PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

GENERIC SOUSLIN SETS

ARNOLD W. MILLER

By iterated forcing we create generic Souslin sets, which we use to answer questions of Ulam, Hansell, and Mauldin. For X a topological space a set $Y \subseteq X$ is analytic in X (also called Souslin in X or Σ_1^1 in X) iff there are Borel sets B_s for $s \in \omega^{<\omega}$ such that:

$$Y = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} B_{f \restriction n}\,.$$

For $X = 2^\omega$ (the Cantor space) a set $Y \subseteq X$ is analytic iff it is the projection of a Borel subset of $2^\omega \times 2^\omega$. Given $R \subseteq P(X)$ (the power set of X) let $B(R)$ be the smallest family of subsets of X including R and closed under countable union and complementation (i.e., the σ-algebra generated by R). If X is a topological space and R the family of open sets then $B(R)$ is the family of Borel subsets of X. The following question was raised by Ulam.

(1) Does there exist $R \subseteq P(2^\omega)$ such that R is countable and every analytic set in 2^ω is an element of $B(R)$?

Rothberger showed that assuming CH there is such a R. We will show that it is consistent with ZFC that there is no such R.

(2) Does there exist a separable metric space X in which every subset is analytic but not every subset is Borel?

This was raised by R. W. Hansell. Clearly CH implies no such X exists. We show that it is consistent with ZFC that such a X exists.

Let $R = \{A \times B\colon A, B \subseteq 2^\omega\}$, the abstract rectangles in the plane. Let $S(R)$ be the family of subsets of $2^\omega \times 2^\omega$ obtained by applying the Souslin operation to sets in $B(R)$. The next question was asked by D. Mauldin.

(3) Does $S(R) = P(2^\omega \times 2^\omega)$ imply $B(R) = P(2^\omega \times 2^\omega)$?

We show that the answer to this question is no.

Preliminaries. Recall the following definitions:

(1) $\omega = \{0, 1, 2, \cdots\}$ and $\forall n < \omega$, $n = \{m \mid m < n\}$;

(2) $\omega^n = \{s \mid s\colon n \to \omega\}$;

(3) for $s \in \omega^m$ and $n < \omega$, $s^\frown n$ is that $t \in \omega^{m+1}$ such that $t \restriction m = s$ and $t(m) = n$;

(4) ϕ denotes the empty sequence;

(5) $\omega^{<\omega} = \bigcup \{\omega^n\colon n < \omega\}$;

(6) $T \subseteqq \omega^{<\omega}$ is a tree iff $\forall s, t \in \omega^{<\omega}(s \subseteqq t \in T \to s \in T)$;

(7) T is a well founded tree iff $\forall f \in \omega^\omega \exists n < \omega f \restriction n \notin T$;

(8) for $s \in T$ a well founded tree $|s|_T$ is defined inductively by:

$$|s|_T = \sup\{|s^\frown n|_T + 1 : \exists n\, s^\frown n \in T\}\ ;$$

(9) for $\alpha < \omega_1$, T is a normal α-tree iff

(a) T is a well founded tree such that $|\phi|_T = \alpha$;

(b) if $s \in T$ and $|s|_T > 0$, then $\forall n\ s^\frown n \in T$;

(c) if $s \in T$ and $|s|_T = \beta + 1$, then $\forall n\, |s^\frown n|_T = \beta$;

(d) if $s \in T$ and $|s|_T = \lambda$ where λ is a limit ordinal, then $\forall \beta < \lambda$, $\{n : |s^\frown n|_T < \beta\}$ is finite (see [9]);

(10) for $T \subseteq \omega^{<\omega}$ a tree define:
$\boldsymbol{P}(T) = \{p \mid \exists F \in [T]^{<\omega},\ p : F \to 2,\ \forall n < \omega,\ \forall s \in \omega^{<\omega},$ if $s, s^\frown n \in F$, then $p(s) = 1$ implies $p(s^\frown n) = 0\}$, $\boldsymbol{P}(T)$ is ordered by inclusion.

(11) A notion of rank on a partial order $\boldsymbol{P}$ is a function whose domain is a subset of $\boldsymbol{P}$ and whose range is the ordinals. For α an ordinal and $p \in \boldsymbol{P}$, we let $|p| = \alpha$ mean that p is in the domain of this function and its value is α. The following property must be satisfied. For every $p \in \boldsymbol{P}$ and $\beta \geqq 1$, there exists $\hat{p} \in \boldsymbol{P}$ compatible with p such that $|\hat{p}| \leqq \beta$ and for every $q \in \boldsymbol{P}$ if $|q| < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.

(12) Given a notion of rank on $\boldsymbol{P}$ if τ is a term such that $\Vdash$ "$\tau \in 2^\omega$", then we say that $|\tau| = 0$ iff for any $p \in \boldsymbol{P}$ and $n < \omega$ there exists $q \in \boldsymbol{P}$ compatible with p such that $|q| = 0$ and $s \in 2^n$ such that $q \Vdash$ "$\check{s} \subseteq \tau$".

(13) For T a normal α-tree and $p \in \boldsymbol{P}(T)$ define $|p|$ to be the maximum $|s|_T$ for $s \in \operatorname{dom}(p)$.

(14) $T^* = \{s \in T : |s|_T = 0\}$.

The following lemma is key. It implies that $|p|$ is a rank on $\boldsymbol{P}(T)$.

LEMMA 1. $\forall \beta \geqq 1\ \forall p \in \boldsymbol{P}(T) \exists \hat{p} \in \boldsymbol{P}(T)$ *such that*

(a) p *and* $\hat{p}$ *are compatible;*

(b) $p \restriction T^* = \hat{p} \restriction T^*$;

(c) $|\hat{p}| \leqq \beta$;

(d) $\forall q \in \boldsymbol{P}(T)$ *if* $|q| < \beta$, *then* $\hat{p}$ *and* q *are compatible implies* p *and* q *are compatible.*

Proof. This is essentially Lemma 2 of [10]. We reprove it here for completeness. Let $F = \{s^\frown n : s \in \operatorname{dom}(p),\ p(s) = 1,\ |s|_T = \lambda$ a limit ordinal $> \beta$, and $|s^\frown n|_T < \beta\}$. By normality of T, F is finite, and $\forall t \in F$, $|t|_T \geqq 2$. Thus we can find $r \geqq p\ \forall t \in F \exists m\ t^\frown m \in \operatorname{dom}(r)$ and $r(t^\frown m) = 1$. Let $D = \{s \in \operatorname{dom}(r) : |s|_T \leqq \beta\}$ and $\hat{p} = r \restriction D$. p and $\hat{p}$ are compatible since r extends them both. $p \restriction T^* = \hat{p} \restriction T^*$ since $\forall t \in F \forall m\, |t^\frown m|_T \geqq 1$.

Now we check (d). Suppose $|q| < \beta$ and p and q are not compatible. Then there are $s \in \operatorname{dom}(p)$ and $t \in \operatorname{dom}(q)$ which demonstrate that $p \cup q$ is not a condition.

Case 1. $s = t$ and $p(s) \neq q(t)$. Since $|q| < \beta$ it follows $|t|_T < \beta$ and so $s \in \mathrm{dom}(\hat{p})$.

Case 2. $s = t^\frown m$ for some m and $p(s) = q(t) = 1$. But then $|s|_T < |t|_T < \beta$ and so again $s \in \mathrm{dom}(\hat{p})$.

Case 3. $t = s^\frown m$ for some m and $p(s) = q(t) = 1$. Since $|t|_T < \beta$ either $|s|_T \leqq \beta$ and so $s \in \mathrm{dom}(\hat{p})$ or $|s|_T = \lambda$ a limit ordinal $> \beta$ in which case $t \in F$ so there exists $n < \omega$ such that $r(t^\frown n) = 1$ and so $t^\frown n \in \mathrm{dom}(\hat{p})$ and so $\hat{p}$ and q are incompatible. In all three cases $\hat{p}$ and q are incompatible. □

The next lemma asserts the fact that statements of small rank should be forced by conditions of small rank. M is the ground model of ZFC and $\boldsymbol{P}$ is any partial order with a notion of rank.

LEMMA 2. *Let $B(r)$ be any $\boldsymbol{\Sigma}^0_\beta$ predicate with parameter in M, $1 \leqq \beta$, $\Vdash_P "\tau \in 2^\omega"$, $|\tau| = 0$, and $p \in \boldsymbol{P}$ such that $p \Vdash "B(\tau)"$. Then $\exists \hat{p} \in \boldsymbol{P}$, $|\hat{p}| < \beta$, p and $\hat{p}$ are compatible and $\hat{p} \Vdash "B(\tau)"$.*

Proof. The proof is by induction β.

Case 1. $\beta = 1$. Then $p \Vdash "\exists n R(\tau \restriction n, x \restriction n)"$ where R is primitive recursive and $x \in M \cap 2^\omega$. Find q extending p and $s \in 2^n$ for some n such that $q \Vdash "\tau \restriction n = \check{s}"$ and $R(s, x \restriction n)$ holds. By the definition of $|\tau| = 0$, $\exists \hat{p}$ compatible with q (and hence with p) such that $|\hat{p}| = 0$ and $\hat{p} \Vdash "\tau \restriction n = \check{s}"$. Thus $\hat{p} \Vdash "\exists n R(\tau \restriction n, x \restriction n)"$.

Case 2. β a limit ordinal. Then $p \Vdash "\exists n B_n(\tau)"$ where each $B_n(r)$ is a $\boldsymbol{\Sigma}^0_{\beta_n}$ predicate for some $\beta_n < \beta$. Let p_0 extend p such that $\exists n_0 < \omega\ p_0 \Vdash "B_{n_0}(\tau)"$. By induction $\exists \hat{p}$ compatible with p_0 (and hence with p) such that $|\hat{p}| < \beta_{n_0} < \beta$ and $\hat{p} \Vdash "B_{n_0}(\tau)"$ (and hence $\hat{p} \Vdash "\exists n B_n(\tau)"$).

Case 3. $\beta = \gamma + 1$ and $\gamma > 0$. As in Case 2 we may as well assume $p \Vdash "B(\tau)"$ where $B(r)$ is a $\boldsymbol{\Pi}^0_\gamma$ predicate. By Lemma 1, $\exists \hat{p} \in \boldsymbol{P}$, $\hat{p}$ and p compatible, $|\hat{p}| \leqq \gamma$, and $\forall q \in \boldsymbol{P}$ if $|q| < \gamma$ and q and $\hat{p}$ are compatible, then q and p are compatible. Then $\hat{p} \Vdash "B(\tau)"$. Otherwise $\exists r$ extending $\hat{p}$, $r \Vdash "\neg B(\tau)"$. Since $\neg B(r)$ is a $\boldsymbol{\Sigma}^0_\gamma$ predicate, by induction $\exists \hat{r} \in \boldsymbol{P}$, $|\hat{r}| < \gamma$, $\hat{r}$ and r compatible, and $\hat{r} \Vdash "\neg B(\tau)"$. But $\hat{r}$ and p are incompatible (since $p \Vdash "B(\tau)"$) and so by choice of

$\hat{p}$, $\hat{r}$ and $\hat{p}$ are incompatible a contradiction. □

Next we describe almost disjoint forcing (similar to the way it is done in [2]). Given $X = \{x_\alpha : \alpha < \omega_1\} \subseteqq 2^\omega$ distinct and $\langle Y_\alpha : \alpha < \omega_1 \rangle = Y$ where each $Y_\alpha \subseteqq \omega^{<\omega}$, we want to force a sequence of G_δ sets $\langle G_s : s \in \omega^{<\omega} \rangle$ such that $\forall s \forall \alpha (x_\alpha \in G_s \leftrightarrow s \in Y_\alpha)$. Let $\boldsymbol{B}$ be the family of all clopen subsets of 2^ω. Define $\boldsymbol{P}(X, Y)$ as follows:

it is the set of all r such that

(a) r is a finite subset of $\omega^{<\omega} \times \omega \times (\boldsymbol{B} \cup X)$;

(b) if $\langle s, n, B \rangle, \langle s, n, x_\alpha \rangle \in r$ then $x_\alpha \notin B$;

(c) if $\langle s, n, x_\alpha \rangle \in r$ then $s \notin Y_\alpha$.

As usual r extends p, $(r \geqq p)$ iff $r \supseteqq p$. It is well known that $\boldsymbol{P}(X, Y)$ satisfies the c.c.c. and also for any G which is $\boldsymbol{P}(X, Y)$-generic if we define $G_s = \bigcap_n \cup \{B : \{\langle s, n, B \rangle\} \in G\}$ then $\forall s \forall \alpha (x_\alpha \in G_s \leftrightarrow s \in Y_\alpha)$.

1. Forcing a Souslin set. We now describe how to force Souslin sets. Let M be our ground model of ZFC. Working in M let F^* be some standard fixed bijection between $\omega^{<\omega}$ and ω, and define $F: 2^\omega \to 2^{(\omega^{<\omega})}$ by $F(x)(s) = x(F^*(s))$. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be a fixed subset of 2^ω such that for all $\alpha < \omega_1$, $F(x_\alpha)$ is the characteristic function of a normal α-tree T_α. Let

$$\boldsymbol{P}_0 = \sum_{\alpha < \omega_1} \boldsymbol{P}(T_\alpha) ,$$

note that $\boldsymbol{P}_0$ has c.c.c. since it is equivalent to adding ω_1 Cohen reals. Note that any G which is $\boldsymbol{P}(T_\alpha)$-generic over M determines (and is determined by) a map $G_\alpha : T_\alpha \to 2$. $G_\alpha \upharpoonright T^*_\alpha$ in fact determines G_α by the rule $G_\alpha(s) = 1$ iff $\forall n \, G_\alpha(s\frown n) = 0$. Given G^0 $\boldsymbol{P}_0$-generic over the ground model M, let $G^0 = \langle G_\alpha : \alpha < \omega_1 \rangle$ and let $y_\alpha = \{s \in T^*_\alpha : G_\alpha(s) = 0\}$. Let $\boldsymbol{P}_1 = \boldsymbol{P}(X, Y)$ where $Y = \langle y_\alpha : \alpha < \omega_1 \rangle$. (So $\boldsymbol{P}_1 \in M[G^0]$.) Let $\boldsymbol{P} = \boldsymbol{P}_0 {}^* \boldsymbol{P}_1$.

Working in $M[G]$ for G $\boldsymbol{P}$-generic over M (so $G = (\langle G_\alpha : \alpha < \omega_1 \rangle, \langle G^s : s \in \omega^{<\omega} \rangle)$) let:

$$A = \{x_\alpha \in X : G_\alpha(\phi) = 1\} .$$

To see that A is analytic in X we will define $\hat{A}$ a $\boldsymbol{\Sigma}^1_1$ set such that $\hat{A} \cap X = A$. Define $x \in \hat{A}$ iff $\exists T \subseteqq \omega^{<\omega}$, $\exists p : \omega^{<\omega} \to 2$, $\exists T^* \subseteqq \omega^{<\omega}$ such that

(a) $F(x)$ is the characteristic function of T;

(b) T is a tree;

(c) $T^* = \{s \in T : \exists n \ s\frown n \notin T\} = \{s \in T : \forall n \ s\frown n \notin T\}$;

(d) $\forall s \in T^* \ p(s) = 1$ iff $x \in G_s$;

(e) $\forall s \in T - T^* \ p(s) = 1$ iff $\forall n \ p(s\frown n) = 0$;

(f) $p(\phi) = 1$.

(a) thru (f) are easily seen to be a Borel predicate of x, T, T^*, and p, and hence $\hat{A}$ is $\boldsymbol{\Sigma}_1^1$.

In order to show A is a new Souslin set we first want to extend our notion of rank to $\boldsymbol{P}$. Let $\boldsymbol{Q} = \{r \mid r$ satisfies (a) and (b) in the definition of $\boldsymbol{P}(X, Y)\}$ (thus $\boldsymbol{Q} \in M$). Then

$$\{(p, q)\colon p \in \boldsymbol{P}_0,\ q \in \boldsymbol{Q}\ , \quad \text{and} \quad p \Vdash \text{"}\check{q} \in \boldsymbol{P}(X, Y)\text{"}\}$$

ordered by $(\hat{p}, \hat{q}) \geqq (p, q)$ iff $\hat{p} \geqq p$ and $\hat{q} \geqq q$, is clearly dense in $\boldsymbol{P}$, so for simplicity assume it is $\boldsymbol{P}$. Let us unravel $p \Vdash \text{"}\check{q} \in \boldsymbol{P}(X, Y)\text{"}$. This means that whenever $\langle s, n, x_\alpha\rangle \in q'$ then $p \Vdash \text{"}s \notin Y_\alpha''$. But $p \Vdash$ "$s \notin Y_\alpha''$ iff $s \notin T_\alpha^*$ or ($s \in T_\alpha^*$, $s \in \operatorname{dom}(p_\alpha)$, and $p_\alpha(s) = 1$). The fact which we note is that if $p, p' \in \boldsymbol{P}_0$ and $\forall \alpha < \omega_1\ p_\alpha \restriction T_\alpha^* = p_\alpha' \restriction T_\alpha^*$, then $\forall r \in \boldsymbol{Q}\ \langle p, r\rangle \in \boldsymbol{P}$ iff $\langle p', r\rangle \in \boldsymbol{P}$.

For any $\alpha < \omega_1$, we define the following rank function on $\boldsymbol{P}$:

$$|(p, q)|_\alpha = \max\{|s|_{T_\gamma}\colon \gamma > \alpha \quad \text{and} \quad s \in \operatorname{dom}(p_\gamma)\}\ .$$

Note that the rank depends only on the part of the condition in $\boldsymbol{P}_0$. To see that it is a rank function, let (p, q) be any condition and $\beta \geqq 1$. For each $\gamma > \alpha$ by Lemma 1 $\exists \hat{p}_\gamma \in \boldsymbol{P}(T_\gamma)$ such that $\hat{p}_\gamma \restriction T_\gamma^* = p \restriction T_\gamma^*$, $\hat{p}_\gamma$ and p_γ are compatible, $|\hat{p}_\gamma| \leqq \beta$, and $\forall q \in \boldsymbol{P}(T_\gamma)$ if $|q| < \beta$ and $\hat{p}_\gamma$ and q are compatible, then p_γ and q are compatible. Let $\hat{p} \in \boldsymbol{P}_0$ be defined by:

$$\hat{p}_\gamma = \begin{cases} p_\gamma & \text{if} \quad \gamma \leqq \alpha \\ \hat{p}_\gamma & \text{if} \quad \gamma > \alpha\ . \end{cases}$$

By what we have already remarked

$$(\hat{p}, q) \in \boldsymbol{P},\ |(\hat{p}, q)|_\alpha \leqq \beta,\ (p, q) \text{ and } (\hat{p}, q) \text{ are compatible}\ ,$$
$$\forall (p', q') \in \boldsymbol{P} \text{ if } |(p', q')|_\alpha < \beta \text{ and}$$

(p', q') is compatible with $(\hat{p}, q)$, then (p', q') is compatible with (p, q).

Let G be $\boldsymbol{P}$-generic over M, and let A be the generic Souslin subset of X determined by G. We first show that $M[G] \models \text{"}A$ is not Borel in X". Suppose on the contrary that $\exists \tau$, $wB(v, \omega)a\boldsymbol{\Sigma}_\beta^0$ predicate with parameters in M, and $r \in \boldsymbol{P}$ such that

$$r \Vdash \text{"}\forall x \in X(x \in A \text{ iff } B(\tau, x))\text{"}\ .$$

By c.c.c. we can find $\alpha < \omega_1$ such that $|\tau|_\alpha = 0$, $|r|_\alpha = 0$, and $\beta < \alpha$. Let γ be any countable ordinal greater than $\alpha + \omega$. Extend $r = (p, q)$ by adding $p_\gamma(\phi) = 1$ to p, and call the result r_1. By this addition, $r_1 \Vdash \text{"}x \in A\text{"}$, so $r_1 \Vdash \text{"}B(\tau, x_\gamma)\text{"}$, so there exists r_2 compatible with r_1 such that $|r_2|_\alpha < \beta$ and $r_2 \Vdash \text{"}B(\tau, x_\gamma)\text{"}$. But since $\gamma > \alpha + \omega$ and $|r_2|_\alpha < \beta < \alpha$, it follows that $\exists r_3 \geqq r_2$ such that $p_\gamma^3(\phi) = 0$ and thus $\Vdash$ "$x_7 \notin A$". This is a contradiction since r_3 and r_1 are compatible (since r_2 and r_1 are compatible).

Now let us prove something a little stronger. Let $M \models$ "$H \subseteq P(X)$, $|H| \leqq \omega$", then, we claim $M[G] \models$ "$A \notin B(H)$ (the σ-algebra generated by H)".
Work in M. Let $H = \{A_n : n < \omega\}$ and define $K: X \to 2^\omega$ by $K(x)(n) = 1$ iff $x \in A_n$. Let Y be the range of K, then K has the property that it maps the σ-algebra generated by H into the Borel subsets of Y.
For any $C \in B(H)^{M[G]} \exists B$ Borel subset of Y, and $p \in \boldsymbol{P}$ such that

$$p \Vdash \text{"}\forall x \in X(x \in C \quad \text{iff} \quad K(x) \in B)\text{"} .$$

The preceding proof now goes through. Finally we are ready to state the theorem.

THEOREM 3. *It is consistent with* ZFC *that there does not exist* $H \subseteq P(2^\omega)$ *countable such that every analytic set is in the* σ-*algebra generated by* H.

Proof. Let M, X, and $\boldsymbol{P}$ be as above. Working in M let $\{\boldsymbol{P}_\alpha : \alpha < \omega_2^M\}$ be a set of isomorphic copies of $\boldsymbol{P}$. Force with $\Sigma\{\boldsymbol{P}_\alpha : \alpha < \omega_2^M\}$. Let $\langle G_\alpha . \alpha < \omega_2^M \rangle$ be generic over M. If $M[G_\alpha : \alpha < \omega_2^M] \models$ "$H \subseteq P(2^\omega)$, $|H| \leqq \omega$" then by c.c.c. $\exists \alpha_0 < \omega_2^M$ such that $\{B \cap X : B \in H\} \in M[G_\alpha : \alpha \neq \alpha_0]$. Let $M[G_\alpha : \alpha \neq \alpha_0]$ be the new ground model and $\hat{A}$ the analytic set created by $\boldsymbol{P}_{\alpha_0}$. Note that although $\boldsymbol{P}_{\alpha_0}$ is not the same as adding Cohen reals, because of its finite nature it is the same partial order whether defined in M or any extension of M (e.g., $M[G_\alpha : \alpha \neq \alpha_0]$). We have already noted that $\hat{A} \cap X$ is not in the σ-algebra generated by $\{B \cap X : B \in H\}$ and therefore $\hat{A}$ is not in the σ-algebra generated by H. $\square$

2. Making subsets generic Souslin sets. Let Σ be the set of countable successor ordinals greater than two. As in §1 let $X^* = \{x_\alpha : \alpha \in \Sigma\} \subseteq 2^\omega$ and $F: 2^\omega \to 2^{(\omega^{<\omega})}$ be the map such that $\forall \alpha \in \Sigma$, $F(x_\alpha)$ is a normal α-tree T_α. For $i = 0$ or 1 and $T \subseteq \omega^{<\omega}$ define:

$$\boldsymbol{P}^i(T) = \{p \in \boldsymbol{P}(T) : \exists \hat{p} \text{ an extension of } p,\ \hat{p}(\phi) = i\} .$$

It is easy to check that for any G which is $\boldsymbol{P}^i(T)$-generic over M, $G(\phi) = i$. Given $Z \subseteq \Sigma$ define $\boldsymbol{P}(Z)$ a suborder of $\boldsymbol{P}$ by $(p, q) \in \boldsymbol{P}(Z)$ iff $(p, q) \in \boldsymbol{P}$ and $\forall \alpha \in \Sigma$

(a) if $\alpha \in Z$ then $p_\alpha \in \boldsymbol{P}^0(T_\alpha)$;

(b) if $\alpha \notin Z$ then $p_\alpha \in \boldsymbol{P}^1(T_\alpha)$.

As before for G $\boldsymbol{P}(Z)$-generic over M, in $M[G]$, $\{x_\alpha : \alpha \in Z\}$ is analytic in X^*. The reason for Σ will be evident in the proof of Lemma 5.

THEOREM 4. *There exist a generic extension N of M such that $N \models$ "Every subset of X^* is analytic in X^* but some subset of X^* is not Borel in X^*".*

Proof. N will be obtained by iterating with finite support $\boldsymbol{P}(Z)$. Since each $\boldsymbol{P}(Z)$ is a relatively simple suborder of $\boldsymbol{P}$ we can give the following simpler definition. We assume $M \models "2^{\omega_1} = \omega_2"$. Let $\boldsymbol{Q} = \sum_{\alpha<\omega_2} \boldsymbol{P}_\alpha$ as in §1 and for $p \in \boldsymbol{Q}$ define $\mathrm{supp}(p) = \{\alpha < \omega_2 : p(\alpha) \neq 0\}$. Let A_α for $\alpha < \omega_2$ list with ω_2 repetitions all maps $A: \omega_1 \to [\boldsymbol{Q}]^{\leqq\omega}$. Inductively define $\boldsymbol{Q}_\alpha \subseteqq \boldsymbol{Q}$ for $\alpha < \omega_2$. For $\alpha = 0$ let $\boldsymbol{Q}_\alpha = \{p \in \boldsymbol{Q}$: $\mathrm{supp}(p) = \{0\}\}$ (i.e., $\boldsymbol{Q}_0 = \boldsymbol{P}$). For all $\alpha \boldsymbol{Q}_\alpha \subseteqq \{p \in \boldsymbol{Q}: \mathrm{supp}(p) \subseteqq \alpha\}$. For α a limit ordinal let $\boldsymbol{Q}_\alpha = \cup \{\boldsymbol{Q}_\beta : \beta < \alpha\}$. For $\alpha + 1$ let G_α be $\boldsymbol{Q}_\alpha$-generic over M and let $Z_\alpha = \{\beta \in \Sigma : A_\alpha(\beta) \cap G_\alpha \neq \phi\}$. Then

$$\begin{aligned} \boldsymbol{Q}_{\alpha+1} = \{p \in \boldsymbol{Q} \mid p \restriction \alpha \in \boldsymbol{Q}_\alpha,\ p \restriction \alpha \Vdash_{\boldsymbol{Q}_\alpha} "p(\alpha) \in \boldsymbol{P}(Z_\alpha)"\ , \\ \text{and} \quad \mathrm{supp}(p) \subseteqq \alpha + 1\}\ . \end{aligned}$$

(Of course by $p \restriction \alpha$ here we mean that condition in $\boldsymbol{Q}$ whose restriction to α is the same as p's and whose support is contained in α.)

Thus if G_{ω_2} is $\boldsymbol{Q}_{\omega_2}$ generic over M then $M[G_{\omega_2}] \models$ "Every subset of X^* is analytic in X^*". Work in M. Given $\alpha < \omega_1$ recall the definition $|p|_\alpha$ for $p \in \boldsymbol{P}$ given in §1. Given $K \subseteqq \omega_2$ and $\alpha < \omega_1$ define a map $F: \boldsymbol{Q}_{\omega_2} \to \alpha \cup \{\infty\}$ by $F(p) = \max\{|p(\delta)|_\alpha : \delta \in K\}$ if $\mathrm{supp}(p) \subseteqq K$ and the max is less than α, and otherwise let $F(p) = \infty$. Denote $F(p)$ by $|p|(K, \alpha)$. For suitably chosen K and α we will show $|p|(K, \alpha)$ is a rank function. Given $\Gamma \subseteqq \boldsymbol{Q}_{\omega_2}$ and θ a sentence we say Γ decides θ iff $\forall p \in \boldsymbol{Q}_{\omega_2} \exists q \in \Gamma$ p and q are compatible, and $q \Vdash "\theta"$ or $q \Vdash "\neg\theta"$.

LEMMA 5. *Suppose that $\forall \delta \in K \forall \beta < \alpha \{p \in \boldsymbol{Q}_\delta : |p|(K, \alpha) = 0\}$ decides $"\beta \in Z_\delta"$. Then $|p|(K, \alpha)$ is a rank function.*

Proof. We must show that given $p \in \boldsymbol{Q}_{\omega_2}$ and $1 \leqq \beta \leqq \alpha$ there exists $\hat{p} \in \boldsymbol{Q}_{\omega_2}$ compatible with p, $|\hat{p}|(K, \alpha) \leqq \beta$, and $\forall q \in \boldsymbol{Q}_{\omega_2}$ if $|q|(K, \alpha) < \beta$ and $\hat{p}$ and q are compatible, then p and q are compatible.

Recall that in the proof that $|\ \ |_\alpha$ is a rank function on $\boldsymbol{P}$ we obtained for each $p \in \boldsymbol{P}$ a $\hat{p} \in \boldsymbol{P}$ such that:

(a) $|\hat{p}|_\alpha \leqq \beta$;

(b) $\hat{p}$ and p are compatible;

(c) $\forall q \in \boldsymbol{P}$ if $|q|_\alpha < \beta$ and q and $\hat{p}$ are compatible, then q and p are compatible;

(d) $\forall\gamma < \alpha$, $\hat{p}(\gamma) = p(\gamma)$.

Given $p \in \boldsymbol{Q}_{\omega_2}$ define $\hat{p}$ by letting $\forall\delta \notin K$, $\hat{p}(\delta) = 0$ and $\forall\delta \in K$, $\hat{p}(\delta)$ is the condition in $\boldsymbol{P}$ obtained above for $p(\delta)$. We show that $\hat{p} \in \boldsymbol{Q}_{\omega_2}$. Suppose not and let δ be the least such that $\hat{p} \restriction \delta$ does not force "$\hat{p}(\delta) \in \boldsymbol{P}(Z_\delta)$". Clearly $\delta \in K$. Let $\hat{p}(\delta) = (p', q)$. Then there must be some $\gamma \in \Sigma$ such that $p'_\gamma \notin \boldsymbol{P}^0(T_\gamma)$ or $p'_\gamma \notin \boldsymbol{P}^1(T_\gamma)$, and $\hat{p} \restriction \delta$ does not force "$\gamma \notin Z''_\delta$" respectively "$\gamma \in Z''_\delta$". If $p'_\gamma \notin \boldsymbol{P}^0(T_\gamma)$ then $\phi \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\phi) = 1$. If $p'_\gamma \notin \boldsymbol{P}^1(T_\gamma)$ then either $\phi \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\phi) = 0$ or $\exists n < \omega$, $\langle n\rangle \in \mathrm{dom}(p'_\gamma)$ and $p'_\gamma(\langle n\rangle) = 1$. Since $\gamma \in \Sigma$ it is a successor ordinal. Since $|p(\delta)|_\alpha \leqq \beta < \alpha$ and $|\langle n\rangle|_{T_\gamma} \geqq \gamma - 1$ it must be that $\gamma < \alpha$. By the properties of K and α, $\exists q \in \boldsymbol{Q}_\delta$, $|q|(K, \alpha) = 0$, $q \Vdash$ "$\gamma \notin Z''_\delta$" (respectively "$\gamma \in Z''_\delta$"), and q is compatible with $\hat{p} \restriction \delta$. But since q is compatible with $\hat{p} \restriction \delta$, it is compatible with $p \restriction \delta$. This is a contradiction, since by (d) $q \Vdash$ "$\hat{p}(\delta) \notin \boldsymbol{P}(Z_\delta)$". □

If A is the analytic subset of X^* which is created at the first step, then A is not Borel in X^* in the model $M[G_{\omega_2}]$. To see this suppose not and $\exists p \in \boldsymbol{Q}_{\omega_2}$

$$p \Vdash \text{"}\forall x \in X^*(x \in A \text{ iff } x \in B_\tau)\text{"}$$

where B_τ is a $\boldsymbol{\Sigma}^0_\beta$ set with parameter $\tau \in 2^\omega$. Using the c.c.c. of $\boldsymbol{Q}_{\omega_2}$ it is easy to obtain $K \subseteqq \omega_2$ countable, $0 \in K$, and $\alpha < \omega_1$ with $\beta < \alpha$, such that $|p|(K, \alpha) = 0$, $|\tau|(K, \alpha) = 0$, and K and α satisfy the requirements set down in Lemma 5. As in §1 this leads to a contradiction.

3. Abstract Souslin sets. Recall that $R = \{A \times B: A, B \subseteqq 2^\omega\}$, $B(R)$ is the σ-algebra generated by R, and $S(R)$ the family of sets which are gotten by applying the Souslin operation to sets in $B(R)$.

THEOREM 6. *It is consistent with* ZFC *that* $S(R) = P(2^\omega \times 2^\omega) \neq B(R)$.

The model used will be a minor modification of the one obtained in §2.

LEMMA 7. *Suppose* $X \subseteqq 2^\omega$, $|X| = |2^\omega|$, *and every subset of* X *of cardinality less than* $|2^\omega|$ *is analytic in* X. *Then* $S(R) = P(2^\omega \times 2^\omega)$.

Proof. Let $\kappa = |2^\omega|$ and $X = \{x_\alpha: \alpha < \kappa\}$. Since $S(R)$ is closed under finite union, it is enough to show that any $Y \subseteqq \kappa^2$ with the property that $\langle\alpha, \beta\rangle \in Y \to \alpha \leqq \beta$, is in $S(R)$. For each β let $X_\beta = \{x_\alpha: \langle\alpha, \beta\rangle \in Y\}$. For each β and $s \in \omega^{<\omega}$ let C^β_s be a closed subset of X such that $X_\beta = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} C^\beta_{f \restriction n}$.

For each $s \in \omega^{<\omega}$ define $B_s = \{\langle \alpha, \beta \rangle : x_\alpha \in C_s^\beta\}$. Since $Y = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} B_{f \restriction n}$ it is enough to check that each $B_s \in B(R)$. Fix $s \in \omega^{<\omega}$ and let $\{D_n : n < \omega\}$ be an open basis for X. For each β define $y_\beta(n) = 1$ iff $D_n \cap C_s^\beta = \phi$. It follows that $\alpha \in C_s^\beta$ iff $\forall n$ (if $y_\beta(n) = 1$ then $\alpha \notin D_n$). Letting $E_n = (D_n \times X) \cup (D_n \times \{\beta : y_\beta(n) = 0\})$ we have that $B_s = \bigcap_{n<\omega} E_n$. □

LEMMA 8. *Suppose $F: X \to Y$ is $1 - 1$ and $\forall U$ open in Y $F^{-1}(U)$ is Borel in X. If every subset of Y is analytic in Y then every subset of X is analytic in X.*

Proof. Given $A \subseteq X$ let $B = F''A$. Then there are Borel subsets of Y, B_s for $s \in \omega^{<\omega}$ such that $B = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} B_{f \restriction n}$. Let $A_s = F^{-1}(B_s)$, then A_s is Borel in X and $A = \bigcup_{f \in \omega^\omega} \bigcap_{n<\omega} A_{f \restriction n}$. □

We now prove Theorem 2. Let M, the ground model of ZFC in §2, be a model of $MA + 2^\omega = \omega_2$. We first show that for G_{ω_2} $\boldsymbol{Q}_{\omega_2}$-generic over M, $M[G_{\omega_2}]$ models that $S(R) = P(2^\omega \times 2^\omega)$. Working in M for any $Z, W \subseteq 2^\omega$ with $|Z| = |W| = \omega_1$, if $F: Z \to W$ is any $1 - 1$ map then by Silver's lemma (see [6]) for every U open in W, $F^{-1}(U)$ is Borel in Z. F still has this property in any extension of M since W is second countable and M contains an open basis for W. Working in M there exists $X \subseteq 2^\omega$ such that $|X| = \omega_2$ and $\forall Y \subseteq X$ if $|Y| \leqq \omega_1$ then Y is Borel in X (a generalized Luzin set is such an example, see [9]). We claim that in $M[G_{\omega_2}]$ every subset of X of size $\leqq \omega_1$ is analytic in X and thus by Lemma 7, $S(R) = P(2^\omega \times 2^\omega)$. Working in $M[G_{\omega_2}]$ for any $Z \subseteq X$ if $|Z| \leqq \omega_1$ then $\exists Y \in M Z \subseteq Y$ and $|Y| \leqq \omega_1$. Letting $F: Y \to X^*$ be any $1 - 1$ map in M we have by Lemma 8 that every subset of Y is analytic in Y, and since Y is Borel in X, Z is analytic in X.

We next want to show that in $M[G_{\omega_2}]$, $P(2^\omega \times 2^\omega) \neq B(R)$. It is enough to show that in $M[G_{\omega_2}]$ there does not exist a countable $H \subseteq P(X^*)$ such that $B(H) = P(X^*)$. To see that this suffices let $\{X_\alpha : \alpha < \omega_2\} = P(X^*)$ and let $Y = \{(x, \alpha) : x \in X_\alpha\} \subseteq X^* \times \omega_2$. If Y is in the σ-algebra generated by $\{A_n \times B_n : n < \omega\}$ then $B(\{A_n : n < \omega\}) = P(X^*)$. Just show by induction that $\forall K \in B(\{A_n \times B_n : n < \omega\}) \forall \beta < \omega_2$ $\{x \in X^* : (x, \beta) \in K\} \in B(\{A_n : n < \omega\})$.

By the technique of §1 and §2 we note that in M there is no countable $H \subseteq P(X^*)$ such that the generic Souslin set created at the first step is in $B(H)$. Note that for $Z = \phi$ and G $\boldsymbol{P}(Z)$-generic over M the set $A = \{x_\alpha \in X^* : G_\alpha(\langle 0 \rangle) = 1\}$ is also a generic Souslin set over M. This is because the requirement that $G_\alpha(\phi) = 0$ puts no constraint on the value of $G_\alpha(\langle 0 \rangle)$. □

4. Remarks. (1) In the model used for Theorem 1 one can show that there does not exist any $H \subseteq P(2^\omega)$, $|H| < |2^\omega|$, such that every analytic subset of 2^ω is in $B(H)$. Note also that ω_2 can be replaced by any $\kappa > \omega_1$ of uncountable cofinality. Also in this model it is true that the universal Σ^1_1 subset of $2^\omega \times 2^\omega$ is not in the σ-algebra generated by the abstact rectangles.

(2) It is not hard to modify the technique of §2 to get it consistent with ZFC that $\exists X \subseteq 2^\omega |X| = \omega_2$ (or even $|X| = \aleph_{\omega_1}$) such that every subset of X is analytic in X but not every subset of X is Borel in X.

(3) X^* in §2 has Baire order ω_1 in $M[G_{\omega_2}]$.

(4) In [5] Kunen showed that if one adds ω_2 Cohen reals to a model of CH then $\{(\alpha, \beta): \alpha < \beta < \omega_2\}$ is not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_2, B \subseteq \omega_2\}$. In the same model (actually CH is not necessary in ground model) there is a subset of $\omega_1 \times \omega_2$ not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To prove this it is enough to find $F \subseteq P(\omega_1) |F| = \omega_2$ such that there does not exist $H \subseteq P(\omega_1)$ countable with $F \subseteq B(H)$. Let $\boldsymbol{P} = \{p \mid p: F \to 2$, for some $F \in [\omega_1]^{<\omega}\}$ and suppose G is $\boldsymbol{P}$-generic over M. Let

$$X = \{\alpha < \omega_1 \mid G(\alpha) = 1\}$$

and note that for any $H \subseteq P(\omega_1)$ countable and in M, $M[G] \Vdash$ "$X \notin B(H)$". This is because for any $Y \in B(H)$ $\exists t \in 2^\omega$ $Y \in M[t]$.

(5) In [12] Rothberger showed that $2^\omega = \omega_2 + 2^{\omega_1} = \aleph_{\omega_2}$ implies that not every subset of $\omega_1 \times \omega_2$ is in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To see this let G_α for $\alpha < \aleph_{\omega_2}$ list all countable subsets of $P(\omega_1)$. Since $|B(G_\alpha)| \leqq 2^\omega = \omega_2$ we can pick $K_\alpha \in P(\omega_1)$ for $\alpha < \omega_2$ such that $K_\alpha \notin \bigcup_{\beta < \omega_\alpha} B(G_\beta)$. It follows as in (4) that $\{(\beta, \alpha): \beta \in K_\alpha\}$ is not in the σ-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$.

References

1. R. H. Bing, W. W. Bledsoe and R. D. Mauldin, *Sets generated by rectangles*, Pacific J. Math., **51** (1974), 27-36.

2. W. Fleissner and A. W. Miller, *On Q-sets*, Proc. Amer. Math. Soc., **78** (1980), 280-284.

3. R. W. Hansell, *Some consequences of* $(V=L)$ *in the theory of analytic sets*, to Proc. Amer. Math. Soc., **80** (1980), 311-319.

4. ———, letter to the author, July 1979.

5. K. Kunen, *Inaccessibility properties of cardinals*, Doctoral Dissertation, Stanford University (1968).

6. D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic, **2** (1970), 143-178.

7. R. D. Mauldin, letter to author, March 1979.

8. ———, *On rectangles and countably generated families*, Fund. Math., **95** (1977), 129-139.

9. A. W. Miller, *On the length of Borel hierachies*, Ann. Math. Logic, **16** (1979), 233-267.
10. ———, *On generating the category algebra and the Baire order problem*, Bull. Acad. Polonaise, **27** (1979), 751-755.
11. F. Rothberger, *A remark on the existence of a denumerable base for a family of functions*, Canad. J. Math., **4** (1952), 117-119.
12. ———, *On families of real functions with a denumerable base*, Annals of Math., **45** (1944), 397-406.
13. S. Ulam, *Problem* 74, Fund. Math., **30** (1938), 365.

Received February 18, 1980 and in revised form July 28, 1980. Research partially supported by an NSF grant.

UNIVERSITY OF TEXAS
AUSTIN, TX 78712