PACIFIC JOURNAL OF MATHEMATICS
Vol. 96, No. 1, 1981

SOME HOMOLOGY LENS SPACES WHICH BOUND RATIONAL HOMOLOGY BALLS

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A homology lens space is a smooth closed 3-manifold M^3 with $H_k(M^3)=H_k(L(p,1))$ for all k (p some nonnegative integer). When $p=1$ M^3 is a homology 3-sphere. It is an open question which of these homology lens spaces bound rational homology balls and of special interest which homology 3-spheres bound contractible manifolds. In this note we answer this question for certain Seifert fibre spaces, each with three exceptional fibres.

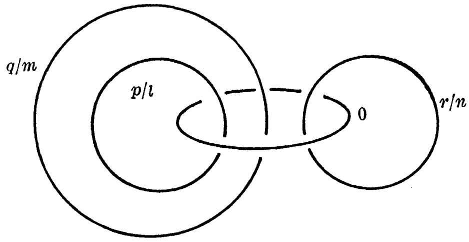
Let p, q, r and d be integers. Whenever l, m, and n can be defied by the relation $lqr + mpr + npq = d$, there is a well-defined orientable Seifert fibre space $M^3(p, q, r; d)$ with three exceptional fibres of type (p, l), (q, m) and (r, n). When $d = 1$ and p, q, r are coprime and positive, M^3 is the Brieskorn manifold $\Sigma(p, q, r)=\{(x, y, z) \in C^3\colon\ x^p + y^q + z^r = 0;\ x\bar{x} + y\bar{y} + z\bar{z} = 1\}$; and when $p = 1$ (so that the corresponding fibre is no longer exceptional) M^3 is a genuine lens space.

THEOREM. *Let $(p, q, r; d)$ be in one of the following six classes:*
(1) $(p, ps \pm k, ps \pm 2k; k^2)$ *for p odd*
(2) $(p, ps - k, ps + k;\ k^2)$ *for p even and s odd*
(3) $(p, q, s^2(p + q - pq) + s(2p - pq) + p; (s(p + q - pq) + p)^2)$
(4) $(p, st + s + 1, pt(s + 1) - (st + s + 1); (pst - (st + s + 1))^2)$
(5) $(p, p, 1 - s; p^2s^2)$
(6) $(p, p, 4s + 1; 4p^2s^2)$.
If M^3 is the associated Seifert fibre space, M^3 can be realized as the boundary of a Mazur-type manifold obtained by adding a 2-handle to $S^1 \times B^3$. In particular, the Brieskorn homology spheres which arise when $d = 1$ bound contractible 4-manifolds.

These Brieskorn classes include $\Sigma(2, 3, 13)$, $\Sigma(2, 5, 7)$, and $\Sigma(3, 4, 5)$ which are shown to bound Mazur manifolds in (1). Along the way we recover also the fact that the lens spaces $L(t^2, qt + 1)$, for q and t relatively prime, bound Mazur-type 4-manifolds of the kind mentioned above.

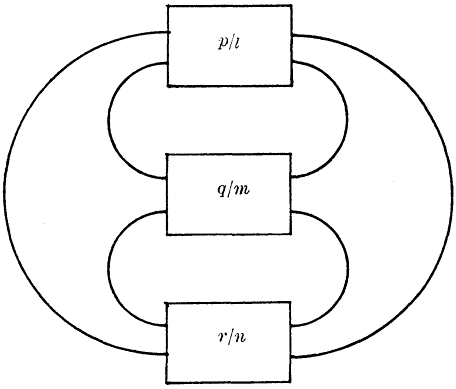
At present we have three methods of constructing these 4-manifolds. We will sketch two of them and prove the theorem with the third. We wish to thank A. Goalby and P. Melvin for helpful discussions.

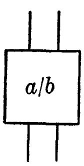
1. Construction of $M(p, q, r; d)$. Determine integers l, m, and n by the equation $lqr + mpr + npq = d$. The $M(p, q, r; d)$ has the following three descriptions:



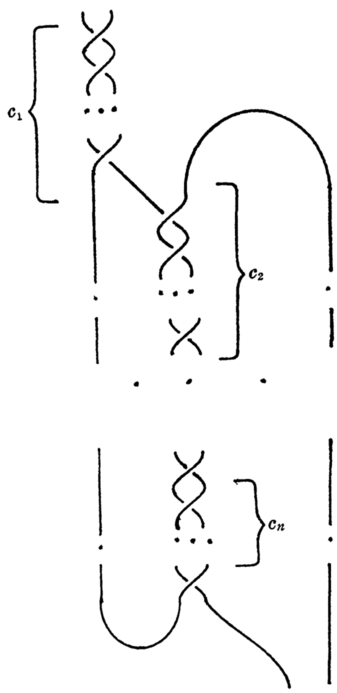
DESCRIPTION 1

$K(p, q, r; d) =$





denotes the rational tangle



(c_i denotes the number of half twists, right handed if $c_i > 0$, left if $c_i < 0$.)

DESCRIPTION 2

Description 1. Dehn surgery on S^3.

M can be obtained by removing tubular neighborhoods of the above circles in S^3 and reattaching them by a map of the boundary torus. The map, using p/l for example, is determined by the requirement that p times the meridian $+l$ times the longitude dies in π_1.

Description 2. Branched covers.

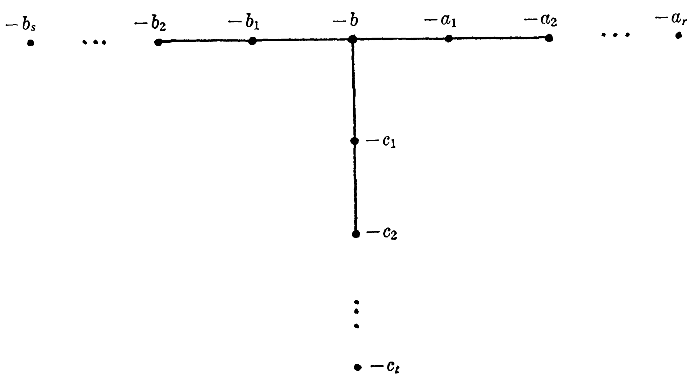
Let $K(p, q, r; d)$ be the link drawn (of either one or two components) where, $a/b = c_1 - 1/(c_2 - 1/(c_3 \cdots - 1/c_t) \cdots)$.
Then $M(p, q, r; d)$ is the two fold cover of S^3 branched along K.

Description 3. Plumbing of B^2 bundles over S^2.

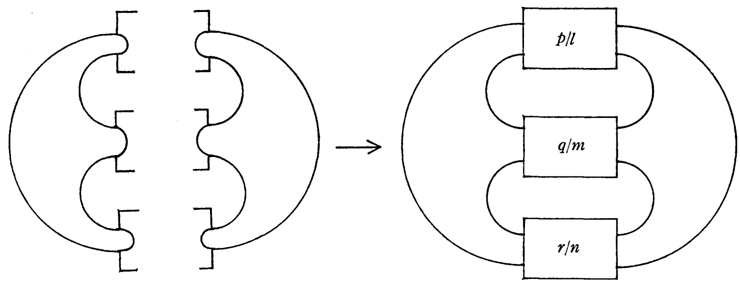
Find β_0, β_1, and β_2 by the equivalences:

$$\beta_0 \equiv l(p) \quad 1 \leqq \beta_0 < p$$
$$\beta_1 \equiv m(q) \quad 1 \leqq \beta_1 < q$$
$$\beta_2 \equiv n(r) \quad 1 \leqq \beta_2 < r.$$

Put $b = d/pqr + 1/p + 1/q + 1/r$ and
$p/\beta_0 = a_1 - 1/(a_2 - 1/(a_3 - \cdots - 1/a_r) \cdots)) = [a_1, \cdots, a_r]$,
$q/\beta_1 = [b_1, \cdots, b_s]$,
$r/\beta_2 = [c_1, \cdots, c_t]$.
(Note that if we put $\alpha_0 = p$, $\alpha_1 = q$ and $\alpha_2 = r$, we recover the notation of [5].) Then $M(p, q, r; d)$ is the boundary of the plumbing described by the diagram:



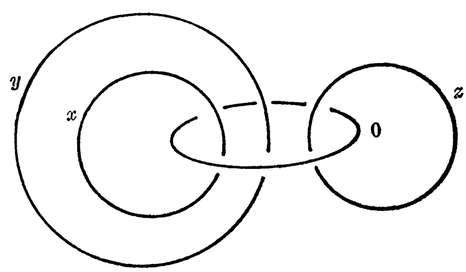
The first description is derived directly from the definition of $M(p, q, r; d)$. The S^1 action given by $\theta(x, y, z) = (\theta^{qr}x, \theta^{pr}y, \theta^{pq}z)$ turns the Brieskorn manifold $\Sigma(p, q, r) \subset C^3$ into the Seifert fibre space $M(p, q, r; 1)$ when p, q and r are co-prime. The second description is obtained from the first as follows: First note that the two-fold cover of S^3 bronched along the unlink of two circles is $S^1 \times S^2$. The link $K(p, q, r; d)$ is obtained from this unlink by introducing the three tangles:



In the cover the result is the three Dehn surgeries of description 1 (see (4)). Finally, when $d = 1$, the plumbing of description 3 is a resolution of the singularity at 0 of the local Brieskon variety $\{(x, y, z) \in C^3: x^p + y^q + z^r = 0, x\bar{x} + y\bar{y} + z\bar{z} \leqq 1\}$ (see e.g., [5]). For general d it can be derived from description 1 by a process which transforms rational surgeries into integral ones ([6], p. 262 for example or [3]).

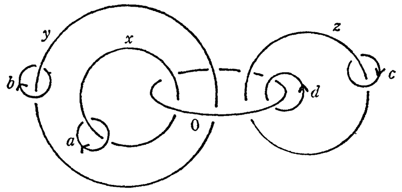
2. Proof of the theorem. Suppose we attach a 2-handle to $M^3 \times I$ along a circle in $M^3 \times \{1\}$. If the result has boundary the disjoint union of M^3 and $S^1 \times S^2$, we can then attach a 3-handle and a 4-handle to get a 4-manifold with boundary M^3. When turned upside down, this 4-manifold will have the handlebody structure predicted by the theorem. Since adding a 2-handle to a 4-manifold performs an integral surgery to its boundary, to prove the theorem we only need to show how such a surgery can change $M(p, q, r; d)$ into $S^1 \times S^2$ for the six classes mentioned.

Consider the 3-manifold obtained by performing these three rational surgeries to $S^1 \times S^2$.



LEMMA. *This manifold is homeomorphic to* $S^1 \times S^2$ *if and only if*
(1) $1/x + 1/y + 1/z = 0$ *and*
(2) *one of* $1/x, 1/y, 1/z$ *is an integer.*

Proof. Put $x = p/l$, $y = q/m$, $z = r/n$ with p, q, r, l, m, n all integers. Suppose none of the reciprocals in (2) is an integer. If we let a, b, c, and d be the generators of $\pi_1(M^3)$ sketched below, then we have relations:

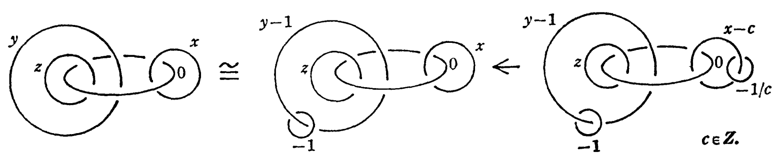


$$a^p = d^l$$
$$b^q = d^m$$
$$c^r = d^n$$
$$abc = 1$$
$$d \in \text{center}(\pi_1(M^3)).$$

Setting $d = 1$ gives a map from $\pi_1(M^3)$ to the triangle group $\{a, b, c\colon a^p = b^q = c^r = abc = 1\}$ which is known to be nonabelian when p, q and r are not 1. Hence $\pi_1(M^3)$ cannot be $\boldsymbol{Z}$. Furthermore, the order of $H_1(M^3)$ is $|(xy + xz + yz)lmn|$, where order 0 means infinite order. This is of course 0 if and only if (1) is satisfied.

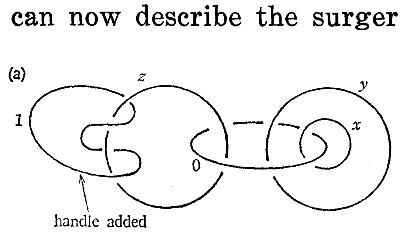
Conversely, suppose conditions (1) and (2) hold. Then, say, $p = 1$ so $1/x = 1$. This means that the corresponding exceptional fiber is in fact not exceptional with the result that M^3 is a Seifert fibre space with only two exceptional fibres. Each such space is easily seen to admit a Heegard splitting of genus one. By condition (1) $H_1(M)$ is infinite, so M must be $S^1 \times S^2$.

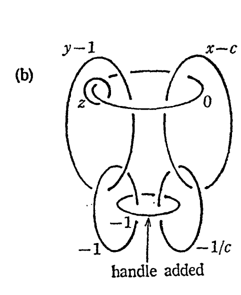
The next thing to do is look at two particular integral surgeries we may perform on our 3-manifold. First observe the following equivalence:



The homeomorphism on the right may be constructed by applying the matrix $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ to the complement of the Hopf link $(= S^1 \times S^1 \times I)$ and then calculating the composition with the maps which sow the solid tori back in.

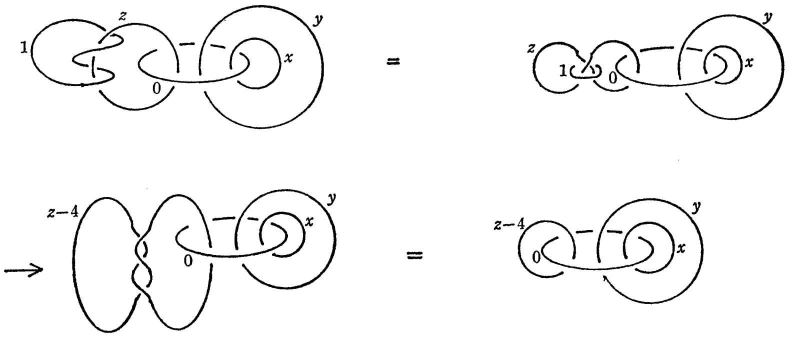
We can now describe the surgeries:



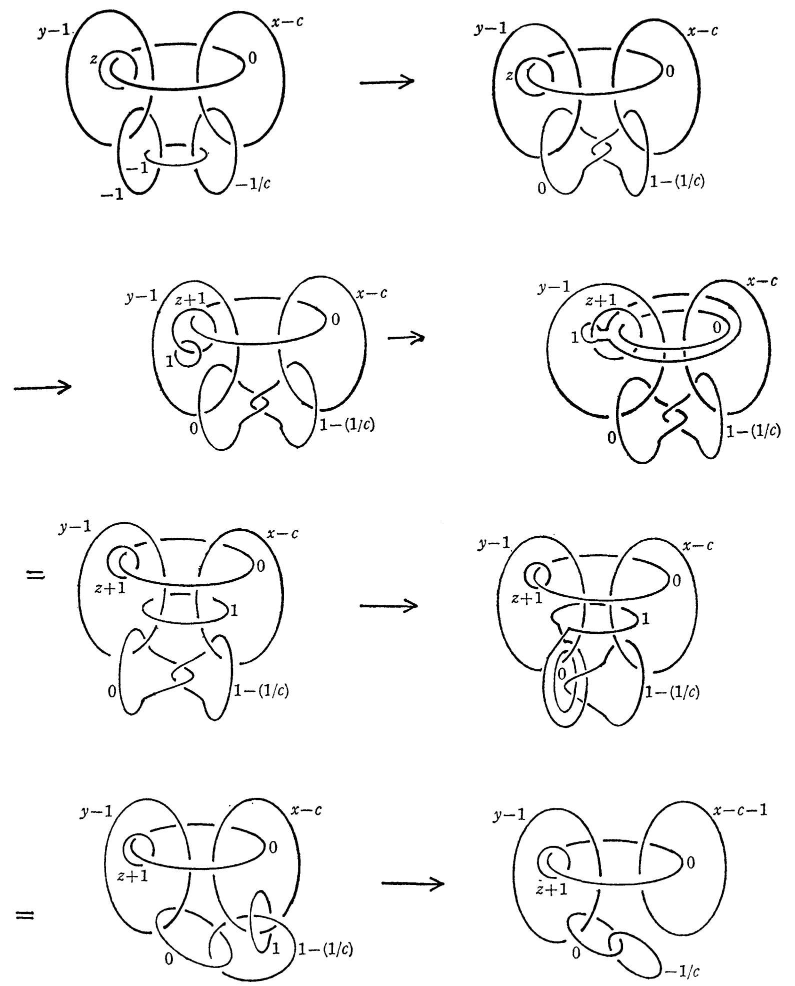


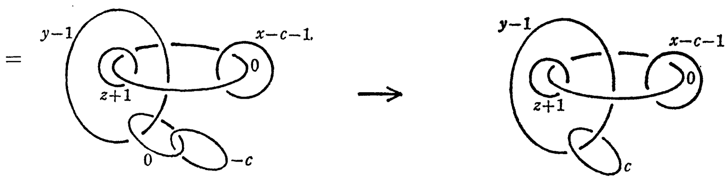
To recognize the resulting 3-manifolds it is necessary to perform some operations on these links. The reader is refered to (6) for an explanation.

(a)

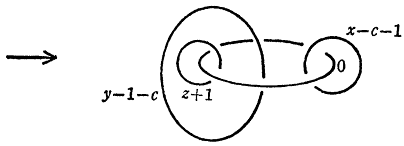


(b)

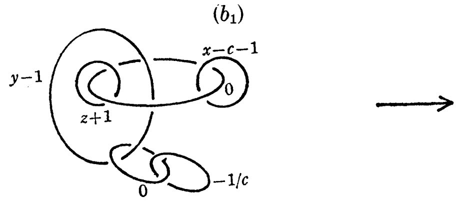


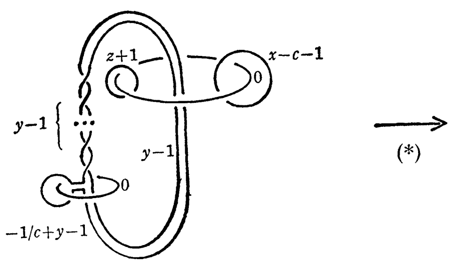


(b_1)

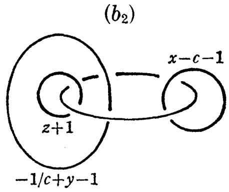


(b_1)



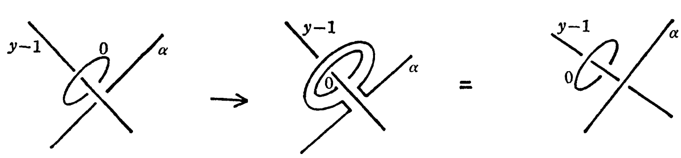


(b_2)



for $y \in \boldsymbol{Z}$

(*) because everything linking the $y-1$ slides over the 0 and off:



).

According to the lemma, we have obtained $S^1 \times S^2$ for (a) if and only if $1/x + 1/y + 1/z - 4 = 0$ and one of $1/x$, $1/y$ or $1/z - 4$ is an integer; for (b_1) if $1/x - c - 1 + 1/y - c - 1 + 1/z + 1 = 0$ and one of $1/x - c - 1$, $1/y - c - 1$ or $1/z + 1$ is an integer, $c = +1$; and for (b_2) if $1/x - c - 1 + 1/(y - 1 - 1/c) + 1/z + 1 = 0$ and $1/x - c - 1$, $1/(y - 1 - 1/c)$ or $1/z + 1$ is an integer, $c, y \in \boldsymbol{Z}$.

Recall that $x = p/l$, $y = q/m$, $z = r/n$ and $(^*)$ $qrl + prm + pqn = \pm d$. Since M^3 will be the boundary of a rational homology ball, a simple argument shows that d must be a square, say $d = k^2$. For (a), let $1/x + 1/y + 1/z - 4 = 0$.

(i) Suppose $1/z - 4 = s \in \boldsymbol{Z}$. Then $r = 4s + 1$ and $n = s$. Solving gives $p = q$ and $1 = -(m + qs)$. By $(^*)$ $\pm k = 2qs$ so we get the classes $(p, p, 4s + 1; 4p^2s^2)$.

(ii) If $1/x \in \boldsymbol{Z}$, $p = 1$, $r = q(4l - 1) + 4m$ and $n = ql + m$. Using $(^*)$ gives $\pm k = 2(ql + m)$ so if we put $s = ql + m$ we get the classes $(1, q, 4s - q; 4s^2)$. Note that $M(1, q, 4s - q\colon 4s^2) = L(4s^2, t)$ where $t = 4sq + 1$.

We shall omit the calculations from here on. For (b_1), $1/x - c - 1 + 1/y - c - 1 + 1/z + 1 = 0$.

(i) When $c = -1$ and $1/x \in \boldsymbol{Z}$ we get $(1, q, r; (q - r)^2)$. Putting $t = q - r$, this is $L(t^2, qt + 1)$.

(ii) $c = -1$ and $1/z + 1 \in \boldsymbol{Z}$ gives $(p, p, 1 - s; p^2s^2)$.

(iii) $c = 1$ and $1/x - 2 \in \boldsymbol{Z}$ gives $(p, ps \pm k, ps \pm 2k; k^2)$.

(iv) $c = 1$ and $1/z + 1 \in \boldsymbol{Z}$ gives $(p, ps - k, ps + k; k^2)$.

Finally, for (b_2), $1/x - c - 1 + 1/(y - 1 - 1/c) + 1/z + 1 = 0$ $(c, y \in \boldsymbol{Z})$.

(i) $1/x - c - 1 \in \boldsymbol{Z}$ gives $(p, st + s + 1, pt(s + 1 - (st + s + 1); (pst - (st + s + 1))^2)$.

(ii) $1/(y - 1 - 1/c) \in \boldsymbol{Z}$ gives subclasses of (b_1) (i), (iii) and (iv).

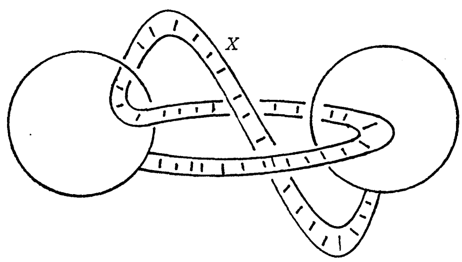
(iii) $1/z + 1 \in \boldsymbol{Z}$ gives $(p, q, s^2(p + q - pq) + s(2p - pq) + p; (s(p + q - pq) + p)^2)$.

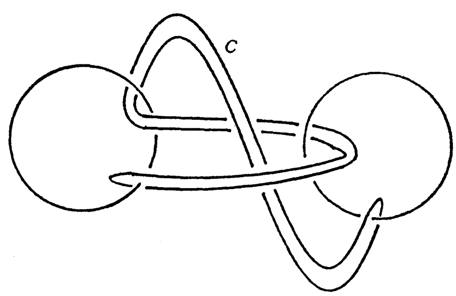
3. Outline of methods 2 and 3. To prove the theorem using description 2, we need the following:

LEMMA. *Let D be a 2-disk or the disjoint union of a 2-disk and a Möbius band. Let $f\colon D \to B^4$ be a smooth proper imbedding. Denote by $p\colon B^4 \to I$ the function which assigns to a point in B^4 its distance from 0 ($B^4 \subset R^4$ with the usual metric). Suppose that $pf\colon D \to I$ is a Morse function on D with three critical points (two index 0 and one index 1). If we put $W^4 =$ the 2-fold cover of B^4 branched along $f(D)$, then W^4 is diffeomorphic to $S^1 \times B^3$ with a 2-handle attached.*

Sketch of Proof. Isotope f to a critical level imbedding and arrange that the index 0 critical values lie in $(0, 1/2)$ while the index 1 value lies in $(1/2, 1)$. Write W_1 for the part of W^4 lying over $p^{-1}([0, 1/2])$; W_1 is diffeomorphic to $S^1 \times B^3$. We need to see that W is obtained from W_1 by attaching a 2-handle.

Let X be the band that appears at the critical level. Thicken X to $\bar{X}$ and thinking of p as a height function, denote by B everything lying above and below $\bar{X}$ (down to $p^{-1}(1/2)$). X is standardly imbedded in B so B and its cover $\hat{B}$ are both 4-balls. $\hat{B}$ is attached to $\overline{W - \hat{B}}$ along a neighborhood of the lift of C, where C is a circle "parallel" to x:

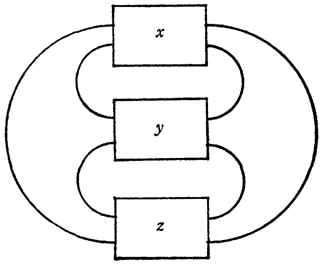




Finally, W_1 is diffeomorphic to $\overline{W - \hat{B}}$ since $B \cap p^{-1}(t)$ is unknotted in each $p^{-1}(t)$.

The theorem is proved (for classes (1)-(5)) by showing that the link $K(p, q, r; d)$ bounds one of the 2-manifolds in B^4 of the type mentioned in the lemma. To do this consider the following ribbon move (RM):

When the result is the unlink of two circles, they can be capped off by disjoint disks to give a critical-level imbedding of D in B^4. After smoothing, this 2-manifold satisfies the hypothesis of the lemma. It is not hard to see that (1) is the unlink of two components if and only if (a) one of $x - c - 1$, $y - c - 1$, $z + 1$ is $1/n$



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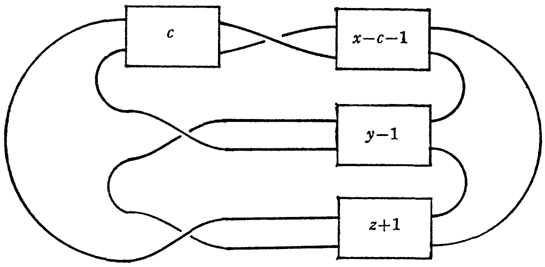
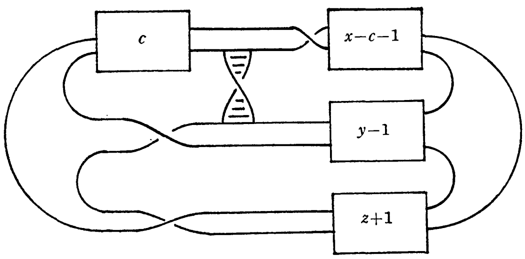
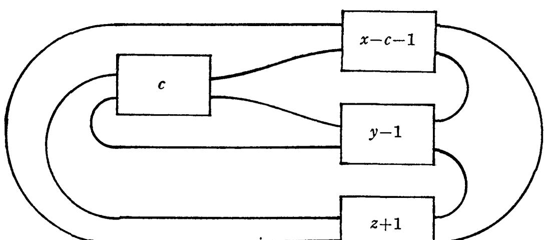


(*RM*)



$\longrightarrow$



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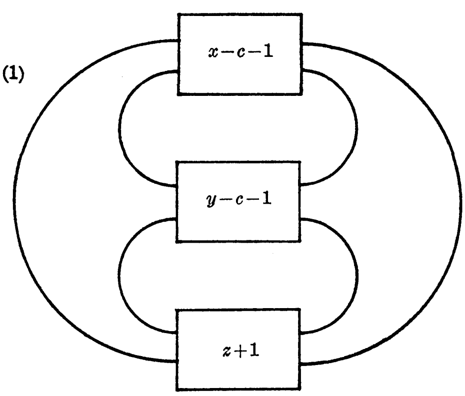
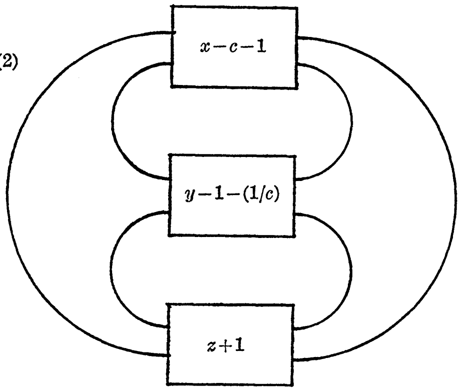
(1)

$=$

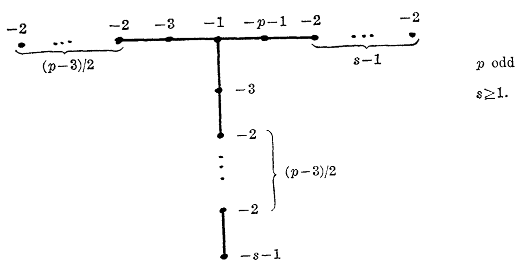


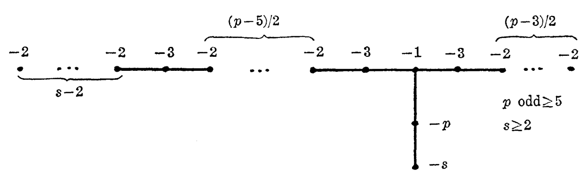
or

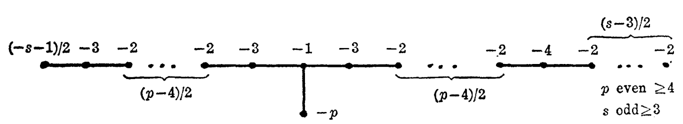
(2)


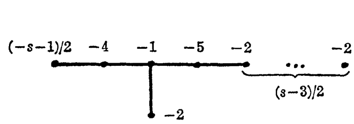
for $n \in \boldsymbol{Z}$ and (*b*) $1/x - c - 1 + 1/y - c - 1 + 1/z + 1 = 0$ while (2) is the unlink when (a) $x - c - 1$, $y - 1 - 1/c$ or $z + 1$ is $1/n$, $n \in \boldsymbol{Z}$ and (b) $1/x - c - 1 + 1/(y - 1 - 1/c) + 1/z + 1 = 0$. These equations are then solved as in § 2 to give the first five classes.

Next, using description 3, we demonstrate the existence of a contractible 4-manifold for the Brieskorn homology 3-spheres in classes 1 and 2 (with $d = 1$) by performing moves from the calculus of framed links. In fact, if $T(p, q, r)$ denotes the 4-manifold of

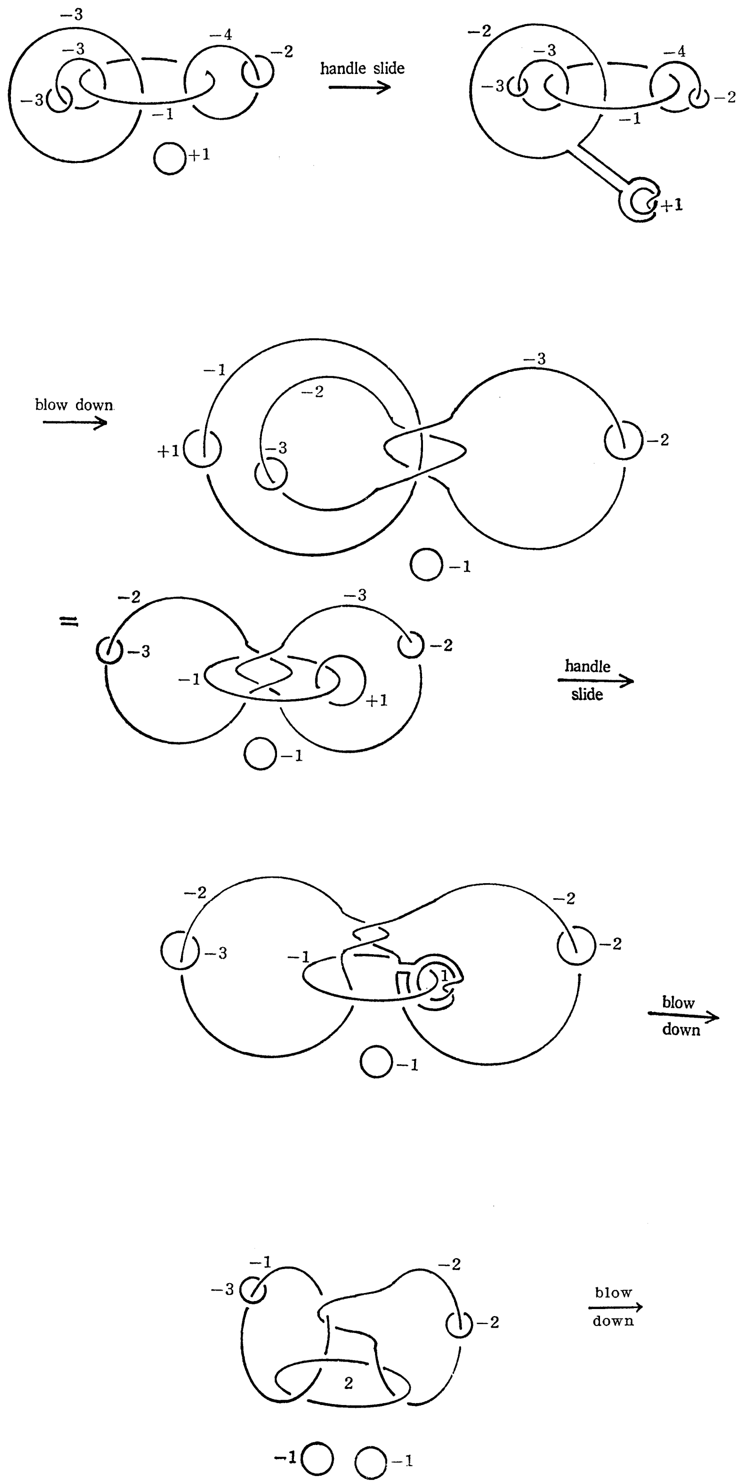








(**) $T(3,7,8)\# CP^2 =$

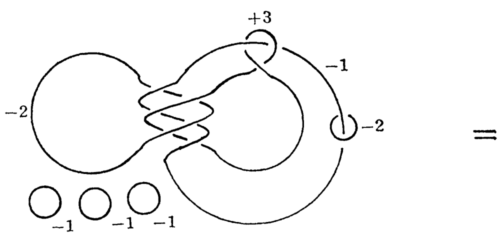


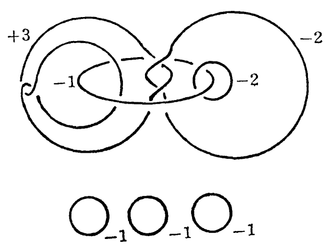
description 3 (a resolution of the singularity for $x^p + y^q + z^r = 0$) we have the

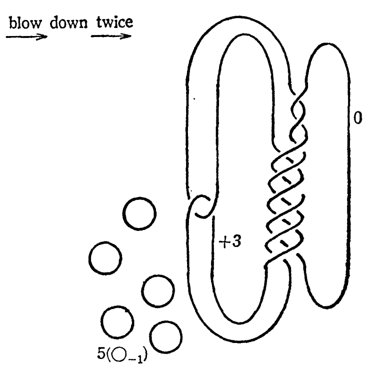
PROPOSITION. *Let $(p, q, r; 1)$ be in class 1 or 2. Then $T(p, q, r)\#$ CP^2 is diffeomorphic to $(-CP^2)\underbrace{\#\cdots\#}_{\beta_2(T)-1}(-CP^2)\#\hat{W}$ ($\beta_2(T) = \text{rank } H_2(Y)$) where $\partial\hat{W} = M(p, q, r; 1)$ and $\hat{W}$ may be surgered (one surgery) to obtain a contractible 4-manifold W.*

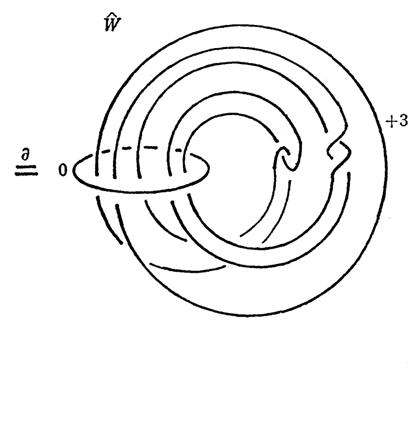
Outline of Proof:

The explicit calculations for all of these are rather long so we will give the proof for only one example. The techniques of the calculus of framed links will be used throughout (see e.g., [2]). The proof begins at (**)

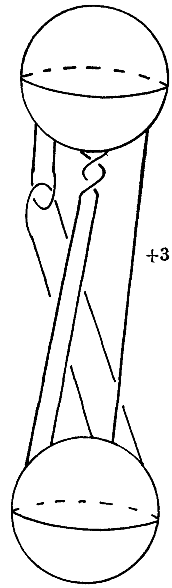








Consider the 2-sphere obtained by taking the core of the handle whose attaching map has framing 0, together with a trivial 2-disk in B^4. This 2-sphere has a trivial normal bundle, so we may surger it. After surgery we get a contractible manifold:



REMARK. It is not hard to see that the 4-manifolds constructed by methods 1 and 2 are diffeomorphic. In fact, if one considers the two-fold over of $S^3\times I$ branched along the trace of the ribbon move used for method 2, one sees where to put the 2-handle for method 1 (and the left half-twist in the band gives the -1 framing for the circle). We have verified for a few simple cases that the 4-manifolds built by method 3 are the same as those of method 1, but we do not know if this is true in general.

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Received April 19, 1978.
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