PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 2, 1981

BAER RINGS AND QUASI-CONTINUOUS RINGS HAVE A MDSN

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The notion of a direct summand of a ring containing the set of nilpotents in some "dense" way has been considered by Y. Utumi, L. Jeremy, C. Faith, and G. F. Birkenmeier. Several types of rings including right self-injective rings, commutative FPF rings, and rings which are a direct sum of indecomposable right ideals have been shown to have a MDSN (i.e., the minimal direct summand containing the nilpotent elements). In this paper, the class of rings which have a MDSN is enlarged to include quasi-Baer rings and right quasi-continuous rings. Also, several known results are generalized. Specifically, the following results are proved: (Theorem 3) Let R be a ring in which each right annihilator of a reduced (i.e., no nonzero nilpotent elements) right ideal is essential in an idempotent generated right ideal. Then $R = A \oplus B$ where B is the MDSN and an essential extension of N_t (i.e., the ideal generated by the nilpotent elements of index two), and A is a reduced right ideal of R which is also an abelian Baer ring. (Corollary 6) Let R be an AW^*-algebra. Then $R = A \oplus B$ where A is a commutative AW^*-algebra, and B is the MDSN of R and B is an AW^*-algebra which is a rational extension of N_t. Furthermore, A contains all reduced ideals of R. (Theorem 12) Let R be a ring such that each reduced right ideal is essential in an idempotent generated right ideal. Then $R = A \oplus B$ where B is the densely nil MDSN, and A is both a reduced quasi-continuous right ideal of R and a right quasi-continuous abelian Baer ring.

From [8 & 14], a ring R is *(quasi-) Baer* if it has unity and the right annihilator of every (right ideal) nonempty subset of R is generated by an idempotent. A Baer ring is *abelian* if all its idempotents are central. The following examples will give some indication of the wide application of these rings: (i) von Neumann algebras, such as the algebra of all bounded operators on a Hilbert space, are Baer rings [2, pp. 21 & 24]; (ii) the commutative C^*-algebra $C(T)$ of continuous complex valued functions on a Stonian space is a Baer ring [2, p. 40]; (iii) the ring of all endomorphisms of an abelian group G with $G = D \oplus E$, where $D \neq 0$ is torsion-free divisible and E is elementary, is a Baer ring [16]; (iv) any right self-injective von Neumann regular ring is Baer [17, p. 253]; (v) any prime ring is quasi-Baer; (vi) since a $n \times n$ matrix ring over a quasi-Baer ring is quasi-Baer [15], the $n \times n$ $(n > 1)$ matrix ring over a non-Prufer commutative domain is a prime quasi-Baer ring which is not a Baer ring [14, p. 17]; (vii) since a $n \times n$ lower triangular matrix ring over a quasi-Baer ring is a quasi-Baer ring [15], the $n \times n$ $(n > 1)$ lower triangular matrix ring over a domain, which is not a division ring, is quasi-Baer but not Baer [14, p. 16]; (viii) semiprime right FPF rings are quasi-Baer [10, p. 168]. The examples show that the class of Baer rings does not contain the class of prime rings and is not closed under extensions to matrix rings or triangular matrix rings. However the notion of a quasi-Baer ring overcomes these shortfalls.

Throughout this paper, all rings are associative; R denotes a ring with unity; $N(X)$ is the set of nilpotent elements of X (N will be used when $X = R$) and N_t is the ideal generated by the nilpotent elements of index two. The word ideal will mean a two-sided ideal unless it is preceded by the words left or right. A *reduced* ring is one without nonzero nilpotent elements. Note that in a reduced ring all idempotents are central. A right ideal X of R is *densely nil* (DN) if either $X = 0$; or $X \neq 0$ and every nonzero right ideal of R which is contained in X has nonzero intersection with N. Equivalently, a right ideal X of R is densely nil if either $X = 0$; or $X \neq 0$ and for every nonzero $x \in X$ there exists $r \in R$ such that $xr \neq 0$ but $(xr)^2 = 0$. From [3], the minimal direct summand (idempotent generated right ideal) containing the nilpotent elements *MDSN* is a semicompletely prime ideal (i.e., if $x^n \in \text{MDSN} \Rightarrow x \in \text{MDSN}$) which equals the intersection of all idempotent generated right ideals containing the set of nilpotent elements of the ring. Also any nonzero direct summand of the MDSN has a nonzero nilpotent element, and the MDSN contains both the right singular ideal of R and the generalized nil radical [1 & 3]. There are nonreduced rings which do not have a MDSN [3, Example 2.6]. The complement of the MDSN is both a maximal reduced idempotent generated right ideal which is unique up to isomorphism and a reduced ring with unity.

LEMMA 1. *Let R be reduced. Then R is a quasi-Baer ring if and only if R is an abelian Baer ring. In particular, a commutative quasi-Baer ring is a reduced Baer ring.*

Proof. In a reduced ring every right annihilator is an ideal. Hence the right annihilator of any subset will equal the right annihilator of the right ideal generated by the subset. Therefore R is quasi-Baer if and only if R is Baer. Commutative quasi-Baer rings are reduced because a quasi-Baer ring contains no nonzero central nilpotent elements. This completes the proof.

LEMMA 2. *If X is an ideal of R such that $X \cap N_t \neq 0$, then X contains a nonzero nilpotent element of index two.*

Proof. From [1 & 9], we have that the generalized nil radical N_g is hereditary and contains N_t. Thus, if $X \cap N_t \neq 0$ then $X \cap N_g(R) = N_g(X) \neq 0$ contains a nonzero nilpotent element of index two.

A right R-module B is an *essential extension* of a submodule X (equivalently, X is essential in B) in case for every submodule L of B, $X \cap L = 0$ implies $L = 0$. The next theorem presents a decomposition in terms of a reduced Baer ring and the MDSN, furthermore the MDSN is "essentially" generated by the nilpotents of index two.

THEOREM 3. *Let R be a ring in which each right annihilator of a reduced right ideal is essential in an idempotent generated right ideal. Then $R = A \oplus B$ where B is the* MDSN *and an essential extension of N_t, and A is a reduced right ideal of R which is also an abelian Baer ring.*

Proof. Let $X = \bigoplus_{i \in I} e_i R$ be maximal among reduced direct sums of idempotent generated right ideals [3, p. 714]. Let Y be the right annihilator of X. From [3, Prop. 1.2], $XN = 0$, so $N \subseteq Y$. Thus, if $Y = 0$ then $R = A$. If $Y \neq 0$ then there exists $y = y^2$ such that yR is an essential extension of Y. By [3, Prop. 1.2], yR is an ideal. Then $Y = yR$ because $X(yR) \subseteq X \cap yR = 0$ since X is reduced. To show that yR is the MDSN, let $0 \neq t = t^2 \in yR$. Assume $tR \cap N = 0$. By the maximality of X, $(X \oplus tR) \cap N \neq 0$. Hence there exists $x \in X$ and $c \in tR$ such that $0 \neq x + c$ and $(x + c)^2 = 0$. Thus $(x + c)^2 = x^2 + xc + cx + c^2 = x^2 + cx + c^2 = 0$. Then $x^2 = (-cx - c^2) \in X \cap Y = 0$. Hence $x = 0$ and $c = 0$ because X and tR are reduced. Contradiction! Therefore $yR = B$ is the MDSN [3, Thrm. 1.4] and $(1 - y)R = A$ is a reduced ring with unity.

Let $0 \neq s \in B$ and assume $sR \cap N_t = 0$. Hence $(sR)N_t = 0$. Therefore N_t is contained in the right annihilator of sR. There exists $e = e^2$ such that eR is an essential extension of the right annihilator of sR. Hence $N_t \subseteq eR$. Thus $B \subseteq eR$, since B is the MDSN [3, Thrm. 1.4]. But this is a contradiction because sR, which is reduced, cannot be contained in eR. Therefore N_t is essential in B.

Let D be a right ideal of A. By [3, Lem. 1.1], D is a right ideal of R. Let U be the right annihilator of D in R. There exists $u = u^2$ such that uR is an essential extension of U. Now $(1 - y)U$ is the right annihilator of D in A, and $(1 - y)uR$ is an essential extension of $(1 - y)U$ in A with $((1 - y)u)^2 = (1 - y)u$. Thus $D \cap (1 - y)uR = 0$. Since A is reduced $(1 - y)u$ is a central idempotent in A. Hence $(1-y)uR$ is an ideal in A. Thus $D((1-y)uR) \subseteqq D \cap (1-y)uR = 0$. Therefore $(1-y)U = (1-y)uR$. Hence A is a quasi-Baer ring. By Lemma 1, A is an abelian Baer ring. In fact, any idempotent generated reduced right ideal of R is an abelian Baer ring. This completes the proof.

From Lemma 2, it can be seen that every nonzero ideal of R contained in B has nonzero nilpotent elements. Hence every reduced ideal of R is contained in A. Also, we note that commutative FPF rings (e.g., integers mod 12) are not necessarily quasi-Baer but satisfy the hypothesis of Theorem 3 [10, p. 168 Lem. 3A]. In fact, Theorem 3 is a generalization of the decomposition for commutative FPF rings obtained by C. Faith [10, p. 184].

COROLLARY 4. *Let R be a quasi-Baer ring. Then $R = A \oplus B$ where B is the* MDSN *and an essential extenison of N_t, and A is a reduced right ideal of R which is also an abelian Baer ring.*

Letting R be the 2×2 lower triangular matrix ring over a domain (e.g., example vii) and applying Corollary 4, we have

$$A = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} R \quad \text{and} \quad B = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} R\,.$$

A right R-module B is a *rational extension* of a submodule X if whenever $x, y \in B$ with $x \neq 0$, there exists an element $r \in R$ such that $xr \neq 0$ and $yr \in X$. Note that if B is a rational extension of X then B is an essential extension of X. A *MDSN ring* is a ring which equals its MDSN.

COROLLARY 5. *Let R be a semiprime (quasi-)Baer ring. Then $R = A \oplus B$ (ring direct sum) where B is a* MDSN *(quasi-)Baer ring which is a rational extension of N_t, and A is a reduced abelian Baer ring.*

Proof. From Corollary 4, $R = A \oplus B$ where B is the MDSN. Let $B = bR$ where $b = b^2$ and $A = (1-b)R$. Then $bR(1-b) = 0$ because R is semiprime. Since $1-b$ is a unity for A, it follows that $1-b$ is central in R. Hence A and B are rings with unity. Let $0 \neq x \in B$. Then $xR \cap N_t \neq 0$ since N_t is essential in B. Now $xN_t \neq 0$ because R is semiprime. Let $r \in N_t$ such that $xr \neq 0$. Then, for any $y \in B$, $yr \in N_t$. Hence B is a rational extension of N_t. This completes the proof.

Corollaries 4 and 5 generalize Proposition 5.7 of [17, p. 255] and Rangaswamy's decomposition of a von Neumann regular Baer ring which is the endomorphism ring of an abelian group [16]. Furthermore, according to Kaplansky's classification of Baer rings [14], types II and III and semiprime type I_{inf} are MDSN rings. Hence reduced Baer rings are type I_{fin}.

Three important classes of semiprime quasi-Baer rings are the class of semiprime right FPF rings [10, p. 168], the class of von Newmann regular Baer rings (e.g., Examples (i), (iii) and (iv)) and the class of Baer *-rings (i.e., a ring with involution *, such that the right annihilator of any subset is generated by a projection [14, p. 27]). A useful subclass of Baer *-rings are the AW^*-algebras (i.e., a C^*-algebra which is Baer * [2, p. 21]). Examples (i) and (ii) are AW^*-algebras [2, pp. 21, 24, 40]. From Lemma 1 and [14, p. 10], we note that an AW^*-algebra is commutative if and only if it is reduced. Thus for AW^*-algebras we have

COROLLARY 6. *Let R be an AW^*-algebra. Then $R = A \oplus B$ (ring direct sum) where A is a commutative AW^*-algebra and B is a* MDSN *AW^*-algebra which is a rational extension of N_t. Furthermore, A contains all reduced ideals of R.*

We remark that a quasi-Baer ring which is a rational extension of N_t is not necessarily *DN*. Let R be a 2×2 matrix ring over a domain which is not a left Ore domain. By [15], R is a prime quasi-Baer ring; and from [5] R is MDSN. Hence, by Corollary 5, R is a rational extension of N_t. However, R is not *DN* [5].

The next lemma and theorem show that under mild finiteness conditions (e.g., no infinite direct sums of index two nilpotent left ideals or no infinite direct sums of reduced subrings with unity) a nonsemiprime (quasi-)Baer ring can be decomposed in terms of a reduced Baer ring, a nilpotent ring and a (quasi-) Baer MDSN ring.

LEMMA 7. *Let R be a ring which is neither reduced nor* MDSN *and in which at least one of the following conditions holds:*

(i) *R is a direct sum of indecomposable right ideals;*

(ii) *R has no infinite sets of orthogonal idempotents $\{e_1, e_2, \cdots\}$ such that each of e_iRe_i is reduced;*

(iii) *R has no infinite direct sums of nilpotent left ideals of the form bRa where a and b are orthogonal idempotents with aRa reduced and the ring eRe has a* MDSN *for every nonzero idempotent $e \in R$.*

Then there exists a positive integer n such that for each $k = 1, \cdots, n$ there is an idempotent b_k where $Rb_k = b_kRb_k$, a reduced ring S_k with unity, and a left ideal X_k of R such that $X_k^2 = 0$. Also, Rb_{k-1} is ring isomorphic to

$$\begin{vmatrix} S_k & 0 \\ X_k & Rb_k \end{vmatrix}$$

with $b_0 = 1$ *and* $X_k \oplus Rb_k$ *(left ideal direct sum) is the* MDSN *of* Rb_{k-1}. *Consequently,* $R = S \oplus X \oplus Rb_n$ *(additive group direct sum) where* $S = \bigoplus_{k=1}^{n} S_k$ *(ring direct sum) is a reduced ring,* $X = \bigoplus_{k=1}^{n} X_k$ *is a nilpotent left ideal of* R, *and* Rb_n *is a reduced ring with unity or a* MDSN *ring with unity.*

Proof. The proof of the lemma using condition (i) is in [4]. Suppose that condition (ii) holds. Then by [3, Thrm. 2.4] R has a MDSN. Hence there exists $e = e^2$ such that $R = eR \oplus (1 - e)R$ where $e \neq 0$ and $e \neq 1$, and $eR = eRe$ is a reduced ring with unity and $(1 - e)R$ is the MDSN of R. Let $b_1 = 1 - e$, $X_1 = b_1 \mathrm{Re}$, and $S_1 = eR$. By [4, Lem. 1] $R = S_1 \oplus X_1 \oplus Rb_1$ (additive group direct sum), X_1 is a left ideal of R such that $X_1^2 = 0$, $Rb_1 = b_1Rb_1$ is a ring with unity, and $X_1 \oplus Rb_1 = b_1R$ (left ideal direct sum) where b_1R is the MDSN of R. If Rb_1 is reduced or if Rb_1 is MDSN then we are finished; otherwise we will continue the decomposition with $R_1 = b_1Rb_1 = Rb_1$. Observe if $a = a^2 \in R_1$ then $aR_1a = a(b_1Rb_1)a = aRa$ since b_1 is the unity of R_1. Thus R_1 has no infinite sets of orthogonal idempotents $\{a_1, a_2, \cdots\}$ such that each $a_iR_1a_i$ is reduced. Again by [3, Thrm. 2.4], R_1 has a MDSN. Hence there exists $e_1 = e_1^2$ such that $e_1R_1 = e_1R_1e_1 = e_1Re_1$ is a reduced ring with unity and $(b_1 - e_1)R_1$ is the MDSN of R_1. Let $b_2 = b_1 - e_1$, $S_2 = e_1R_1$, and $X_2 = b_2R_1e_1 = b_2Re_1$. Note $\{e, e_1, b_2\}$ is a set of orthogonal idempotents of R. From [4, Lem. 1] $b_2R_1b_2 = R_1b_2 = Rb_2 = b_2Rb_2$ is a ring with unity and thus $R = S_1 \oplus X_1 \oplus S_2 \oplus X_2 \oplus Rb_2$ (additive group direct sum) where $S_1 \oplus S_2$ is a ring direct sum of reduced rings with unity. Since b_1R is an ideal in R, it can be shown that $r = ere + b_1re + b_1rb_1$ for $r \in R$. Thus $rX_2 = (b_1rb_1)X_2 \subseteq X_2$ because X_2 is a left ideal of R_1 [4, Lem. 1]. Therefore X_2 is a left ideal of R. Hence $X_1 \oplus X_2$ is a nilpotent left ideal of R. Also $X_2 \oplus Rb_2$ (left ideal direct sum) is the MDSN of R_1. If Rb_2 is reduced or if Rb_2 is MDSN, then we are finished; otherwise, we will continue with $R_2 = Rb_2$. Since $\{e, e_1\}$ is a set of orthogonal idempotents such that eRe and e_1Re_1 are reduced, one can see that the above procedure will terminate after, say, n steps. Thus it follows that R has the desired group direct sum decomposition. The triangular matrix characterization for Rb_{k-1} follows from [4, Lem. 1].

The proof of the theorem using condition (iii) is similar to the above proof except that "the procedure will terminate after, say, n steps because there can be only finitely many X_i." This completes the proof.

COROLLARY 8. *Let* $R = S \oplus X \oplus Rb_n$ *as in Lemma* 7. *Then* $N(R) = \{k \in R \mid k = x + y$ *where* $x \in X$ *and* $y \in N(Rb_n)\}$, *and* $N(R)$ *is an ideal of* R *if and only if* $N(Rb_n)$ *is a left ideal of* Rb_n. *Furthermore, if* $N(R)$ *is an ideal of* R *then* $N(R) = X \oplus N(Rb_n)$ (*direct sum of left ideals of* R).

Proof. From repeated use of [4, Lem. 3], $N(R) = \{k \in R \mid k = x + y$ where $x \in X$ and $y \in N(Rb_n)\}$. Suppose $N(R)$ is an ideal of R. $N(Rb_n) = N(R) \cap Rb_n$. Thus $N(Rb_n)$ is a left ideal of R. Hence $N(Rb_n)$ is a left ideal of Rb_n, in fact $N(Rb_n)$ is an ideal of Rb_n; and $N(R) = X \oplus N(Rb_n)$ (direct sum of left ideals of R).

Conversely, assume $N(Rb_n)$ is a left ideal of Rb_n. From the triangular representation of Rb_k, it follows that a left ideal of Rb_n is also a left ideal of R. Hence $N(Rb_n)$ is a left ideal of R. Thus $N(R) = X \oplus N(Rb_n)$ (direct sum of left ideals of R). Consequently $N(R)$ is a left ideal of R, hence $N(R)$ is an ideal of R. This completes the proof.

We note that if Rb_n is reduced then $N(R) = X$ is a ideal of R.

THEOREM 9. *Let* R *be a (quasi-)Baer ring which is neither reduced nor* MDSN *and in which at least one of the following conditions holds*:

(i) *R is a direct sum of indecomposable right ideals*;

(ii) *R has no infinite sets of orthogonal idempotents* $\{e_1, e_2, \cdots\}$ *such that each* e_iRe_i *is reduced*;

(iii) *R has no infinite direct sums of nilpotent left ideals of the form* bRa *where* a *and* b *are orthogonal idempotents with* aRa *reduced.*

Then there exists a positive integer n *such that for each* $k = 1, \cdots, n$ *there is an idempotent* b_k *where* $Rb_k = b_kRb_k$, *a reduced Baer ring* S_k, *and a left ideal* X_k *of* R *such that* $X_k^2 = 0$. *Also* Rb_{k-1} *is ring isomorphic to*

$$\begin{vmatrix} S_k & 0 \\ X_k & Rb_k \end{vmatrix}$$

with $b_0 = 1$ *and* $X_k \oplus Rb_k$ (*left ideal direct sum*) *is the* MDSN *of* Rb_{k-1}. *Consequently,* $R = S \oplus X \oplus Rb_n$ (*additive group direct sum*) *where* $S = \bigoplus_{k=1}^n S_k$ (*ring direct sum*) *is a reduced Baer ring,* $X = \bigoplus_{k=1}^n X_k$ *is a nilpotent left ideal of* R, *and* Rb_n *is a reduced Baer ring or* Rb_n *is a* MDSN (*quasi-*)*Baer ring. Furthermore,* $N(R)$ *is an ideal of* R *if and only if* $N(Rb_n)$ *is a left ideal of* Rb_n.

Proof. The proof follows from Lemma 7, Corollary 4, Corollary

8 and the fact that if $e = e^2$ and R is (quasi-)Baer then eRe is (quasi-) Baer [8] and [14, p. 6].

By observing that an indecomposable reduced Baer ring is a domain, it follows that if R satisfies condition (i) or (ii) of Theorem 9, then each S_k is a ring direct sum of domains; consequently S is a ring direct sum of domains.

From [13], a right R-module M is *quasi-continuous* (also known as π-injective [11]) if it satisfies the following two conditions:

(i) each submodule of M is essential in a direct summand of M.

(ii) if P and Q are direct summands of M such that $P \cap Q = 0$, then $P \oplus Q$ is a direct summand of M.

From [7], a right R-module M is a *CS module* if and only if each complement submodule of M is a direct summand of M, equivalently each submodule of M is essential in a direct summand of M (i.e., condition (i) in the definition above). A ring is right (quasi-continuous) CS if it is (quasi-continuous) CS as a right R-module [6]. Right self-injective rings and products of right Ore domains are right quasi-continuous [13 & 19]. A $n \times n (n > 1)$ lower triangular matrix ring over a field is a right CS ring which is not a right quasi-continuous ring.

PROPOSITION 10. *Let R be a semiprime ring such that the right annihilator of every ideal is essential in an idempotent generated right ideal of R. Then R is quasi-Baer.*

Proof. Let X be an ideal of R and Y is the right annihilator of X. Then $X \cap Y = 0$ since R is semiprime. Let eR be an essential extension of Y with $e = e^2$. Hence $X \cap eR = 0$. Now $(XeR)^2 = (XeR)(XeR) = X(eRX)eR = 0$. Thus $X(eR) = 0$. Therefore $Y = eR$. By [8, Lem. 1], R is quasi-Baer.

COROLLARY 11. *Let R be a semiprime ring. If R is a CS ring or a right quasi-continuous ring, then R is a quasi-Baer ring.*

Corollary 11 has no converse since a domain which is not a right Ore domain is quasi-Baer, but such a domain is not a right CS ring. Also the semiprime condition is necessary since the integers mod 4 form a quasi-continuous ring which is not quasi-Baer.

The next theorem and corollaries generalize several results [3, Thrm. 3.9], [11, Thrm. 1.15], and [13, Prop. 5.2 & Prop. 5.5].

THEOREM 12. *Let R be a ring such that each reduced right ideal is essential in an idempotent generated right ideal. Then $R = A \oplus B$*

where B is the densely nil MDSN, *and A is both a reduced quasi-continuous right ideal of R and a right quasi-continuous abelian Baer ring.*

Proof. By Zorn's lemma there exists a maximal reduced right ideal K. From [13, Lem. 5.1], any right ideal which is an essential extension of K is also reduced. From the maximality of K, $aR = K$ where $a = a^2$. By [3, Prop. 1.2 & Prop. 1.7], $A = K$ and $B = (1 - a)R$ is the densely nil MDSN. Since every right ideal of A is a right ideal of R [3, Lem. 1.1] we need only show that A is a right quasi-continuous ring. We note that part (ii) of the definition of a quasi-continuous module is satisfied since every idempotent is central in a reduced ring. To show part (i) of the definition of a quasi-continuous module, let X be a nonzero right ideal of A. Then there exists $e = e^2$ such that X is essential in eR as an R-module. Now $X \subseteq aeR$ and $(ae)^2 = ae \in A$. Let $0 \neq aer \in aeR$. Then $er \neq 0$, hence there exists $s \in R$ such that $0 \neq ers \in X \subseteq A$. Thus $ers = aers = (aer)(as)$ since a is a unity for A. Therefore X is essential in aeR as an A-module and as an R-module. From Lemma 1 and Corollary 11, A is a reduced abelian Baer ring. This completes the proof.

COROLLARY 13. *Let R be a right (quasi-continuous) CS ring. Then $R = A \oplus B$ where B is the (quasi-continuous) CS densely nil* MDSN, *and A is both a quasi-continuous reduced right ideal of R and a right quasi-continuous abelian Baer ring*

From the proof of Theorem 12, one can see that if the reduced right ideals of a ring are essential in idempotent generated right ideals then every idempotent generated reduced right ideal of R is a quasi-continuous reduced abelian Baer ring. Furthermore, A contains every reduced ideal of R since B is DN. Also, any condition on R (such as semiprime) which forces $(1 - a)Ra = 0$ will make the decomposition a ring decomposition. From Corollary 11 and Corollary 13 we have:

COROLLARY 14. *Let R be a semiprime right (quasi-continuous) CS ring. Then $R = A \oplus B$ where A is a right quasi-continuous reduced abelian Baer ring, and B is a (quasi-continuous) CS densely nil quasi-Baer ring.*

Any $n \times n (n > 1)$ lower triangular matrix ring over the integers provides an example for Theorem 9 and Theorem 12 which is a quasi-Baer ring but not a right CS ring [6, p. 73]. Example (iv) satisfies the hypothesis of Corollary 14. Also, from [12, Thrm. 2.3] and [13, p. 219], any strongly modular Baer*-ring is a semiprime quasi-continuous ring and thus satisfies the hypotheses of Corollaries 5 and 14. In particular, any finite AW^*-algebra (e.g., example (ii) is a strongly modular Baer*-ring [12, p. 14].

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Received July 10, 1979 and in revised form September 29, 1980.

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