DERIVATIONS OF OPERATOR ALGEBRAS INTO SPACES OF UNBOUNDED OPERATORS

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This paper is to study the spatiality of unbounded derivations in operator algebras. Let $\mathcal{M}$ be a von Neumann algebra ($C^*$-algebra) on a Hilbert space $\mathcal{G}$ and $\delta$ be an unbounded derivation in $\mathcal{M}$. In this paper, extending $\delta$ to a derivation $\hat{\delta}$ of $\mathcal{M}$ into a certain space of unbounded operators, we study the spatiality of $\delta$ by investigating the property of $\hat{\delta}$.

1. Introduction. Unbounded derivations in operator algebras ($C^*$-algebras and von Neumann algebras) have recently been investigated by many authors, since they are appeared as infinitesimal generators of strongly continuous one-parameter groups of *-automorphisms on $C^*$-algebras [see; 12]. In particular, the infinitesimal generator mentioned above is implemented by a symmetric operator by giving some representation of its $C^*$-algebra on a Hilbert space, and there exist many closed derivations in $C^*$-algebras which possess such a property [2]. In this point of view, we shall study the spatiality of unbounded derivations in operator algebras (see [2]; Problem). Our method is, roughly speaking, to examine the spatiality of an unbounded derivation $\delta$ in an operator algebra $\mathcal{M}$ by extending $\delta$ to a derivation of $\mathcal{M}$ into some space of unbounded operators containing $\mathcal{M}$.

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{G}$ and let $\delta$ be a *-derivation in $\mathcal{M}$ with $\sigma$-strongly dense domain $\mathcal{D}(\delta)$. Let $\mathcal{D}$ be a dense subspace of $\mathcal{G}$. We introduce various locally convex topologies in the space $L^*(\mathcal{D}, \mathcal{G})$ which is the set of all linear operators $T$ of $\mathcal{D}$ into $\mathcal{G}$ with $\mathcal{D}(T^*) \supset \mathcal{D}$, and extend $\delta$ to a *-derivation $\hat{\delta}$ of $\mathcal{M}$ into $L^*(\mathcal{D}, \mathcal{G})$ assuming corresponding continuity of $\delta$ in these topologies.

We shall then examine under what conditions the continuous *-derivation $\hat{\delta}$ of $\mathcal{M}$ into $L^*(\mathcal{D}, \mathcal{G})$ with some specified topology is spatial, i.e., there exists an element $H$ of $L^*(\mathcal{D}, \mathcal{G})$ such that $\hat{\delta}(A)\xi = [H, A]\xi = (HA - AH)\xi$ for all $A \in \mathcal{M}$ and $\xi \in \mathcal{D}$. We call the dense subspace $\mathcal{D}$ countably dominated by a sequence $\{T_n\}$ of closed operators if $\mathcal{D} = \bigcap_{n=1}^\infty \mathcal{D}(T_n)$ and $\|T_n\xi\| \leq \|T_{n+1}\xi\|$ for each $\xi \in \mathcal{D}$ and $n = 1, 2, \ldots$.

Our first result (Theorem 4.11) shows that if $\mathcal{M}$ is a left von Neumann algebra of a Hilbert algebra $\mathfrak{A}$ with identity and $\mathcal{D}$ is countably dominated by $\{T_n\}$ of closed operators then $\delta$ is spatial.
The second purpose of this paper is to show (Theorem 4.15) that if \( \mathcal{M} \) has certain property (Definition 4.2) and \( \mathcal{D} \) is countably dominated by \( \{T_n\} \) of closed operators \( \eta \mathcal{M} \) then \( \delta \) is a spatial \( \ast \)-derivation of \( \mathcal{M} \) into \( \mathcal{L}^1(\mathcal{D}, \mathcal{G}) \).

2. Spaces of unbounded operators. Let \( \mathcal{G} \) be a Hilbert space with inner product (\( | \cdot | \)) and let \( \mathcal{D} \) be a dense subspace of \( \mathcal{G} \). We denote by \( \mathcal{L}(\mathcal{D}, \mathcal{G}) \) (resp. \( \mathcal{L}_c(\mathcal{D}, \mathcal{G}) \)) the space of all (resp. closable) linear operators of \( \mathcal{D} \) into \( \mathcal{G} \) and by \( \mathcal{L}^1(\mathcal{D}, \mathcal{G}) \) the space of operators \( A \) in \( \mathcal{L}(\mathcal{D}, \mathcal{G}) \) for which there exists the adjoints \( A^* \) whose domains \( \mathcal{D}(A^*) \) contain \( \mathcal{D} \). For each \( T \in \mathcal{L}(\mathcal{D}, \mathcal{G}) \) we define

\[
\| A \|_T = \sup_{\xi \in \mathcal{D}} \frac{\| A\xi \|}{\| T_\xi \|}, \quad A \in \mathcal{L}(\mathcal{D}, \mathcal{G}),
\]

where \((\lambda/0) = \infty \) for \( \lambda > 0 \) and \((0/0) = 0\),

\[
\mathcal{M}_T = \{ A \in \mathcal{L}(\mathcal{D}, \mathcal{G}); \| A \|_T < \infty \}
\]

and

\[
\mathcal{M}_T^\sharp = \{ A \in \mathcal{L}^1(\mathcal{D}, \mathcal{G}); \| A \|_T < \infty \}.
\]

Then it is easily seen that \( \mathcal{M}_T \) is a Banach space equipped with the norm \( \| \cdot \|_T \) and \( \mathcal{M}_T^\sharp \) is a subspace of \( \mathcal{M}_T \).

The following lemma is an immediate consequence of the definitions of the spaces of \( \mathcal{M}_T \) and \( \mathcal{M}_T^\sharp \).

**Lemma 2.1.** Let \( T \) be an element of \( \mathcal{L}^1(\mathcal{D}, \mathcal{G}) \) such that \( \mathcal{D} = \bigcap_{n=1}^{\infty} \mathcal{D}(T_n) \) is dense in \( \mathcal{D} \), where \( \mathcal{B}(\mathcal{H}) \) denotes the algebra of all bounded linear operators on \( \mathcal{H} \). We set

\[
\mathcal{B}_r = \{ AT^{-1}; A \in \mathcal{M}_T \} \quad \text{and} \quad \mathcal{B}_r^\sharp = \{ AT^{-1}; A \in \mathcal{M}_T^\sharp \}.
\]

Then the map \( \phi: A \to AT^{-1} \) is an isometric isomorphism of the Banach space \( \mathcal{M}_T \) onto the Banach space \( \mathcal{B}(\mathcal{H}) \).

**Lemma 2.2.** Let \( \mathcal{H} \) be a Hilbert space with inner product (\( | \cdot | \)). If there exists a sequence \( \{T_n\} \) of closed operators on \( \mathcal{H} \) such that

1. \( \mathcal{D} = \bigcap_{n=1}^{\infty} \mathcal{D}(T_n) \) is dense in \( \mathcal{D} \);
2. \( \| T_n \xi \| \leq \| T_{n+1} \xi \| \) for all \( \xi \in \mathcal{D} \) and \( n = 1, 2, \ldots \),

then \( \mathcal{L}^1(\mathcal{D}, \mathcal{G}) = \bigcup_{n=0}^{\infty} \mathcal{M}_T^\sharp \), where \( T_0 = I \).

**Proof.** For each \( \xi \in \mathcal{D} \) we set

\[
\| \xi \|_{T_n} = \| T_n \xi \| \quad \text{for} \quad n = 0, 1, 2, \ldots .
\]

We consider the locally convex topology \( t_{T_n} \) on \( \mathcal{D} \) generated by
family of the seminorms $\| \cdot \|_{T_n} (n = 0, 1, 2, \cdots)$. Suppose that $\{ \xi_k \}$ is a Cauchy sequence in $(\mathcal{D}, t_{(T_n)})$. Then we have

$$
\lim_{k \to \infty} \| \xi_k - \xi_1 \| = 0 \quad \text{and} \quad \lim_{k \to \infty} \| T_n \xi_k - T_n \xi_1 \| = 0
$$

for $n = 1, 2, \cdots$.

Since $T_n$ is a closed operator, it follows that $x \in \mathcal{D}(T_n)$ and $\lim_{k \to \infty} T_n \xi_k = T_n x$ for $n = 1, 2, \cdots$. Hence we have $x \in \bigcap_{n=1}^{\infty} \mathcal{D}(T_n) = \mathcal{D}$ and $\lim_{k \to \infty} T_n \xi_k = T_n x$ for $n = 1, 2, \cdots$. This implies that $(\mathcal{D}, t_{(T_n)})$ is a Fréchet space.

Suppose $S \in \mathcal{L}^t(\mathcal{D}, \mathcal{G})$. We show that the graph of $S: G(S) \equiv \{ \langle \xi, S \xi \rangle; \xi \in \mathcal{D} \}$ is closed in $(\mathcal{D}, t_{(T_n)}) \times \mathcal{G}$. Suppose that a sequence $\{ \langle \xi_n, S \xi_n \rangle \}$ in $G(S)$ converges to an element $\langle \xi, y \rangle$ of $\mathcal{D} \times \mathcal{G}$. It then follows that $\xi_n - \xi \in \mathcal{D}$, $\lim_{n \to \infty} \| \xi_n - \xi \| = 0$ and $\lim_{n \to \infty} \| S(\xi_n - \xi) - (y - S \xi) \| = 0$. Since $S$ is closable, we have $y = S \xi$. This implies that $G(S)$ is closed in $(\mathcal{D}, t_{(T_n)}) \times \mathcal{G}$. By the closed graph theorem it follows that the map $S: (\mathcal{D}, t_{(T_n)}) \to \mathcal{G}$ is continuous. Hence there exist a number $n$ and a constant $\gamma > 0$ such that

$$
\| S \xi \| \leq \gamma \| T_n \xi \| \quad \text{for all} \quad \xi \in \mathcal{D}.
$$

Therefore, $S \in \mathcal{L}^{t*}_{T_n}$. This implies that $\mathcal{L}^t(\mathcal{D}, \mathcal{G}) = \bigcup_{n=1}^{\infty} \mathcal{L}^{t*}_{T_n}$.

DEFINITION 2.3. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{G}$. If there exists a sequence $\{ T_n \}$ of closed operators in $\mathcal{G}$ such that $\mathcal{D} = \bigcap_{n=1}^{\infty} \mathcal{D}(T_n)$ and $\| T_n \xi \| \leq \| T_{n+1} \xi \|$ for all $\xi \in \mathcal{D}$ and $n = 1, 2, \cdots$, then $\mathcal{D}$ is said to be countably dominated by $\{ T_n \}$. If there exists a sequence $\{ S_n \}$ in $\mathcal{L}^t(\mathcal{D}, \mathcal{G})$ such that $\mathcal{L}^t(\mathcal{D}, \mathcal{G}) = \bigcup_{n=1}^{\infty} \mathcal{L}^{t*}_{S_n}$ and $\| S_n \xi \| \leq \| S_{n+1} \xi \|$ for all $\xi \in \mathcal{D}$ and $n = 1, 2, \cdots$, then $\mathcal{L}^t(\mathcal{D}, \mathcal{G})$ is said to be countably dominated by $\{ S_n \}$.

REMARK. (1) Lemma 2.2 implies that if a pre-Hilbert space $\mathcal{D}$ is countably dominated then $\mathcal{L}^t(\mathcal{D}, \mathcal{G})$ is also countably dominated.

(2) It will be seen, by a simple calculation, that if $\mathcal{L}^t(\mathcal{D}, \mathcal{G}) = \bigcup_{n=1}^{\infty} \mathcal{L}^{t*}_{S_n}$ for $S_n \in \mathcal{L}^t(\mathcal{D}) \equiv \mathcal{L}^t(\mathcal{D}, \mathcal{D})(n = 1, 2, \cdots)$, then $\mathcal{L}^t(\mathcal{D}, \mathcal{G})$ is countably dominated.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{G}$. We now introduce some locally convex topologies on $\mathcal{L}^t(\mathcal{D}, \mathcal{G})$. We put

$$
P_{t, \alpha}(A) = \| A \xi \|,
$$

$$
P_{t}(A) = \| A \xi \|,
$$

where $A \in \mathcal{L}(\mathcal{D}, \mathcal{G}), \xi \in \mathcal{D}$ and $x \in \mathcal{G}$. The locally convex topology on $\mathcal{L}(\mathcal{D}, \mathcal{G})$ generated by the seminorms $\{ P_{t, \alpha}(\cdot); \xi, \eta \in \mathcal{D} \}$ (resp.
\{P_{t,x}(\cdot); \xi \in \mathcal{D}, x \in \mathcal{G}\}, \{P_t(\cdot); \xi \in \mathcal{D}\}\) is said to be the weak topology (resp. quasi-weak topology, strong topology) and is simply denoted by \(t^w\) (resp. \(t^q_w, t^s\)).

Let \(\mathcal{G}_n\) be the Hilbert direct sum of the Hilbert spaces \(\mathcal{G}_n \equiv \mathcal{G}(n = 1, 2, \ldots)\) and let

\[\mathcal{D}_\infty(\mathcal{D}) = \{\{\xi_n\} \in \mathcal{G}_n; \xi_n \in \mathcal{D} \quad \text{for} \quad n = 1, 2, \ldots\}\]

and \(\sum_{n=1}^{\infty} \|A\xi_n\|^2 < \infty\) for all \(A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G})\).

We set

\[P_{\{t_n, \|x_n\}|}(A) = \left| \sum_{n=1}^{\infty} (A\xi_n|x_n) \right|,
\]

\[P_{\{t_n\}}(A) = \left[ \sum_{n=1}^{\infty} \|A\xi_n\|^2 \right]^{1/2},\]

where \(A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G}), \{\xi_n\} \in \mathcal{D}_\infty(\mathcal{D})\) and \(\{x_n\} \in \mathcal{D}_\infty\). We equip \(\mathcal{L}^t(\mathcal{D}, \mathcal{G})\) with the locally convex topology \(t^w\) (resp. \(t^w_q, t^s\)) induced by the seminorms \(\{P_{\{t_n, \|x_n\}|}(\cdot); \{\xi_n\} \in \mathcal{D}_\infty(\mathcal{D})\}\) (resp. \(\{P_{\{t_n\}|}(\cdot); \{\xi_n\} \in \mathcal{D}_\infty(\mathcal{D})\}\)). The topology \(t^w\) (resp. \(t^w_q, t^s\)) is said to be the \(\sigma\)-weak topology (resp. quasi-\(\sigma\)-weak topology, \(\sigma\)-strong topology) on \(\mathcal{L}^t(\mathcal{D}, \mathcal{G})\).

We next define the uniform topology and the quasi-uniform topology. A subset \(\mathcal{M}\) of \(\mathcal{D}\) is said to be \(\mathcal{D}\)-bounded if

\[\sup_{\xi \in \mathcal{M}} \|A\xi\| < \infty\]

for each \(A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G})\).

We then define

\[P_{\mathcal{M}}(A) = \sup_{\xi \in \mathcal{M}} |(A\xi|\gamma)|,
\]

\[P^\mathcal{M}(A) = \sup_{\xi \in \mathcal{M}} \|A\xi\|,
\]

where \(\mathcal{M}\) is \(\mathcal{D}\)-bounded and \(A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G})\). The locally convex topology generated by the seminorms \(\{P_{\mathcal{M}}(\cdot); \mathcal{M}\) is \(\mathcal{D}\)-bounded\} (resp. \(\{P^\mathcal{M}(\cdot); \mathcal{M}\) is \(\mathcal{D}\)-bounded\}) is said to be the uniform topology (resp. quasi-uniform topology) on \(\mathcal{L}^t(\mathcal{D}, \mathcal{G})\) and is simply denoted by \(t^u\) (resp. \(t^u_q\)).

We next define the \(\rho\)-topology and \(\lambda\)-topology on \(\mathcal{L}^t(\mathcal{D}, \mathcal{G})\). For each \(T \in \mathcal{L}^t(\mathcal{D}, \mathcal{G})\) we put

\[\rho_T(A) = \sup_{\xi \in \mathcal{D}} \frac{|(A\xi|\xi)|}{\|T\xi\|^2}, \quad A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G}),\]

where \((\lambda/0) = \infty\) for \(\lambda > 0\) and \(0/0 = 0\), and

\[\mathcal{M}_T^\rho = \{A \in \mathcal{L}^t(\mathcal{D}, \mathcal{G}); \rho_T(A) < \infty\}.\]
Then it is easily seen that $\mathcal{R}_1$ is a normed space equipped with the norm $\rho_T(\cdot)$ and $L^4(\mathcal{D}, \mathcal{G}) = \bigcup_{T \in \mathcal{R}} L^4_T$. The inductive limit topology on $L^4(\mathcal{D}, \mathcal{G})$ with respect to the normed spaces $\{\mathcal{R}_1, \rho_T(\cdot); T \in L^4(\mathcal{D}, \mathcal{G})\}$ is said to be the $\rho$-topology (resp. $\lambda$-topology) on $L^4(\mathcal{D}, \mathcal{G})$ and is denoted by $t_\rho$ (resp. $t_\lambda$).

Now one may easily see the following lemma by the definitions of the topologies.

**Lemma 2.4.** The relation among the topologies introduced here are as follows:

$$
\begin{align*}
t_\rho &\leq t_\sigma \\
t_\rho &\leq t_\omega \\
t_\omega &\leq t_\sigma \\
t_\omega &\leq t_\omega \\
\end{align*}
$$

where the symbols $\tau_1 \leq \tau_2$, $\tau_2 \geq \tau_1$, $\forall_1$ and $\forall_2$ mean the topology $\tau_2$ is finer than the topology $\tau_1$.

**Remark.** The topologies $t_\sigma$ and $t_\omega$ (resp. the topologies $t_\rho$ and $t_\sigma$) on $L^4(\mathcal{D}, \mathcal{G})$ are generalizations of the uniform topology and quasi-uniform one (resp. the $\rho$-topology and $\lambda$-topology) introduced by G. Lassner [8] (resp. D. Arnal and J. P. Jurzak [1]), for an unbounded operator algebra respectively. We denote by $t_u$ (resp. $t_w$, $t_s$, $t_{ow}$, $t_{os}$) the usual uniform (resp. weak, strong, $\omega$-weak, $\sigma$-strong) topology on $\mathcal{B}(\mathcal{G})$. The relations between the topologies on $\mathcal{B}(\mathcal{G})$ are as follows: $t_u = t_{wu} = t_{w_s} = t_{s_w} = t_w = t_{sw} = t_{w_s} = t_{sw} = t_{w_s} = t_{sw}$ and $t_{os} = t_{os}$.

**Lemma 2.5.** Suppose that $L^4(\mathcal{D}, \mathcal{G})$ is countably dominated by $\{T_n\}$ and $\mathcal{R}$ is a subset of $L^4(\mathcal{D}, \mathcal{G})$. Then the following statements are equivalent:

1. $\mathcal{R}$ is $t_\rho$-bounded;
2. $\mathcal{R}$ is $t_\sigma$-bounded;
3. there exist a number $n$ and a constant $\gamma > 0$ such that

$$
|\langle A\xi, \xi \rangle| \leq \gamma \|\langle I + |T_n|\rangle \xi \| \quad \text{for all } A \in \mathcal{R} \text{ and } \xi \in \mathcal{D},
$$

where $T_n = U |T_n|$ is the polar decomposition of $T_n$.

**Proof.** This is proved in the same way as in ([13] Lemma 2.1).
Lemma 2.6. Suppose that $\mathcal{L}(\mathcal{D}, \mathcal{G})$ is countably dominated by \( \{T_n\} \) and \( \mathcal{R} \) is a subset of $\mathcal{L}^s(\mathcal{D}, \mathcal{G})$. Then the following statements are equivalent:

1. \( \mathcal{R} \) is $t^s$-bounded;
2. \( \mathcal{R} \) is $t^u$-bounded;
3. \( \mathcal{R} \) is $t^{su}$-bounded;
4. there exists a number \( n \) and a constant \( \gamma > 0 \) such that
   \[ ||A\xi|| \leq \gamma \left( \sum_{k=1}^{\infty} ||T_k\xi||^2 \right)^{1/2} \]
   for all \( A \in \mathcal{R} \) and \( \xi \in \mathcal{D} \).

Furthermore, if \( \mathcal{D} = \bigcap_{T \in \mathcal{L}^s(\mathcal{D}, \mathcal{G})} \mathcal{D}(\bar{T}) \), then the statements (1)−(4) are equivalent to the following statements (5) and (6):

5. \( \mathcal{R} \) is $t^s$-bounded;
6. \( \mathcal{R} \) is $t^{su}$-bounded.

Proof. Since $t^s \geq t^u$ and $t^{su} \geq t^{su}$, one can see the implications (4) $\Rightarrow$ (1), (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). We show the implication (3) $\Rightarrow$ (4).

Suppose that the statement (4) is not true. Then there exists a sequence \( \{A_n\} \) in \( \mathcal{R} \) and a sequence \( \{\xi_n\} \) of nonzero elements of \( \mathcal{D} \) such that

\[ ||A_n\xi_n|| \leq n^2 \left( \sum_{k=1}^{\infty} ||T_k\xi_n||^2 \right)^{1/2} \]

for \( n = 1, 2, \ldots \).

Putting

\[ \eta_n = \frac{\xi_n}{n||\sum_{k=1}^{\infty} ||T_k\xi_n||} \]

for \( n = 1, 2, \ldots \), we have

\[ ||A_n\eta_n|| \geq n \quad \text{and} \quad ||T_k\eta_n|| < \frac{1}{n} \]

We now show \( \{\eta_n\} \in \mathcal{D}(\mathcal{D}) \). Since $\mathcal{L}^s(\mathcal{D}, \mathcal{G}) = \bigcup_{n=1}^{\infty} \mathcal{M}^s(T_n)$, it follows that for each \( A \in \mathcal{L}^s(\mathcal{D}, \mathcal{G}) \) there exists a number \( k \) and a constant \( \gamma > 0 \) such that

\[ ||A\xi|| \leq \gamma ||T_k\xi|| \]

for all \( \xi \in \mathcal{D} \).

Then we have

\[ \sum_{n=1}^{\infty} ||A\eta_n||^2 \leq \gamma \sum_{n=1}^{\infty} ||T_k\eta_n||^2 \]

\[ \leq \gamma \left\{ \sum_{n=1}^{k-1} ||T_k\eta_n||^2 + \sum_{n=k+1}^{\infty} ||T_k\eta_n||^2 \right\} \]

\[ \leq \gamma \left\{ \sum_{n=1}^{k-1} ||T_k\eta_n||^2 + \sum_{n=k+1}^{\infty} ||T_k\eta_n||^2 \right\} \]
This means \( \{ \eta_n \} \in \mathcal{D}_w(\mathcal{D}) \). Furthermore, we have
\[
\sup_{A \in \mathcal{A}} P_{\{\eta_n\}}(A) = \sup_{A \in \mathcal{A}} \left[ \sum_{n=1}^{\infty} \| A \eta_n \|^2 \right]^{1/2} \geq \| A \eta_n \| \geq n.
\]
This contradicts that \( \mathcal{A} \) is \( t^\infty_c \)-bounded. This completes the proof of the implication (3) \( \Rightarrow \) (4).

The implication (2) \( \Rightarrow \) (4) is proved in the same way as in ([13] Lemma 2.2).

If \( \mathcal{D} = \bigcap_{t \in \mathcal{D}(\mathcal{D}), t \in \mathcal{D}(\mathcal{D}), t \in \mathcal{D}(\mathcal{D})} \mathcal{D}(T) \), the equivalence of the statements (1) \( \sim \) (6) follows from ([1] Proposition 1.6).

3. Extension of derivations. Let \( \mathcal{M} \) be a \( C^* \)-algebra (or a von Neumann algebra). A linear map \( \delta: \mathcal{D}(\mathcal{D}) \subset \mathcal{M} \rightarrow \mathcal{M} \) is said to be a \( * \)-derivation in \( \mathcal{M} \) if it satisfies the following conditions:

1. the domain \( \mathcal{D}(\mathcal{D}) \) of \( \delta \) is a dense \( * \)-subalgebra of \( \mathcal{M} \) (i.e., \( \mathcal{D}(\mathcal{D}) \) is norm-dense if \( \mathcal{M} \) is a \( C^* \)-algebra, and weak-dense if \( \mathcal{M} \) is a von Neumann algebra);
2. \( \delta(AB) = \delta(A)B + A\delta(B) \) for each \( A, B \in \mathcal{D}(\mathcal{D}) \);
3. \( \delta(A^*) = \delta(A)^* \) for each \( A \in \mathcal{D}(\mathcal{D}) \).

We begin with the following lemma.

**Lemma 3.1.** Let \( \mathcal{M} \) be a unital \( C^* \)-algebra acting on a Hilbert space \( \mathfrak{H} \) and let \( \delta \) be a \( * \)-derivation in \( \mathcal{M} \) with domain \( \mathcal{D}(\mathcal{D}) \). If there exists a dense subspace \( \mathcal{D} \) of \( \mathfrak{H} \) such that \( \mathcal{M} \mathcal{D} \subset \mathcal{D} \) and \( \delta \) is a continuous map of \( (\mathcal{D}(\mathcal{D}), t_w) \) into \( (\mathcal{M}, t^\infty_w) \), then \( \delta \) is extended to a continuous linear map \( \delta \) of \( (\mathcal{M}, t_w) \) into \( (\mathcal{D}(\mathcal{D}), \mathfrak{H}), t^\infty_w \) such that

1. \( \delta(AB)\xi = \delta(A)B\xi + A\delta(B)\xi \);
2. \( \delta(A)^*\xi = \delta(A^*)\xi \);
3. \( \delta(A^*)C\xi = C\delta(A)\xi \)

for each \( A, B \in \mathcal{M}, C \in \mathcal{M}' \) and \( \xi \in \mathcal{D} \). Namely, the following diagram holds:

\[
\begin{array}{ccc}
\delta: (\mathcal{M}, t_w) & \xrightarrow{\text{continuous}} & (\mathcal{D}(\mathcal{D}), \mathfrak{H}), t^\infty_w \\
\cup & \xrightarrow{U} & \cup \\
\delta: (\mathcal{D}(\mathcal{D}), t_w) & \xrightarrow{\text{continuous}} & (\mathcal{M}, t^\infty_w)
\end{array}
\]

By Lemma 3.1 we define a derivation of a \( C^* \)-algebra into a space of unbounded operators as follows:
DEFINITION 3.2. Let $D$ be a dense subspace in a Hilbert space $G$ and let $M$ be a unital $C^*$-algebra acting on $G$ with $MD \subset D$. A linear map $\delta$ of $M$ into $L(D, G)$ is said to be a derivation of $M$ into $L(D, G)$ if

$$\delta(AB)\xi = \delta(A)B\xi + A\delta(B)\xi \quad \text{for each } A, B \in M \text{ and } \xi \in D.$$ 

In particular, a derivation $\delta$ is said to be a $^*$-derivation if the range of $\delta$ is contained in $L^2(D, G)$ and

$$\delta(A^*)\xi = \delta(A^*)\xi \quad \text{for each } A \in M \text{ and } \xi \in D.$$

If a derivation $\delta$ of $M$ into $L(D, G)$ is a continuous map of $(M, \tau_1)$ into $(L(D, G), \tau_2)$, where $\tau_1$ and $\tau_2$ are topologies on $M$ and $L(D, G)$ respectively, then it is said to be $(\tau_1 \to \tau_2)$-continuous.

We also have the following result:

**Lemma 3.3.** Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $G$ and let $\delta$ be a $^*$-derivation in $\mathcal{M}$. If $\delta$ is $(\tau_w \to \tau_w)$-continuous (resp. $(\tau_2 \to \tau_2)$-continuous), then $\delta$ is extended to a $(\tau_w \to \tau_w)$-continuous (resp. $(\tau_2 \to \tau_2)$-continuous) $^*$-derivation $\hat{\delta}$ of $\mathcal{M}$ into $L^2(D, G)$ satisfying $\hat{\delta}(A^*)\xi = C\delta(A)\xi$ for each $A \in \mathcal{M}$, $C \in \mathcal{M}'$ and $\xi \in D$.

DEFINITION 3.4. Let $D$ be a dense subspace of a Hilbert space $G$ and let $\delta$ be a $^*$-derivation of a $C^*$-algebra $M$ on $G$. If $\delta(M) \subset M^+$ for some $T \in L^2(D, G)$, then $\delta$ is said to be a $^*$-derivation of $M$ into $M^+$. If there exists an element $T$ of $L^2(D, G)$ such that $\delta(M)$ is a bounded subspace of the normed space $M^+$, then $\delta$ is said to be quasi-bounded.

**Lemma 3.5.** Let $\mathcal{M}$ be a unital $C^*$-algebra acting on a Hilbert space $G$ and let $\delta$ be a $^*$-derivation in $\mathcal{M}$. If there exist a dense subspace $D$ of $G$ and an element $T$ of $L^2(D, G)$ such that $MD \subset D$ and $\|\delta(A)\|_T \leq \|A\|$ for all $A \in D(\delta)$, then $\delta$ is extended to a quasi-bounded $^*$-derivation $\hat{\delta}$ of $\mathcal{M}$ into $M^+$ satisfying $\hat{\delta}(A^*)\xi = C\delta(A)\xi$ for each $A \in \mathcal{M}$, $C \in \mathcal{M}'$ and $\xi \in D$.

We now give some examples of quasi-bounded $^*$-derivations.

**Example 3.6.** Let $\delta$ be a spatial derivation in a $C^*$-algebra $M$ acting on a Hilbert space $G$ with domain $\mathcal{D}(\delta)$, i.e., there exists a symmetric operator $H$ on $G$ such that $\mathcal{D}(\delta) \subseteq \mathcal{D}(H)$ and $\delta(A)\xi = i[H, A]\xi$ for each $A \in \mathcal{D}(\delta)$ and $\xi \in \mathcal{D}(H)$. If there exists a closed...
operator $T\eta^M$ and a constant $\gamma > 0$ such that $\|H_\xi\| \leq \gamma \|T_\xi\|$ for all $\xi \in D(T)$, then $\delta$ is extended to a quasi-bounded $*$-derivation $\hat{\delta}$ of $M$ into $L^1(D(T), G)$.

2. Let $M_i$ be a von Neumann algebra on a Hilbert space $G_i$ and let $\delta_i$ be a bounded $*$-derivation on $M_i (i = 1, 2, \cdots)$. Let $M$ be a direct sum of the von Neumann algebras $M_i$ and let $G$ be the direct sum of the Hilbert spaces $G_i$. We define

$$D(\delta) = \{ A = (A_i) \in \prod_i M_i; A_i \neq 0 \text{ for only finite coordinates} \},$$

$$\delta(A) = (\delta_i(A_i)), \quad A = (A_i) \in D(\delta).$$

Then $\delta$ is a $*$-derivation in $M$ with the weakly dense domain $D(\delta)$, but it is not generally bounded. However, $\delta$ is $(t_u \rightarrow t_\varphi)$-continuous (and $(t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi)$-continuous), where

$$D = \{ (\xi_i) \in G; \xi_i \neq 0 \text{ for only finite coordinates} \}.$$

Putting

$$T = (\| \delta_i \| I_i)$$

where $\| \delta_i \|$ is the norm of $\delta_i$ and $I_i$ is the identity operator on $G_i$, we have

$$\| \delta(A) \xi \| \leq \| A \| \| T_\xi \| \text{ for each } A \in D(\delta) \text{ and } \xi \in D.$$

Hence, $\delta$ is extended to a quasi-bounded $*$-derivation of $M$ into $M^*_G$.

3. Let $\delta$ be a $(t_u \rightarrow t_\varphi)$-continuous $*$-derivation of $M$ into $L^1(D)\equiv L^1(D, D)$. If $D(M)$ is a finite dimensional subspace of $L^\infty(D)$, then $\delta$ is a quasi-bounded $*$-derivation of $M$ into $L^1(D, G)$.

4. Let $\delta$ be a $*$-derivation in a $C^*$-algebra $M$ acting on a Hilbert space $G$. If there exists a densely defined closed operator $T$ on $G$ such that $M D(T) \subset D(T)$ and $\delta$ is $(t_u \rightarrow t_\varphi^{(r)})$-continuous (or $(t_u \rightarrow t_\varphi^{(r)})$-continuous), then $\delta$ is extended to a quasi-bounded $*$-derivation of $M$ into $L^1(D(T), G)$. This follows immediately from Lemma 2.2.

As a slight generalization of Example 3.6, 4 we have the following result:

**Lemma 3.7.** Let $D$ be a countably dominated subspace in a Hilbert space $G$ by a sequence $\{ T_\alpha \}$ of closed operators on $G$. If $\delta$ is a $(t_u \rightarrow t_\varphi)$-continuous (or $(t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi)$, $(t_u \rightarrow t_\varphi^*), (t_u \rightarrow t_\varphi^*)$-continuous) $*$-derivation of $M$ into $L^1(D, G)$, then $\delta$ is quasi-bounded.
Proof. Suppose that $\delta$ is $(t_u \to t_u^n)$-continuous. By the continuity of $\delta$, $\delta(\mathcal{M})$ is a bounded subset of $(\mathcal{L}(\mathcal{D}, \mathcal{G}), t_u^n)$, where $\mathcal{M}$ is the unit ball of $\mathcal{M}$. It then follows from Lemma 2.4 that $\delta(\mathcal{M})$ is a bounded subset of the normed space $\mathcal{W}_{t_u^n}^{t_u^n}$ for some $n$. This implies that $\delta$ is quasi-bounded.

4. The spatiality of quasi-bounded $*$-derivations. Throughout this section we may assume that $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{G}$ and $\mathcal{M}$ is a unital $C^*$-algebra with $\mathcal{M} \subseteq \mathcal{D}$. Let $\delta$ be a quasi-bounded $*$-derivation of $\mathcal{M}$ into $\mathcal{L}(\mathcal{D}, \mathcal{G})$, i.e., there exists an element $T$ of $\mathcal{L}(\mathcal{D}, \mathcal{G})$ such that $T$ is a bounded subset of the normed space $\mathcal{W}_T^{t_u^n}$.

**Lemma 4.1.** Suppose that $\mathcal{W}$ is a subspace of $\mathcal{L}(\mathcal{D}, \mathcal{G})$. Then the following statements are equivalent.

1. $f$ is a $t_u^n$-continuous linear functional on $\mathcal{W}$;
2. $f$ is a $t_u^n$-continuous linear functional on $\mathcal{W}$;
3. $f = \sum_{i=1}^n \omega_i x_i$ for $\xi_i \in \mathcal{D}$ and $x_i \in \mathcal{G}$, where $\omega_i(x) = (A \xi_i | x)$ for $A \in \mathcal{L}(\mathcal{D}, \mathcal{G})$, $\xi \in \mathcal{D}$ and $x \in \mathcal{G}$.

**Proof.** This is proved in the same way as in ([1] Theorem 1.3).

Let $T \in \mathcal{L}(\mathcal{D}, \mathcal{G})$ and $T^{-1} \in \mathcal{B}(\mathcal{G})$. Then, by Lemma 2.1 $\mathcal{W}_T^{t_u^n} = \{AT^{-1}; A \in \mathcal{W}_T^{t_u^n}\}$ is a subspace of $\mathcal{B}(\mathcal{G})$. We denote by $\mathcal{W}_T^{t_u^n}$ the $t_u^n$-closure of $\mathcal{W}_T^{t_u^n}$ and denote by $\mathcal{M}_T^{t_u^n}$ the $t_u^n$-closure of $\mathcal{M}_T^{t_u^n}$ in $\mathcal{L}(\mathcal{D}, \mathcal{G})$. Then $\mathcal{W}_T^{t_u^n}$ is a weakly closed subspace of $\mathcal{B}(\mathcal{G})$ and $\mathcal{M}_T^{t_u^n}$ is $t_u^n$-closed subspace of $\mathcal{L}(\mathcal{D}, \mathcal{G})$. Furthermore, the following lemma is seen by a simple calculation.

**Lemma 4.2.** Let $\phi$ be the isomorphism of $\mathcal{M}_T^{t_u^n}$ onto $\mathcal{W}_T^{t_u^n}$ in Lemma 2.1. Then $\phi^{-1}$ is a continuous map of $(\mathcal{B}_T^{t_u^n}, t_u^n)$ onto $(\mathcal{M}_T^{t_u^n}, t_u^n)$, so that it is extended to a continuous linear map $\phi^{-1}$ of $(\mathcal{B}_T^{t_u^n}, t_u^n)$ onto $(\mathcal{M}_T^{t_u^n}, t_u^n)$.

**Lemma 4.3.** Let $\mathcal{K}$ be a subset of $\mathcal{M}_T^{t_u^n}$ and let $\mathcal{Q}$ be the $t_u^n$-closed convex hull of $\mathcal{K}$ in $\mathcal{L}(\mathcal{D}, \mathcal{G})$. If $\mathcal{K}$ and $\mathcal{K}^* = \{A^*; A \in \mathcal{K}\}$ are bounded in $\mathcal{M}_T^{t_u^n}$, where $A^*|\mathcal{D}$ is the restriction of $A^*$ to $\mathcal{D}$, then $\mathcal{Q}$ is a $t_u^n$-compact subset of $\mathcal{M}_T^{t_u^n}$.

**Proof.** Let $\mathcal{K}'$ be the convex hull of $\mathcal{K}$. Then $\mathcal{K}'$ and $(\mathcal{K}')^*$ are bounded in $\mathcal{M}_T^{t_u^n}$. Hence we may assume that $\mathcal{K}$ is convex. We first show that $\mathcal{Q}$ is a bounded subset of the normed space $\mathcal{M}_T^{t_u^n}$. By the boundedness of $\mathcal{K}$ and $\mathcal{K}^*$ there exists a constant $\gamma > 0$ such that $\|A\|_T \leq \gamma$ and $\|A^*\|_T \leq \gamma$ for all $A \in \mathcal{K}$. For each $S \in \mathcal{Q}$ there is a...
net \( \{A_n\} \) in \( B \) which converges to \( S \) with respect to the topology \( t_{q_w} \).

It then follows that for each \( \xi \in D \) and \( x \in \mathcal{G} \)

\[
|\left(S_\xi | x\right)| = \lim_{n \to \infty} |A_n \xi | x| | \\
\leq \lim_{n \to \infty} \|A_n \xi \| \|x\| \\
\leq \gamma \|T_\xi \| \|x\| ,
\]

so that \( \|S\|_\tau \leq \gamma \). Furthermore, for each \( \xi, \eta \in D \) we have

\[
|\left(S_\xi | \eta\right)| = \lim_{n \to \infty} |A_n \xi | \eta| | \\
\leq \lim_{n \to \infty} \|A_n \xi \| \|\xi\| \\
\leq \gamma \|T_\xi \| \|\xi\| .
\]

Hence, \( \eta \in D(S^*) \). Thus we have \( S \in \mathcal{M}_\tau^\delta \) and \( \|S\|_\tau \leq \gamma \).

We show that \( D \) is a \( t_{q_w} \)-compact subset of \( \mathcal{M}_\tau^\delta \). In fact, \( (\mathcal{B}_\tau^\delta)_\tau = \{X \in \mathfrak{B}_\tau^\delta; \|X\| \leq \gamma\} \) is weakly compact, and so Lemma 4.2 implies that \( \tilde{\phi}^{-1}(\mathfrak{B}_\tau^\delta)_\tau \) is \( t_{q_w} \)-compact in \( \mathcal{M}_\tau^\delta \). Since \( D \) is a \( t_{q_w} \)-closed subset of \( \tilde{\phi}^{-1}(\mathfrak{B}_\tau^\delta)_\tau \), it follows that \( D \) is a \( t_{q_w} \)-compact subset of \( \mathcal{M}_\tau^\delta \).

**Notation.** Let \( \mathcal{R}_\delta \) be a set \( \{U^*\delta(U); U \in \mathcal{M}_\delta\} \) and let \( \mathcal{D}_\delta \) be the \( t_{q_w} \)-closed convex hull of \( \mathcal{R}_\delta \) in \( L(D, \mathcal{G}) \).

**Lemma 4.4.** \( \mathcal{D}_\delta \) is a \( t_{q_w} \)-compact subset of \( \mathcal{M}_\tau^\delta \).

**Proof.** It is easily seen that \( \mathcal{R}_\delta \) and \( \mathcal{D}_\delta \) are bounded subsets of \( \mathcal{M}_\tau^\delta \). Hence, the lemma follows from Lemma 4.3.

Furthermore, one may easily see the following lemma.

**Lemma 4.5.** For each \( U \in \mathcal{M}_\delta \) we define

\[
A_U(S) = U^*SU + U^*\delta(U) \quad \text{for} \quad S \in L^\tau(D, \mathcal{G}) .
\]

Then,

1. \( A_U \) is a \( t_{q_w} \)-continuous affine map of \( L^\tau(D, \mathcal{G}) \) into \( L^\tau(D, \mathcal{G}) \);
2. \( A_U(V^*\delta(V)) = (VU)^*\delta(VU) \) for each \( U, V \in \mathcal{M}_\delta \);
3. \( A_U \mathcal{D}_\delta \subset \mathcal{D}_\delta \) for each \( U \in \mathcal{M}_\delta \);
4. \( A_U A_V = A_{UV} \) for each \( U, V \in \mathcal{M}_\delta \).

Hence, \( G_{\delta, \tau} = \{A_U; U \in \mathcal{M}_\delta\} \) is a semigroup of \( t_{q_w} \)-continuous affine maps of \( \mathcal{D}_\delta \) into \( \mathcal{D}_\delta \).

**Definition 4.6.** If for each pair of elements \( S_1 \neq S_2 \) in \( \mathcal{D}_\delta \) the \( t_{q_w} \)-closure of \( \{A_U(S_1) - A_U(S_2); U \in \mathcal{M}_\delta\} \) does not contain 0, then \( G_{\delta, \tau} \) is said to be noncontracting.
DEFINITION 4.7. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{G}$ and let $\mathcal{M}$ be a $C^*$-algebra acting on $\mathcal{G}$ with $\mathcal{M}\mathcal{D} \subset \mathcal{D}$. A $*$-derivation (resp. a derivation) $\delta$ of $\mathcal{M}$ into $\mathcal{L}^2(\mathcal{D}, \mathcal{G})$ (resp. $\mathcal{L}(\mathcal{D}, \mathcal{G})$) is said to be spatial if there exists an element $H$ of $\mathcal{L}^2(\mathcal{D}, \mathcal{G})$ (resp. $\mathcal{L}(\mathcal{D}, \mathcal{G})$) such that

$$\delta(A)\xi = [H, A]\xi \quad \text{for all } A \in \mathcal{M} \text{ and } \xi \in \mathcal{D}.$$ 

PROPOSITION 4.8. If $G_{s, T}$ is noncontracting, then there exists an element $S$ of $\mathcal{D}_z$ such that

$$\delta(A)\xi = [S, A]\xi \quad \text{for all } A \in \mathcal{M} \text{ and } \xi \in \mathcal{D} ;$$

that is, $\delta$ is spatial.

Proof. We consider the locally convex space $\mathcal{L}^\sigma = (\mathcal{L}^2(\mathcal{D}, \mathcal{G}), t^\sigma)$. By Lemma 4.1 we have $\sigma(\mathcal{M}, \mathcal{L}^\sigma) = t^\sigma$, and hence it follows from Lemmas 4.4, 4.5 that $\mathcal{D}_z$ is a weakly compact subset of $\mathcal{L}^\sigma$ and $G_{s, T}$ is a noncontracting semigroup of weakly continuous affine maps of $\mathcal{D}_z$ into $\mathcal{D}_z$. By Ryll-Nardzewski's fixed point theorem [9] there exists an element $S_0$ of $\mathcal{D}_z$ such that $A_\xi(S_0) = S_0$ for all $U \in \mathcal{M}$. Hence, putting $S = -S_0$, we have

$$\delta(A)\xi = [S, A]\xi \quad \text{for all } A \in \mathcal{M} \text{ and } \xi \in \mathcal{D}.$$ 

COROLLARY 4.9. Let $\mathcal{D}$ be a countably dominated subspace of a Hilbert space $\mathcal{G}$ and let $\mathcal{M}$ be a commutative $C^*$-algebra acting on $\mathcal{G}$ with $\mathcal{M}\mathcal{D} \subset \mathcal{D}$. Then there does not exist any nonzero (or $t^w$, $t^v$, $t^u$)-continuous $*$-derivation in $\mathcal{M}$.

Proof. Suppose that $\delta$ is a $*$-derivation which is continuous in one of the above topologies. It then follows from Lemma 3.3 that $\delta$ is extended to a quasi-bounded $*$-derivation $\delta$ of $\mathcal{M}$ into $\mathcal{M}'$ where $T \in \mathcal{L}^2(\mathcal{D}, \mathcal{G})$ and $T^{-1} \in \mathcal{B}(\mathcal{G})$. Since $\mathcal{M}$ is commutative, we can easily see that the semigroup $G_{s, T}$ is noncontracting. Hence it follows from Proposition 4.8 that there exists an element $H$ of $\mathcal{D}_z$ such that $\delta(A)\xi = [H, A]\xi$ for all $A \in \mathcal{M}$ and $\xi \in \mathcal{D}$. By Lemma 3.3 the elements $A$ and $H$ commute, and so $\delta = 0$.

LEMMA 4.10. Let $\mathcal{G}$ be the completion of a maximal Hilbert algebra $\mathcal{A}$ with identity $e$ and let $\mathcal{M}$ be the left von Neumann algebra of $\mathcal{A}$. Let $\mathcal{D}$ be a dense subspace of $\mathcal{G}$ such that $e \in \mathcal{D}$ and
\[ MD \subset D \] (for example, \( \mathcal{A} \) or the maximal unbounded Hilbert algebra \( L^2(\mathbb{S}) \) [5]). If \( \delta \) is a quasi-bounded *-derivation of \( \mathcal{M} \) into \( \mathcal{L}^*(D, \mathfrak{S}) \) such that \( \hat{\delta}(A)\mathcal{M} \) for each \( A \in \mathcal{M} \), then it is spatial.

**Proof.** Since \( \delta \) is quasi-bounded, there is an element \( T \) of \( \mathcal{L}^*(D, \mathfrak{S}) \) such that \( T^{-1} \in \mathcal{B}(\mathfrak{S}) \) and \( \delta(\mathcal{M}) \) is a bounded subset of the normed space \( \mathcal{M} \). It is easily showed that \( \mathcal{A} \subset D \) and \( SB' = B'S \) for all \( S \in \mathcal{S}_\delta, B' \in \mathcal{M}' \) and \( \xi \in \mathcal{A} \). This implies that \( G_{\delta,T} \) is non-contracting. In fact, for each pair of elements \( S_1 \neq S_2 \) in \( \mathcal{S}_\delta \) and \( U \in \mathcal{M} \), we have

\[
\| U^*(S_1 - S_2)Ue \| = \| (S_1 - S_2)\pi'(u)e \| \\
= \| \pi'(u)(S_1 - S_2)e \| \\
= \| (S_1 - S_2)e \| \\
\neq 0,
\]

where \( \pi(\text{resp. } \pi') \) is the left (resp. right) regular representation of \( \mathcal{A} \) and \( U = \pi(u) \) for \( u \in \mathcal{A} \). Hence it follows from Proposition 4.8 that \( \delta \) is spatial.

**Theorem 4.11.** Let \( \mathcal{M} \) be the left von Neumann algebra of a maximal Hilbert algebra \( \mathcal{A} \) with identity \( e, \mathfrak{S} \) the completion of \( \mathcal{A} \) and \( D \) be a countably dominated subspace of \( \mathfrak{S} \) by a sequence \( \{T_n\} \) of closed operators such that \( e \in D \) and \( MD \subset D \). If \( \delta \) is a \( (t_w \to t_w^\sigma) \)-continuous (or \( (t_w \to t_w^\tau), (t_w \to t_w^\sigma), (t_w \to t_w^\sigma) \)-continuous) *-derivation in \( \mathcal{M} \), then it can be extended to a spatial *-derivation \( \hat{\delta} \) of \( \mathcal{M} \) into \( \mathcal{L}^*(D, \mathfrak{S}) \).

**Proof.** This follows from Lemma 3.7 and Lemma 4.10.

We next examine the spatiality of derivations of \( \mathcal{M} \) into \( \mathfrak{M} \) when \( \hat{T}, \mathcal{M}' \) (or \( \hat{T}, \mathcal{M} \)).

Suppose that \( \delta \) is a derivation of \( \mathcal{M} \) into \( \mathfrak{M} \), where \( T \in \mathcal{L}_\delta(D, \mathfrak{S}) \) and \( T^{-1} \in \mathcal{B}(\mathfrak{S}) \). We set

\[
\delta_T(A) = \delta(A)T^{-1} \quad \text{for} \quad A \in \mathcal{M}.
\]

It then follows from Lemma 2.1 that \( \delta_T \) is a linear map of \( \mathcal{M} \) into \( \mathcal{B}(\mathfrak{S}) \), and so we have the following diagram:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\delta} & \mathfrak{M} \\
\uparrow & & \uparrow \\
\mathcal{A} & \xrightarrow{\delta_T} & \delta(A)T^{-1} \in \mathcal{B}(\mathfrak{S})
\end{array}
\]
Furthermore, we have the following result, by a simple calculation

**Lemma 4.12.** If \( T \in \mathcal{L}(\mathfrak{D}, \mathfrak{G}) \) and \( T^{-1} \in \mathfrak{M} \), then the linear map \( \delta_T \) is a derivation of \( \mathfrak{M} \) into \( \mathcal{B}(\mathfrak{G}) \).

**Definition 4.13.** A von Neumann algebra \( \mathfrak{M} \) on \( \mathfrak{G} \) is said to have the property (C) if every derivation \( \delta \) of \( \mathfrak{M} \) into \( \mathcal{B}(\mathfrak{G}) \) is inner; that is, \( \delta \) is implemented by an element of \( \mathcal{B}(\mathfrak{G}) \).

We note [3] that if \( \mathfrak{M} \) is of type I or properly infinite then \( \mathfrak{M} \) has the property (C).

**Proposition 4.14.** Let \( \mathfrak{D} \) be a dense subspace in a Hilbert space \( \mathfrak{G} \) and let \( \mathfrak{M} \) be a von Neumann algebra on \( \mathfrak{G} \) with the property (C) and \( \mathfrak{M} \mathfrak{D} \subset \mathfrak{D} \). If \( \delta \) is a \(*\)-derivation of \( \mathfrak{M} \) into \( \mathfrak{M}_T \) where \( T \in \mathcal{L}(\mathfrak{D}, \mathfrak{G}) \) and \( T^{-1} \in \mathfrak{M} \), then there exists an element \( B_0 \) of \( \mathcal{B}(\mathfrak{G}) \) such that

\[
\delta(A)\xi = [B_0 T, A] \xi
\]

for all \( A \in \mathfrak{M} \) and \( \xi \in \mathfrak{D} \), i.e., \( \delta \) is spatial.

**Proof.** By Lemma 4.12, \( \delta_T \) is a derivation of \( \mathfrak{M} \) into \( \mathcal{B}(\mathfrak{G}) \). Hence it follows by the assumption that there exists an element \( B_0 \) of \( \mathcal{B}(\mathfrak{G}) \) such that

\[
\delta_T(A) = [B_0, A] \quad \text{for all} \quad A \in \mathfrak{M}.
\]

This implies that

\[
\delta(A)\xi = [B_0 T, A] \xi \quad \text{for all} \quad A \in \mathfrak{M} \quad \text{and} \quad \xi \in \mathfrak{D}.
\]

**Theorem 4.15.** Let \( \mathfrak{M} \) be a von Neumann algebra on a Hilbert space \( \mathfrak{G} \) with the property (C) and let \( \delta \) be a \(*\)-derivation in \( \mathfrak{M} \). Suppose that there exists a countably dominated subspace \( \mathfrak{D} \) of \( \mathfrak{G} \) by a sequence \( \{T_n\} \) of closed operators \( T_n \tau \mathfrak{M}' \) such that \( \delta \) is \((t_n \to t_\infty)-\)continuous (or \((t_n \to t_\sigma), (t_\infty \to t_\sigma), (t_\infty \to t_\sigma)-\)continuous). Then there exists an element \( B_0 \) of \( \mathcal{B}(\mathfrak{G}) \) and a closed operator \( T_\delta \mathfrak{M}' \) such that

\[
\delta(A)\xi = [B_0 T, A] \xi \quad \text{for all} \quad A \in \mathfrak{D}(\delta) \quad \text{and} \quad \xi \in \mathfrak{D}.
\]

**Proof.** Since \( T_n^* \mathfrak{M}' \) for \( n = 1, 2, \ldots \), we have \( \mathfrak{M} \mathfrak{D} \subset \mathfrak{D} \). It follows from Lemma 3.3 that \( \delta \) is extended to a \((t_n \to t_\infty)-\)continuous \(*\)-derivation \( \tilde{\delta} \) of \( \mathfrak{M} \) into \( \mathcal{L}^*(\mathfrak{D}, \mathfrak{G}) \). Furthermore, by Lemma 2.6 \( \tilde{\delta} \) is quasi-bounded, i.e., \( \tilde{\delta}(\mathfrak{M}) \subset \mathfrak{M}_n \) for some \( n \). Hence the theorem follows from Proposition 4.14.
**Corollary 4.16.** Let $\mathfrak{A}$ be the completion of a Hilbert algebra $\mathfrak{A}$, $\mathfrak{M}$ the left von Neumann algebra of $\mathfrak{A}$ and let $J$ be the unitary involution on $\mathfrak{A}$. Suppose that $\mathfrak{M}$ has the property $(C)$ and there exists a countably dominated subspace $\mathfrak{D}$ of $\mathfrak{A}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathfrak{M}$ such that $J \mathfrak{D} = \mathfrak{D}$. If $\delta$ is a $(t_w \to t_{aw})$-continuous (or $(t_s \to t_{sw})$, $(t_{aw} \to t_{aww})$)-continuous $*$-derivation in $\mathfrak{M}$, then it is extended to spatial derivation $\delta$ of $\mathfrak{M}$ into $\mathfrak{L}(\mathfrak{D}, \mathfrak{S})$.

**Proof.** We put

$$T_n = J T_n J, \quad n = 1, 2, \ldots$$

It is then proved that $\mathfrak{D}$ is countably dominated by the sequence $\{T_n\}$ of closed operators $T_n \eta \mathfrak{M}$. Hence the corollary follows from Theorem 4.15.

**Proposition 4.17.** Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathfrak{G}$ and let $\delta$ be a $*$-derivation in $\mathfrak{M}$. If there exists a countably dominated subspace $\mathfrak{D}$ of $\mathfrak{G}$ by a sequence $\{T_n\}$ of closed operators $T_n \eta \mathfrak{M} \cap \mathfrak{M}'$ such that $\delta$ is $(t_w \to t_{aw})$-continuous, then $\delta$ is extended to a spatial $*$-derivation $\delta$ of $\mathfrak{M}$ into $\mathfrak{L}(\mathfrak{D}, \mathfrak{G})$.

**Proof.** By Lemma 3.3 and Lemma 2.6, $\delta$ is extended to a quasi-bounded $*$-derivation $\delta$ of $\mathfrak{M}$ into $\mathfrak{M}'$, where $T \in \mathfrak{L}(\mathfrak{D}, \mathfrak{G})$ and $T^{-1} \in \mathfrak{M} \cap \mathfrak{M}'$, satisfying $\delta(A^*)^* C \xi = C \delta(A) \xi$ for each $A \in \mathfrak{M}$, $C \in \mathfrak{M}'$ and $\xi \in \mathfrak{D}$. Since $\mathfrak{M} \mathfrak{D} \subset \mathfrak{D}$ and $\mathfrak{M}' \mathfrak{D} \subset \mathfrak{D}$, we have $\delta(A) \eta \mathfrak{M}$ for each $A \in \mathfrak{M}$. Since $T \in \mathfrak{M} \cap \mathfrak{M}'$, $\delta_T$ is a derivation of $\mathfrak{M}$ into $\mathfrak{M}$. Hence, there exists an element $B_0$ of $\mathfrak{M}$ such that

$$\delta_T(A) = [B_0, A] \quad \text{for each} \quad A \in \mathfrak{M},$$

so that

$$\delta(A) \xi = [B_0 T, A] \xi \quad \text{for all} \quad A \in \mathfrak{M} \quad \text{and} \quad \xi \in \mathfrak{D}.$$

**References**


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