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SOME REMARKS ABOUT C^∞ VECTORS IN REPRESENTATIONS OF CONNECTED LOCALLY COMPACT GROUPS

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Given a continuous representation U of a connected locally compact group G in a quasi-complete locally convex topological vector space E, one may introduce the space E_∞ of C^∞-vectors which contains the dense space F_∞ of regular vectors. Natural questions are then: (1) does $F_\infty = E_\infty$ hold? (2) is the differential U_∞ of U a representation of the Lie algebra of G on E_∞? We here prove that answer to (1) is "yes" when G is a quotient of a direct product of compact connected Lie groups and E has a continuous norm, and that answer to (2) is always "yes". Of special interest are locally compact groups which are almost Lie in the sense that any subgroup algebraically generated by two continuous one-parameter subgroups is a Lie group in a finer connected topology. We prove that a connected locally compact group is almost Lie if and only if its universal covering in the sense of Lashof is $H \times A$ with H simply connected Lie group and A direct product of copies of R.

Let G be a connected locally compact group and $\{H_\alpha, \alpha \in I\}$ a directed decreasing family of normal compact subgroups of G such that

(1) $G_\alpha = G/H_\alpha$ is a Lie group for each $\alpha \in I$ (by a Lie group we shall always mean a finite dimensional real Lie group),

and

(2) $$\bigcap_{\alpha \in I} H_\alpha = \{e\}\,, \quad e \text{ identity of } G\,.$$

We shall identify G to the projective limit of the G_α's. Denote by $\mathfrak{G}$ the Lie algebra of G, which is the projective limit of the Lie algebras $\mathfrak{G}_\alpha$ of the Lie groups G_α. If $X = (X_\alpha) \in \mathfrak{G}$, $t \in \boldsymbol{R}$, denote by $\exp tX$ the element $(\exp tX_\alpha)$ of G. Let U be a continuous representation of G in a quasi-complete locally convex topological vector space E. For $\alpha \in I$, introduce $A_\alpha = \int_{H_\alpha} U(h)d\mu_\alpha(h)$, with μ_α normalized Haar measure of H_α. A_α is a continuous endomorphism of E ([2], Prop. 10(a), p. 17).

LEMMA 1.

(i) *For each $\alpha \in I$, A_α is a projector of E* (i.e., $A_\alpha^2 = A_\alpha$), *orthogonal if E is a Hilbert space and U unitary; its range is the* *closed subspace* $E_\alpha = \{a \in E;\ U(h)a = a \forall h \in H_\alpha\}$ *which is stable under* U.

(ii) $F = \bigcup_{\alpha \in I} E_\alpha$ *is a dense vector subspace of* E *which does not depend on the family* $\{H_\alpha, \alpha \in I\}$.

Proof. (i) $A_\alpha^2 = \int_{H_\alpha} \left(U(k) \circ \int_{H_\alpha} U(h) d\mu_\alpha(h)\right) d\mu_\alpha(k) = \int_{H_\alpha} \left(\int_{H_\alpha} U(kh) d\mu_\alpha(h)\right) d\mu_\alpha(k) = A_\alpha$. If E is a Hilbert space and U a unitary representation, $A_\alpha^* = \int_{H_\alpha} U(h^{-1}) d\mu_\alpha(h) = A_\alpha$. It is obvious that $E_\alpha = A_\alpha E$ and E_α is closed. If $a \in E_\alpha$, $x \in G$, $h \in H_\alpha$, $U(h)U(x)a = U(x)U(x^{-1}hx)a = U(x)a$, so E_α is U-stable.

(ii) Let $\varepsilon > 0$, $a_0 \in E$, q any continuous seminorm on E and $\alpha \in I$ such that $q(U(h)a_0 - a_0) < \varepsilon \forall h \in H_\alpha$. Then $q(A_\alpha a_0 - a_0) \leqq \int_{H_\alpha} q(U(h)a_0 - a_0) d\mu_\alpha(h) \leqq \varepsilon$, hence F is dense. The remaining assertion of (ii) follows from ([6], p. 45, Lemma 1).

COROLLARY (*R. Lipsman, C.C. Moore*). *If* U *is irreducible, or if* E *is a Hilbert space and* U *unitary factorial, there exists* $\alpha \in I$ *such that* $U(h) = 1\ \forall h \in H_\alpha$, i.e., $F = E$.

Proof. The irreducible case is clear. Now suppose E Hilbert and U unitary factorial, and choose $\alpha \in I$ such that $E_\alpha \neq \{0\}$. The restriction U_α of U to E_α is quasiequivalent to U, hence there exists cardinals m, n with $mU_\alpha \cong nU$, and the result follows.

This result is well-known ([11] Th. 2.1, Th. 3.1) ([14] Prop. 2.2), but the foregoing proof based on the projector A_α, though probably well-known, does not appear in the literature, to the author's knowledge.

DEFINITION 1. A vector $a \in E$ is said to be C^∞ for U if for every $X \in \mathfrak{G}$ the mapping $t \to U(\exp tX)a$ from $\boldsymbol{R}$ into E is C^∞. If E is a Hilbert space, $a \in E$ is said to be analytic for U if the above mapping is analytic.

In the sequel, when considering analytic vectors, we shall always implicitly assume that E is a Hilbert space.

Definition 1 generalizes the classical definition when G is a Lie group, by Goodman's theorem in the C^∞ case and by [8] in the analytic case. We shall denote by E_∞(resp. E_ω) the space of C^∞ (resp. analytic) vectors. Introduce the space $(E_\alpha)_\infty$ (resp. $(E_\alpha)_\omega$) of C^∞ (resp. analytic) vectors of E_α for the representation of G_α defined by U, and define $F_\infty = \bigcup_{\alpha \in I}(E_\alpha)_\infty = F \cap E_\infty$ (resp. $F_\omega = \bigcup_{\alpha \in I}(E_\alpha)_\omega = F \cap E_\omega$).

F_∞ may be called the space of regular vectors for U; it is the Gårding domain introduced in [13], by ([7] Th. 3.3), if E is a Fréchet space.

For $X \in \mathfrak{G}$ and $a \in E_\infty$, define

$$U_\infty(X)a = \left[\frac{d}{dt}U(\exp t\,X)a\right]_{t=0} .$$

PROPOSITION 1. *F_∞ (resp. F_ω) is dense in E and U_∞ is a representation of the Lie algebra $\mathfrak{G}$ on F_∞ (resp. F_ω). If E is a Hilbert space and U unitary, $U_\infty(X)$ is essentially skew-adjoint on F_∞ and F_ω for every $X \in \mathfrak{G}$, and its closure is the generator of the 1-parameter group $t \to U(\exp t\,X)$.*

This proposition is straightforward since F is dense in E and stable under $U(\exp t\,X)$ $t \in \boldsymbol{R}$, $X \in \mathfrak{G}$.

Now, E_∞ and E_ω are U-stable, since for $a \in E_\infty$ (resp. E_ω), $X = (X_\alpha) \in \mathfrak{G}$, $x = (x_\alpha) \in G$, $U(\exp t\,X)U(x)a = U(x)U(\exp t\,Ad_G(x^{-1})X)a$, where $Ad_G(x^{-1})X = (Ad_{G_\alpha}(x_\alpha^{-1})X_\alpha) \in \mathfrak{G}$. Hence, if E is a Hilbert space and U unitary, $U_\infty(X)$ is essentially skew-adjoint on E_∞ and E_ω for each $X \in \mathfrak{G}$. Moreover, the continuous mapping $(s, t) \to U(\exp s\,X \exp t\,Y)a$, $X, Y \in \mathfrak{G}$, is separately C^∞ (resp. analytic) from $\boldsymbol{R}^2$ into E.

LEMMA 2. *Let $a \in E_\infty$, $X, Y \in \mathfrak{G}$. Then mapping*

$$(s, t) \longrightarrow U(\exp s\,X \exp t\,Y)a$$

from $\boldsymbol{R}^2$ into E is differentiable, its differential at $(s_0, t_0) \in \boldsymbol{R}^2$ being $(s, t) \to s\,U_\infty(X)U(\exp s_0\,X \exp t_0\,Y)a + t\,U(\exp s_0 X \exp t_0\,Y)U_\infty(Y)a$.

This lemma follows at once from the equality

$U(\exp(s_0 + s)X \exp(t_0 + t)Y)a - U(\exp s_0 X \exp t_0 Y)a - s\,U_\infty(X)U(\exp s_0 X \exp t_0 Y)a - t\,U(\exp s_0 X \exp t_0 Y)U_\infty(Y)a = U(\exp(s_0 + s)X \exp t_0 Y)(U(\exp t\,Y)a - a - t\,U_\infty(Y)a) + t\,U(\exp s_0 X)(U(\exp s\,X \exp t_0\,Y)U_\infty(Y)a - U(\exp t_0\,Y)U_\infty(Y)a) + U(\exp(s_0 + s)X \exp t_0\,Y)a - U(\exp s_0 X \exp t_0\,Y)a - s\,U_\infty(X)U(\exp s_0 X \exp t_0 Y)a$ and ([16] p. 220, (c)).

PROPOSITION 2. *Let $a \in E_\infty$. The mapping $X \to U_\infty(X)a$ from $\mathfrak{G}$ into E is continuous linear.*

Proof. The linearity is proven exactly as in ([1], p. 226, Lemma 2.2) using Lemma 2. Now, the Lie algebra $\mathfrak{G}$ is a direct product of finite dimensional Lie algebras [10], hence a Baire space ([4], p.

114 Ex: 16(a)). This implies that the linear mapping $X \mapsto U_\infty(X)a$ which is the point wise limit of the continuous mapping $X \mapsto n(U(\exp(1/n)X)a - a)$, $n \to +\infty$, is continuous ([4], p. 115, Ex. 20(b)).

DEFINITION 2 ([15]). A vector $a \in E$ is said to be weakly regular for the representation U if for any continuous linear form a' on E the mapping $x \to \langle U(x)a, a' \rangle$ is regular in the sense of [6].

Denote by R the space of weakly regular vectors. Then $F_\infty \subset R \subset E_\infty$ and U_∞ is a representation of $\mathfrak{G}$ on R.

PROPOSITION 3. *Suppose G is any quotient of a direct product of compact connected Lie groups. Then*: (i) $E_\infty = R$; (ii) *if there exists a continuous norm on E, $F_\infty = E_\infty$.*

Proof. Let $a \in E_\infty$. Suppose first $G = \prod_{i \in J} G_i$, G_i compact connected Lie group. Then $\mathfrak{G}$ is the direct product of the Lie algebras $\mathfrak{G}_i$ of the Lie groups G_i. For $X = (X_i) \in \mathfrak{G}$ and J_0 finite subset of J, define

$$X_{J_0} = (Y_i) \in \mathfrak{G},\ Y_i = \begin{cases} X_i & i \in J_0 \\ 0 & i \notin J_0 \,. \end{cases}$$

We have $U_\infty(X_{J_0})a = [d/dt\, \tilde{a}(\exp tX_{J_0})]_{t=0}$ where $\tilde{a}$ denotes the mapping $x \to U(x)a$ from G into E. Choose for each $i \in J$ a coordinate system of the first kind for G_i in a neighborhood of the identity e_i of G_i, and denote by $D_i\tilde{a}$ the differential of the restriction of $\tilde{a}$ to G_i evaluated at e_i. Then $U_\infty(X_{J_0})a = \sum_{i \in J_0} (D_i\tilde{a})(X_i)$. Now, by Proposition 2, $U_\infty(X)a = \lim_{J_0 \in \mathscr{F}} U_\infty(X_{J_0})a$, $\mathscr{F}$ denoting finite subsets of J and lim being in the obvious sense hence the family $((D_i\tilde{a})(X_i))_{i \in J}$ is summable in E, and $\sum_{i \in J}(D_i\tilde{a})(X_i) = U_\infty(X)a$. As $X \in \mathfrak{G}$ is arbitrary, if there exists a continuous norm on E, then there exists a finite subset J_0 of J such that $D_i\tilde{a} = 0\ \forall i \notin J_0$, i.e., $\tilde{a} = \tilde{a} \circ p_{J_0}$ where $p_{J_0}\colon G \to \prod_{i \in J_0} G_i$ is the projection. This proves that $a \in F$. Suppose now that $G = \tilde{G}/H$, with $\tilde{G} = \prod_{i \in J} G_i$ as above and H normal closed subgroup of G, and let $\pi\colon \tilde{G} \to G$ denote the projection. Choose a directed decreasing family of compact normal subgroups $\{H_\alpha; \alpha \in I\}$ of $\tilde{G}$ with properties (1) and (2). Then the family $\{\pi(H_\alpha), \alpha \in I\}$ of compact normal subgroups of G has the same properties. The vector a is C^∞ for the representation $\tilde{U} = U \circ \pi$ of $\tilde{G}$, so there exists $\alpha \in I$ such that $U(\pi(h))a = a\ \forall h \in H_\alpha$, which implies $a \in F$. This proves (ii). (i) follows easily by putting in the above reasoning $a' \circ \tilde{a}$ in place of $\tilde{a}$, with a' any continuous linear form on E.

EXAMPLE. Let $G = \prod_{n=1}^{\infty} G_n$, G_n compact connected Lie group, and U the left regular representation on $L^2(G)$. Then F (resp. E_∞) consists of those $f \in L^2(G)$ such that there exists $n \in N$ and $f_n \in L^2(G_1 \times \cdots \times G_n)$ (resp. $C^\infty(G_1 \times \cdots \times G_n)$) with $f = f_n \circ p_n$ where $p_n: G \to G_1 \times \cdots \times G_n$ is the projection.

PROPOSITION 4. *If G is compact, F contains the set of G-finite vectors.*

Proof. Let $\hat{G}$ the unitary dual of G and $P_\lambda = \int_G \overline{\pi_\lambda(x)} U(x) d\mu(x)$, $\lambda \in \hat{G}$, $\pi_\lambda = \dim(\lambda)\chi_\lambda$, χ_λ character of λ, μ normalized Haar measure of G. P_λ is a projector of E, $E^\lambda = P_\lambda(E)$ is the λ-type isotypic component of E, and the set of G-finite vectors of E is the algebraic direct sum $\sum_{\lambda \in \hat{G}} E^\lambda$. For $\lambda \in \hat{G}$, there exists, by corollary to Lemma 1, $\alpha \in I$ such that $\pi_\lambda(hx) = \pi_\lambda(x) \forall x \in G \forall h \in H_\alpha$; then $U(h)P_\lambda = P_\lambda \forall h \in H_\alpha$ which proves the result.

Any locally compact connected group being locally isomorphic to $H \times K$ with H connected Lie group and K compact connected group, one still has $E_\infty = F_\infty$ when K is a quotient of a direct product of compact connected Lie groups and E has a continuous norm. For other groups like the p-adic solenoïd $\Sigma_p = (\boldsymbol{R} \times \boldsymbol{Z}_p)/B$ (p a prime), where $\boldsymbol{Z}_p$ denotes the additive group of p-adic integers and $B = \{(n, n) \in \boldsymbol{R} \times \boldsymbol{Z}_p, n \in \boldsymbol{Z}\}$, the question of whether or not $E_\infty = F_\infty$ is here left open. We shall see that for general locally compact connected G, U_∞ is a representation of the Lie algebra $\mathfrak{G}$ on E_∞ and E_ω.

DEFINITION 3. A topological group G is said to be almost Lie if the following condition is satisfied: for any two continuous one-parameter subgroups θ_1, θ_2 of G, there exists on the subgroup $G(\theta_1, \theta_2)$ of G algebraically generated by $\theta_1(\boldsymbol{R}) \cup \theta_2(\boldsymbol{R})$ a finer connected topology for which $G(\theta_1, \theta_2)$ is a Lie group.

Such a topology a $G(\theta_1, \theta_2)$ is unique if it exists, and has the same continuous one-parameter subgroups as the topology induced by G. ([12], Lemma 2).

EXAMPLES.

(1) Any connected nilpotent topological group is almost Lie [12].

(2) Any Lie group is almost Lie ([3], p. 177, Prop. 9).

LEMMA 3.

(i) *The direct product of two almost Lie topological groups is almost Lie.*

(ii) *Any quotient of an almost Lie topological group by a locally compact normal subgroup is almost Lie.*

Proof.

(i) results from Example (2), and (ii) follows from ([9], Lemma 1).

If G is connected locally compact almost Lie, then for $a \in E_\infty$ (resp. E_ω), X, $Y \in \mathfrak{G}$, the mapping $(s, t) \to U(\exp sX \exp tY)a$ is jointly C^∞ (resp. analytic) and U_∞ is a representation of the Lie algebra $\mathfrak{G}$ of G on E_∞ (resp. E_ω).

PROPOSITION 5. *Let G be any connected locally compact group. U_∞ is a strongly continuous representation of the Lie algebra $\mathfrak{G}$ of G on E_∞, and on E_ω, by essentially skew-adjoint operators if E is a Hilbert space and U unitary.*

Proof. We already noted that G is locally isomorphic to $H \times K$, with H connected Lie group and K compact connected group. Now, K is isomorphic to $(P \times A)/H$, with P direct product of compact connected Lie groups, A abelian compact connected group and H closed normal subgroup of $P \times A$. Proposition 5 then follows from Proposition 3 and from the fact that A is almost Lie, observing that, for a representation or local representation of a product of two locally compact groups, a vector is C^∞ (resp. analytic) if and only if it is separately C^∞ (resp. analytic) on each factor (this results of [17], p. 186, Ex. 92 in the C^∞ case and of [5] in the analytic case).

We now turn to the characterization of locally compact connected almost Lie groups.

Let $\mathfrak{G}$ a Lie algebra and X, $Y \in \mathfrak{G}$. A formal commutator of X and Y is any expression $Z = [X_1, [X_2, [\cdots[X_{n-1}, X_n]\cdots]$ where $X_i = X$ or Y $\forall i = 1, 2, \cdots, n$. $n = \deg Z$ is the degree of the formal commutator Z. Any formal commutator of X and Y defines an element of $\mathfrak{G}$ which we shall again denote by the same letter Z; the formal commutator Z will be said to be $\neq 0$ if the corresponding element $Z \in \mathfrak{G}$ is $\neq 0$. The set of all formal commutators of X and Y is clearly ordered as follows:

$$X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]], [X, [X, [X, Y]]],$$
$$[Y[X, [X, Y]]], [X, [Y, [X, Y]]], [Y, [Y, [X, Y]]], \cdots$$

LEMMA 4. *Let $\mathfrak{G}$ be a finite dimensional real Lie algebra and $\mathfrak{G}_\infty = \prod_{n=1}^\infty \mathfrak{G}_n$, where $\mathfrak{G}_n = \mathfrak{G} \forall n \geq 1$. Fix X, $Y \in \mathfrak{G}$ and define X^0, $Y^0 \in \mathfrak{G}_\infty$ by $X^0 = (nX)_{n \geq 1}$, $Y^0 = (nY)_{n \geq 1}$. If there is an infinite sequence of formal commutators of X and Y which are $\neq 0$, then the Lie subalgebra of $\mathfrak{G}_\infty$ generated by X^0 and Y^0 is infinite dimen-*

sional.

Proof. We may suppose that $\mathfrak{G}$ is generated as a Lie algebra by $\{X, Y\}$. Denote by $Z_1 = X, Z_2 = Y, Z_3, \cdots, Z_p (p = \dim \mathfrak{G})$ the p first (with respect to the order) formal commutators of X and Y with the property that the family $\{Z_1, \cdots, Z_p\}$ of the corresponding elements of $\mathfrak{G}$ is linearly free. For each $q \in \boldsymbol{N}$, choose $\neq 0$ formal commutators $W_h (h = 1, 2, \cdots, q)$, such that $\deg W_h = \deg Z_p + h$. Denote by Z_i^0, W_h^0 $1 \leqq i \leqq p, 1 \leqq h \leqq q$ the formal commutators of X^0 and Y^0 in $\mathfrak{G}_\infty$ analogous to Z_i, W_h formed with X^0 and Y^0 in place of X and Y. We have $Z_i^0 = (n^{\deg Z_i} Z_i)_{n \geqq 1}$ $1 \leqq i \leqq p$, $W_h^0 = (n^{\deg W_h} W_h)_{n \geqq 1} 1 \leqq h \leqq q$. Let $\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_q \in \boldsymbol{R}$ such that $\sum_{i=1}^p \lambda_i Z_i^0 + \sum_{h=1}^q \mu_h W_h^0 = 0$ in the Lie algebra $\mathfrak{G}_\infty$. If we expand W_h in $\mathfrak{G}$ as $W_h = \sum_{i=1}^p \xi_h^i Z_i (\xi_h^i \in \boldsymbol{R})$, we get $\lambda_i = -\sum_{h=1}^q \xi_h^i \mu_h$, and

$$(*) \qquad \sum_{h=1}^q \mu_h \xi_h^i (n^{\deg W_h} - n^{\deg Z_i}) = 0 \quad \forall i = 1, 2, \cdots, p, \forall n > 1 .$$

Fix i $1 \leqq i \leqq p$. The linear system with q equations and the q unknowns $\mu_h \xi_h^i$ $1 \leqq h \leqq q$ obtained by writing down (*) for $n = 2, 2^2, \cdots, 2^q$ has determinant

$$\Delta_{i,q} = 2^{\deg Z_i q(q+1)/2} \begin{vmatrix} 2^{k_i+1} - 1 & \cdots & 2^{k_i+q} - 1 \\ \vdots & & \vdots \\ 2^{q(k_i+1)} - 1 & \cdots & 2^{q(k_i+q)} - 1 \end{vmatrix}$$

with $k_i = \deg Z_p - \deg Z_i \geqq 0$.

Clearly $\Delta_{i,q} \sim 2^{(\deg Z_i + k_i + 1) q(q+1)/2} \prod_{1 \leqq l \leqq j \leqq q} (2^j - 2^l)$, $q \to +\infty$, so that for large q, $\mu_h \xi_h^i = 0$ $\forall h = 1, 2 \cdots q$, $\forall i = 1, 2 \cdots p$. Then $\lambda_1 = \cdots = \lambda_p = \mu_1 = \cdots = \mu_q = 0$ for large q, i.e., the elements $Z_1^0, \cdots, Z_p^0, W_1^0, \cdots, W_q^0$ are linearly independant in $\mathfrak{G}_\infty$. This completes the proof.

REMARK. There exists $X, Y \in \mathfrak{G}$ satisfying the hypothesis of Lemma 4 if and only if the finite dimensional real Lie algebra $\mathfrak{G}$ is not nilpotent.

LEMMA 5. *The Lie subalgebra of* $\mathfrak{so}(n)$ $n > 2$ *generated by*

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & & & \\ 0 & -1 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & -1 & 0 \end{pmatrix} \textit{ and } \quad Y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

has dimension $\geqq n$.

Proof. It is enough to note that for $1 \leqq p \leqq n - 2$.

$$(adX)^p Y = \begin{pmatrix} \boxed{\begin{matrix} & \\ & A_p & \\ & \end{matrix}} & \begin{matrix} \varepsilon_p & 0 \cdots\cdots 0 \\ 0 & 0 \cdots\cdots 0 \\ \vdots & \vdots \qquad \vdots \end{matrix} \\ \begin{matrix} -\varepsilon_p & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \cdots\cdots 0 \\ 0 & 0 \cdots\cdots 0 \\ \vdots & \vdots \qquad \vdots \\ 0 & 0 \cdots\cdots 0 \end{matrix} \end{pmatrix}$$

where $A_p \in \mathfrak{so}(p+1)$, $\varepsilon_p = \pm 1$.

LEMMA 6. *In the Lie algebra $\mathfrak{sp}(n)$ $n > 2$, let*

$$X = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline -X_2 & X_1 \end{array}\right) \quad \textit{with} \quad X_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 1 & 0 \cdots & 0 & 0 \end{pmatrix}$$

$$Y = \left(\begin{array}{c|c} Y_1 & 0 \\ \hline 0 & Y_1 \end{array}\right) \quad \textit{with} \quad Y_1 = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ -1 & 0 & 0 \cdots & 0 \\ 0 & 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 0 \end{pmatrix}.$$

The Lie subalgebra of $\mathfrak{sp}(n)$ generated by X and Y has dimension $\geqq [n/2] + 1$, where $[\cdot]$ denotes entire part.

Proof. It is enough to note that for $2p \leqq n - 1$

$$(adX)^p Y = \left(\begin{array}{c|c} Z_{p,1} & Z_{p,2} \\ \hline -Z_{p,2} & Z_{p,1} \end{array}\right)$$

$$\text{where} \quad Z_{p,1} = \begin{pmatrix} \overbrace{* \ * \ \cdots \ *}^{p+1} & \varepsilon_p & 0 \cdots 0 & \overbrace{* \ * \ \cdots \ *}^{p-1} \\ * \ * \ \cdots \ * & 0 & 0 \cdots 0 & * \ * \ \cdots \ * \\ \vdots \ \vdots \qquad \vdots & \vdots & \vdots & \vdots \ \vdots \qquad \vdots \\ * \ * \ \cdots \ * & 0 & 0 \cdots 0 & * \ * \ \cdots \ * \end{pmatrix}$$

$$Z_{p,2} = \begin{pmatrix} \overbrace{* \ * \ \cdots \ *}^{p} & 0 & 0 \cdots 0 & \overbrace{* \ * \ \cdots \ *}^{p} \\ * \ * \ \cdots \ * & 0 & 0 \cdots 0 & * \ * \ \cdots \ * \\ \vdots \ \vdots \qquad \vdots & \vdots & \vdots & \vdots \ \vdots \qquad \vdots \\ * \ * \ \cdots \ * & 0 & 0 \cdots 0 & * \ * \ \cdots \ * \\ * \ * \ \cdots \ * & \alpha_p & 0 \cdots 0 & * \ * \ \cdots \ * \end{pmatrix}$$

and $\varepsilon_p = \pm 1$, $\alpha_p \in \boldsymbol{R}$.

Recall now that the universal covering group of a connected locally compact group defined in [10] is a LP-group which is in general not locally compact.

PROPOSITION 6. *A connected locally compact group is almost Lie if and only if its universal covering group is $H \times A$ with H simply connected Lie group and A direct product of copies of the additive group $\boldsymbol{R}$.*

Proof. Let G be a locally compact connected group. Its Lie algebra $\mathfrak{G}$ has the form $\mathfrak{H} \times \mathfrak{A} \times \mathfrak{J}$, with $\mathfrak{H}$ finite dimensional Lie algebra, $\mathfrak{A}$ direct product of 1-dimensional Lie algebras and $\mathfrak{J} = \prod_{i \in J} \mathfrak{G}_i$ direct product of simple compact finite dimensional Lie algebras. Each $\mathfrak{G}_i$ is isomorphic to one of the classical types $A_l = \mathfrak{su}(l+1)$ $l \geqq 1$, $B_l = \mathfrak{so}(2l+1)$ $l \geqq 2$, $C_l = \mathfrak{sp}(l)$ $l \geqq 3$, $D_l = \mathfrak{so}(2l)$ $l \geqq 4$ or one of the exceptional E_6, E_7, E_8, F_4, G_2. Suppose G is almost Lie. From Lemma 4, there is but a finite number of indices $i \in J$, with $\mathfrak{G}_i$ isomorphic to an exceptional type, and for fixed l, there is only a finite number of indices $i \in J$ with $\mathfrak{G}_i$ isomorphic to A_l, B_l, C_l or D_l. From Lemma 5 the set of l's for which there exists a $\mathfrak{G}_i$ of the type A_l, B_l, or D_l is finite, and from Lemma 6 the set of l's for which there exist a $\mathfrak{G}_i$ of the type C_l is finite. Hence $\mathfrak{J}$ is finite dimensional, and the result follows.

Suppose now that the universal covering group of G is $\tilde{G} = H \times A$, with H simply connected Lie group and A direct product of copies of the additive group $\boldsymbol{R}$. The canonical continuous homomorphism $w\colon \tilde{G} \to G$ is in general neither open nor onto, but its kernel D is a central totally disconnected subgroup and its range G_0 is the dense subgroup of G algebraically generated by the set $\{\exp tX,\ X \in \mathfrak{G},\ t \in \boldsymbol{R}\}$. By Lemma 3 (i), $\tilde{G}$ is an almost Lie topological group. Let $\theta_i(t) = \exp tX_i (i = 1, 2)$, $X_i \in \mathfrak{G}$, be continuous one-parameter subgroups of G and $\tilde{\theta}_i$ continuous one-parameter subgroups of $\tilde{G}$ such that $w(\tilde{\theta}_i(t)) = \theta_i(t) \forall t \in \boldsymbol{R}$. Denote by H (resp. $\tilde{H}$) the subgroup of G (resp. $\tilde{G}$) algebraically generated by $\theta_1(\boldsymbol{R}) \cup \theta_2(\boldsymbol{R})$ (resp. $\tilde{\theta}_1(\boldsymbol{R}) \cup \tilde{\theta}_2(\boldsymbol{R})$). Then $H = w(\tilde{H})$ and the map $w^*\colon \tilde{H}/_{\tilde{H} \cap D} \to H$ induced by w is an algebraic isomorphism which is continuous when $\tilde{H}$ is equipped with its Lie group structure. Hence G is almost Lie.

EXAMPLE. The compact group $G = \prod_{n=1}^{\infty} G_n$ where $G_n = \mathrm{SU}(2)$ $\forall n \geqq 1$ is not almost Lie. In particular, G is not of the form $(L \times M)/H$, with L compact connected Lie group, M compact connected abelian group and H normal closed subgroup of $L \times M$.

References

1. H. Boseck and C. Czichowski, *Grundfunktionen und verallgemeinerte Funktionen auf topologischen Gruppen I*, Math. Nachr., **58** (1973), 215-240.
2. N. Bourbaki, *Intégration*, Chap. VI, Hermann, Paris, 1968.
3. ———, *Groupes et algèbres de Lie*, Chap. 2 et 3, Hermann, Paris, 1972.
4. ———, *Topologie Générale*, Chapitres 5 à 10, Hermann, Paris, 1974.
5. F. E. Browder, *Real analytic functions on product spaces and separate analyticity*, Canad. J. Math., **13** (1961), 650-656.
6. F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques*, Bull. Soc. Math. France, **89** (1961), fasc. 1, 43-75.
7. J. Dixmier and P. Malliavin, *Factorisation de fonctions et de vecteurs indéfiniment différentiables*, Bull. Soc. Math., **102** (1978), 305-330.
8. M. Flato and J. Simon, *Separate and joint analyticity in Lie groups representations*, J. Functional Analysis, **13** (1973), 268-276.
9. W. M. Gluschkow, *Lie algebras of locally compact groups*, Usp. Mat. Nauk XII, **74** (1957), 137-142 (Russian).
10. R. K. Lashof, *Lie algebras of locally compact groups*, Pacific J. Math., **7** (1957), 1145-1162.
11. R. Lipsman, *Representation theory of almost connected groups*, Pacific J. Math., **42** (1972), 453-467.
12. L. Magnin and J. Simon, *Lie algebras associated with topological nilpotent groups*, Rep. on Math. Physics, **8** (1975), 171-180.
13. ———, *Vecteurs analytiques dans les représentations unitaires des groupes localement compacts connexes*, C. R. Acad. Sc. Paris, série A, **289** (1979), 715-718.
14. C. C. Moore, *Groups with finite dimensional irreducible representations*, Trans. Amer. Math. Soc., **166** (1972), 401-410.
15. J. Pichaud, 1-*cohomologie de representations induites*, J. Math. Pures et appl., **56** (1977), 339-366.
16. G. Warner, *Harmonic Analysis on Semi-simple Lie Groups I*, Springer Verlag, 1972.
17. C. Zuily, *Problèmes de distributions*, Hermann, Paris, 1978.

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