldsymbol{C}^n$ be a basis in $\boldsymbol{C}^n$ such that the jth coordinate of $\boldsymbol{e}_k$ is δ_{kj}, $k = 1, \cdots, n$, and $\boldsymbol{e}_k = \gamma_{1k}\boldsymbol{a}_1 + \cdots + \gamma_{nk}\boldsymbol{a}_n$, $k = 1, \cdots, n$. Putting

$$B_k = \gamma_{k1}A_1 + \cdots + \gamma_{kn}A_n , \quad k = 1, \cdots, n ,$$

it follows that

$$(*) \quad \overline{W(B_1, \cdots, B_n)} \subset \{(\beta_1, \cdots, \beta_n) : 0 \leqq \arg \beta_k \leqq \theta_k,\ k = 1, \cdots, n\} .$$

We shall apply Hildebrandt's method [7, p. 232] to the argument follows. Let k be any fixed element in the index set $\{1, \cdots, n\}$. We put here

$$e^{i\theta_k} = \lambda_k = \mu_k + i\nu_k ,$$

where μ_k, ν_k are real numbers. Since we can, moreover, assume θ_k to be nonzero, ν_k is assumed to be nonzero. Therefore

$$(**) \qquad i = \frac{1}{\nu_k}(\lambda_k - \mu_k) .$$

Furthermore, we decompose B_k such that

$$B_k = X_k + iY_k ,$$

where X_k, Y_k are self-adjoint. Substituting the formula (**) for i,

$$B_k = X_k - \frac{\mu_k}{\nu_k}Y_k + \lambda_k\left(\frac{1}{\nu_k}Y_k\right) .$$

Here, we put

$$T_k = X_k - \frac{\mu_k}{\nu_k}Y_k \quad \text{and} \quad S_k = \frac{1}{\nu_k}Y_k .$$

Then T_k, S_k are self-adjoint and $B_k = T_k + \lambda_k S_k$. Since $(B_k x, x) = (T_k x, x) + \lambda_k(S_k x, x)$ for every unit vector x, T_k and S_k are positive from (*).

Now, since $z = (0, \cdots, 0) \in \overline{W(A)}$, $(0, \cdots, 0) \in \overline{W(B)}$ and then there exists a sequence $\{x_i\}$ of unit vectors such that $(B_k x_i, x_i) \to 0(k = 1, \cdots, n, i \to \infty)$. And, since $(T_k x_i, x_i)$ and $(S_k x_i, x_i)$ also converge to zero for every k, we have

$$B_k x_i \longrightarrow 0 (i \longrightarrow \infty) , \quad k = 1, \cdots, n .$$

On the other hand, since the matrix $\boldsymbol{M}$:

$$\boldsymbol{M} = \begin{bmatrix} \gamma_{11} & \cdot & \cdot & \cdot & \gamma_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{n1} & \cdot & \cdot & \cdot & \gamma_{nn} \end{bmatrix}$$

is regular,

$$\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} = \boldsymbol{M}^{-1} \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} .$$

Hence we have $A_k x_i \to 0 (i \to \infty)$, $k = 1, \cdots, n$. Thus we get the proof of the first half of the theorem.

Next we assume that $z = (0, \cdots, 0) \in W(A)$. Then, there exists a unit vector x such that $(A_k x, x) = 0$, $k = 1, \cdots, n$. So, if we take x in place of the above sequence $\{x_i\}$, the proof of the latter half of the theorem follows in the same way as the first half.

2. Joint normaloid operator-families.

DEFINITION 2. For any n-tuple $A = (A_1, \cdots, A_n)$ of operators, the following nonnegative numbers:

$$\begin{aligned} \|A\| &= \sup \{(\|A_1 x\|^2 + \cdots + \|A_n x\|^2)^{1/2} : \|x\| = 1\} , \\ w(A) &= \sup \{(|(A_1 x, x)|^2 + \cdots + |(A_n x, x)|^2)^{1/2} : \|x\| = 1\} , \\ r(A) &= \sup \{(|z_1|^2 + \cdots + |z_n|^2)^{1/2} : z \in \sigma(A)\} , \\ r_\pi(A) &= \sup \{(|z_1|^2 + \cdots + |z_n|^2)^{1/2} : z \in \sigma_\pi(A)\} \end{aligned}$$

are called the *joint operator norm, joint numerical radius, joint spectral radius* and *joint approximate point spectral radius respectively, of A.*

DEFINITION 3. An n-tuple $A = (A_1, \cdots, A_n)$ of operators is said to belong to the *joint normaloid* or to be *joint normaloid* if $w(A) = \|A\|$.

In order to show the following propositions we need the following results shown by Dash [5].

First, suppose that $A = (A_1, \cdots, A_n)$ is a commuting n-tuple of normal operators. Then there exists a measure space $(X; \mu)$ and a set of bounded measurable functions $\phi_1, \cdots, \phi_n$ in $L^\infty(X; \mu)$ such that each A_k is unitary equivalent to the multiplication by ϕ_k on $L^2(X; \mu)$,

$k = 1, \cdots, n$. That is,

$$A_k f = \phi_k f \quad \text{for all} \quad f \in L^2(X; \mu) , \quad k = 1, \cdots, n .$$

And the joint spectrum of A is the *joint essential range* of $\phi = (\phi_1, \cdots, \phi_n)$, that is, the set of all the points $z = (z_1, \cdots, z_n)$ in $\boldsymbol{C}^n$ such that for every $\varepsilon > 0$

$$\mu\left(\left\{t \in X: \sum_{k=1}^{n} |\phi_k(t) - z_k| < \varepsilon\right\}\right) > 0 .$$

And the joint spectrum of A is equal to the joint approximate point spectrum of A, i.e., $\sigma(A) = \sigma_\pi(A)$.

Secondly, suppose that $\phi = (\phi_1, \cdots, \phi_n)$ is an n-tuple of bounded measurable function on the unit circle and $L_\phi = (L_{\phi_1}, \cdots, L_{\phi_n})$ and $T_\phi = (T_{\phi_1}, \cdots, T_{\phi_n})$ are the n-tuples of the Laurent operators on L^2 and Toeplitz operators on H^2 respectively induced by ϕ. That is, for each $k = 1, \cdots, n$,

$$L_{\phi_k} f = \phi_k f \text{ for all } f \in L^2, \text{ and } T_{\phi_k} f = PL_{\phi_k} f \text{ for all } f \in H^2 ,$$

where P denotes the projection from L^2 onto H^2. Then the joint spectrum of L_ϕ is a subset of the joint approximate point spectrum of T_ϕ, i.e., $\sigma(L_\phi) \subset \sigma_\pi(T_\phi)$. If, furthermore, all T_{ϕ_k}, $k = 1, \cdots, n$, are analytic, i.e., all ϕ_k belong to H^∞, then the joint spectrum of T_ϕ is the closure of the joint residual spectrum of it. (Consult Dash [5].)

PROPOSITION 1. *If $A = (A_1, \cdots, A_n)$ is a commuting n-tuple of normal operators, then $\|A\| = w(A) = r(A)$, and so A is joint normaloid.*

Proof. Since $\sigma(A) = \sigma_\pi(A)$, it follows that $w(A) \geqq r(A)$. On the other hand, it follows that

$$\|A\|^2 = \sup\left\{\int \sum_{k=1}^{n} |\phi_k(t)|^2 |f(t)|^2 (d\mu)(t): \|f\| = 1\right\}$$

and

$$r(A)^2 = \sup\left\{\sum_{k=1}^{n} |z_k|^2 : \mu\left(\left\{t \in X: \sum_{k=1}^{n} |\phi_k(t) - z_k| < \varepsilon\right\}\right) > 0 \text{ for any } \varepsilon > 0\right\} ,$$

from the definition and the above Dash's results. Since

$$\mu\left(\left\{t \in X: \sum_{k=1}^{n} |\phi_k(t)|^2 > r(A)^2\right\}\right) = 0 ,$$

it follows that

$$\int \sum_{k=1}^{n} |\phi_k(t)|^2 |f(t)|^2 (d\mu)(t) \leqq r(A)^2 \cdot \int |f|^2 d\mu \leqq r(A)^2 \|f\|^2 .$$

Hence $\|A\| \leqq r(A)$. So, the proof is complete.

PROPOSITION 2. *If $T_\phi = (T_{\phi_1}, \cdots, T_{\phi_n})$ is an n-tuple of Toeplitz operators, then $\|T_\phi\| = w(T_\phi) = r_\pi(T_\phi)$, and so T_ϕ is joint normaloid. Moreover, if T_ϕ is an n-tuple of analytic Toeplitz operators, then $\|T_\phi\| = w(T_\phi) = r(T_\phi)$.*

Proof. It holds that $\|T_\phi\| \leqq \|L_\phi\| \leqq r(L_\phi) \leqq r_\pi(T_\phi) \leqq w(T_\phi) \leqq \|T_\phi\|$. So, the first half is proved. Moreover, if $T_{\phi_1}, \cdots, T_{\phi_n}$ are all analytic, then $\sigma(T_\phi) = \overline{\sigma_r(T_\phi)} \subset \overline{W(T_\phi)}$. Hence

$$r_\pi(T_\phi) \leqq r(T_\phi) \leqq w(T_\phi) ,$$

and so the latter half is proved, too.

3. The bare points and extreme points of the joint normaloid operator-family.

DEFINITION 4. Let K be a bounded and connected set in C^n. The point α of K will be called an *extreme point* of K if no line segment joining any two points of K other than α contains α. And the point β of K will be called a *bare point* of K if there exists a spherical surface through β such that no points of K lie outside this spherical surface.

The set of the bare points of K is included in the set of extreme points of K and dense in it (cf. Berberian [2], p. 181).

THEOREM 2. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of operators such that $A - z = (A_1 - z_1, \cdots, A_n - z_n)$ is joint normaloid for every point $z = (z_1, \cdots, z_n)$ in C^n. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an extreme point of $\overline{W(A)}$, then α belongs to the joint approximate point spectrum $\sigma_\pi(A)$ of A. If, moreover, α is a bare point of $W(A)$, then α is a joint eigenvalue of A.*

Proof. Observing the joint approximate point spectrum is closed, it is sufficient for the proof to show that $\alpha \in \sigma_\pi(A)$ if α is a bare point of $\overline{W(A)}$. So, now, let α be a bare point of $\overline{W(A)}$. Then there exists a spherical surface S with the central point $z = (z_1, \cdots, z_n)$ such that no points of $\overline{W(A)}$ lie outside S and α is on S. Thus,

$$\left(\sum_{k=1}^{n} |\alpha_k - z_k|^2\right)^{1/2} = w(A - z) = \|A - z\|$$

and there exists a sequence $\{x_i\}$ of unit vectors such that

$$((A_k - z_k)x_i, x_i) \longrightarrow \alpha_k - z_k (i \longrightarrow \infty) , \quad k = 1, \cdots, n .$$

Consequently,

$$\begin{aligned}\sum_{k=1}^{n} \|(A_k - \alpha_k)x_i\|^2 &= \sum_{k=1}^{n} \|(A_k - z_k)x_i - (\alpha_k - z_k)x_i\|^2 \\ &= \sum_{k=1}^{n} \|(A_k - z_k)x_i\|^2 - 2\cdot\mathrm{Re}\sum_{k=1}^{n} (\overline{\alpha_k} - \overline{z_k})((A_k - z_k)x_i, x_i) \\ &\quad + \sum_{k=1}^{n} |\alpha_k - z_k|^2 \longrightarrow 0 \quad (i \longrightarrow \infty) .\end{aligned}$$

Hence $\alpha \in \sigma_\pi(A)$. The latter half is also proved in the same way as the first half.

REMARK. In the case of a single operator, if $A - z$ is normaloid for every complex number z, Hildebrandt [7] said A to belong to operator-class C_3.

COROLLARY 1. *For any n-tuple $A = (A_1, \cdots, A_n)$ of operators, $w(A) = \|A\|$ if and only if $r_\pi(A) = \|A\|$.*

COROLLARY 2. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of commuting normal operators. If α is an extreme point of $\overline{W(A)}$, then $\alpha \in \sigma_\pi(A)$. If α is a bare point of $W(A)$, then α is a joint eigenvalue of A.*

COROLLARY 3. *Let $A = (A_1, \cdots, A_n)$ be an n-tuple of Toeplitz operators. If α is an extreme point of $\overline{W(A)}$, then $\alpha \in \sigma_\pi(A)$.*

Dash [4] has shown that $W(A)$ is convex, if $A = (A_1, \cdots, A_n)$ is a commuting n-tuple of normal operators or an n-tuple of Toeplitz operators (see Thm. 2.5 and Thm. 2.6 in [4]). Now, we recall that $\sigma(A) = \sigma_\pi(A) \subset \overline{W(A)}$ if A is a commuting n-tuple of normal operators. And if T_ϕ is an n-tuple of analytic Toeplitz operators, then $\sigma(T_\phi) = \overline{\sigma_r(T_\phi)} \subset \overline{W(T_\phi)}$. Consequently, we get the followings.

COROLLARY 4 ([4], Thm. 2.8). *Let $A = (A_1, \cdots, A_n)$ be a commuting n-tuple of normal operators. Then we have*

$$\sum(\sigma(A)) = \overline{W(A)} ,$$

where $\sum(\sigma(A))$ denotes the convex hull of $\sigma(A)$.

COROLLARY 5 ([4], Thm. 2.10). *Let $T_\phi = (T_{\phi_1}, \cdots, T_{\phi_n})$ be an* *n-tuple of Toeplitz operators. Then we have*

$$\sum(\sigma_\pi(T_\phi)) = \overline{W(T_\phi)}\ .$$

COROLLARY 6 ([4], Cor. 2.11). *Let $T_\phi = (T_{\phi_1}, \cdots, T_{\phi_n})$ be an n-tuple of analytic Toeplitz operators. Then we have*

$$\sum(\sigma(T_\phi)) = \overline{W(T_\phi)}\ .$$

In the case of single operators, Klein [9] has shown that the numerical range of a Toeplitz operator has no extreme points if it is nonconstant. Next, we shall generalize his result for the case of operator-families.

PROPOSITION 3. *Let $T_\phi = (T_{\phi_1}, \cdots, T_{\phi_n})$ be an n-tuple of Toeplitz operators. Unless the joint numerical range of T_ϕ consists of only one point, it has no extreme points and so it is an open set.*

Proof. Now, suppose that there exists an extreme point in $W(T_\phi)$ and that $z = (z_1, \cdots, z_n)$ is its point. Then there exists an $n \times n$ unitary matrix $\boldsymbol{U}$:

$$\boldsymbol{U} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

such that the point $\alpha_{11}z_1 + \alpha_{12}z_2 + \cdots + \alpha_{1n}z_n$ in $\boldsymbol{C}$ is the extreme point of the numerical range of the operator $T \equiv \alpha_{11}T_{\phi_1} + \alpha_{12}T_{\phi_2} + \cdots + \alpha_{1n}T_{\phi_n}$ and T is nonconstant. So, since the operator T is also Toeplitz, it is impossible from Klein's results. Therefore, the joint numerical range of T_ϕ has no extreme points.

We would like to express our thanks to the referee for his useful comments.

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Received August 6, 1979 and in revised form March 12, 1980.

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