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QUASIDIAGONAL WEIGHTED SHIFTS

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We characterize the quasidiagonality of a two-way weighted shift solely in terms of the weights. We use the characterization to show that quasidiagonality fails to be an invariant for similarity: an operator similar to the (unweighted) bilateral shift may fail to be quasidiagonal.

Introduction. We given necessary and sufficient conditions for the quasidiagonality of a two-way weighted shift. We use the characterization to show that quasidiagonality fails to be invariant for similarity.

A (bounded) operator A on a separable Hilbert space is *quasidiagonal* if there exists a sequence $\{P_n\}_1^\infty$ of orthogonal projections of finite rank such that $\{P_n\}$ converges strongly to 1 and $\{P_nA - AP_n\}$ converges uniformly to 0. The only facts about quasidiagonal operators that we use in this paper are that each normal operator is quasidiagonal and that each compact perturbation of a quasidiagonal operator is again quasidiagonal. Both facts are proved in [6, § 4], where the concept was introduced.

Throughout this paper $\{e_i\}_{-\infty}^{+\infty}$ is a fixed basis for a complex Hilbert space, and $\{w_i\}_{-\infty}^{+\infty}$ is a fixed, bounded sequence of complex numbers. The operator B determined on the Hilbert space by the equations $Be_i = w_ie_{i+1}$ is a *two-way weighted shift* with weight sequence $\{w_i\}_{-\infty}^{+\infty}$. The main goal of the paper is to prove Theorems 1, 2, and 5, which together characterize quasidiagonal weighted shifts solely in terms of their weight sequences.

1. Sufficiency.

THEOREM 1. *If the weight sequence of a two-way weighted shift has 0 as a limit point in both directions, then it is quasidiagonal.*

Proof. Let $\{w_{i_n}\}_{n=-\infty}^{+\infty}$ be a subsequence of the weights that converges to 0 in both directions. For each positive integer n, define P_n to be the (orthogonal) projection onto the span of $\{e_{i_{-n}+1}, e_{i_{-n}+2}, \cdots, e_{i_n}\}$. It is easy to compute that $\|P_nB - BP_n\|$ is the larger of $|w_{i_{-n}}|$ and $|w_{i_n}|$; therefore $\|P_nB - BP_n\| \to 0$. The P_n are increasing, and the union of their ranges contains all the vectors in the basis; therefore $P_n \to 1$ strongly. By definition, consequently, B is quasidiagonal. $\square$

DEFINITION. A two-way sequence $\{w_i\}_{-\infty}^{+\infty}$ of complex numbers is *block-balanced* if for each positive number ε and for each positive integer n there exist integers p and q with $p+n<0<q$ and $|w_{p+k}-w_{q+k}|<\varepsilon$ for each k with $0 \leqq k \leqq n$.

Roughly, $\{w_i\}_{-\infty}^{+\infty}$ is block-balanced if there exist arbitrarily long blocks (permitted to overlap) on each side of the center number that are arbitrarily near each other. It is obvious that the center number does not have to be at w_0; it could be at any fixed number w_i.

THEOREM 2. *If the weight sequence of two-way weighted shift is block-balanced, then it is quasidiagonal.*

We prove Theorem 2 after two lemmas. Next we describe a projection of the kind to be used in the proof. Let n be a positive integer, and let p and q be integers with $p+n<0<q$. For each integer k with $0 \leqq k \leqq n$, define vectors g_k and h_k by

$$g_k = \sqrt{k/n}\,e_{p+k} + \sqrt{(n-k)/n}\,e_{q+k}\,,$$
$$h_k = \sqrt{(n-k)/n}\,e_{p+k} - \sqrt{k/n}\,e_{q+k}\,.$$

The set $\{g_0, g_1, \cdots, g_n, h_0, h_1, \cdots, h_n\}$ is an orthonormal set whose span is precisely the span of the orthonormal set $\{e_p, e_{p+1}, \cdots, e_{p+n}, e_q, e_{q+1}, \cdots, e_{q+n}\}$. (Note that there may be many vectors e_i between e_{p+n} and e_q.) We define the projection P, depending on n, p, and q, to be the projection onto the span of the orthonormal set $\{g_0, g_1, \cdots, g_n, e_{p+n+1}, e_{p+n+2}, \cdots, e_{q-1}\}$, (that is, the vectors g_k and each vector e_i between e_{p+n} and e_q). The vectors h_k, though not involved directly in the definition of P, are nevertheless useful in the computations to come.

What is needed for Theorem 2 is an estimate for $\|PB-BP\|$; the next two lemmas provide it. Let D be the diagonal operator (basis: $\{e_i\}_{-\infty}^{+\infty}$) with diagonal entries $\{w_i\}_{-\infty}^{+\infty}$, and let W be the unweighted bilateral shift (same basis). Clearly, $B=WD$.

LEMMA A. $\|PD-DP\| \leqq \max|w_{p+k}-w_{q+k}|$, *(over k with $0 \leqq k \leqq n$).*

Proof. For each integer k $(0 \leqq k \leqq n)$ the space $M_k = \bigvee\{g_k, h_k\} = \bigvee\{e_{p+k}, e_{q+k}\}$ ("$\bigvee$" denotes span) reduces $PD-DP$, and $PD-DP$ vanishes on the orthogonal complement of the space spanned by all the vectors g_k and all the vectors h_k. It is therefore sufficient to prove that

$$\|(PD-DP)|M_k\| \leqq |w_{p+k}-w_{q+k}| \quad \text{for} \quad 0 \leqq k \leqq n\,.$$

Using the equations $Pg_k = g_k$ and $Ph_k = 0$, one can easily compute that the matrix for $(PD - DP)|M_k$ with respect to the basis $\{g_k, h_k\}$ is the matrix

$$\begin{pmatrix} 0 & (Dh_k, g_k) \\ -(Dg_k, h_k) & 0 \end{pmatrix},$$

whose norm is the larger of $|(Dg_k, h_k)|$ and $|(Dh_k, g_k)|$. Now

$$\begin{aligned} |(Dg_k, h_k)| &= |(w_{p+k}\sqrt{k/n}e_{p+k} + w_{q+k}\sqrt{(n-k)/n}e_{q+k}, \\ &\qquad \sqrt{(n-k)/n}e_{p+k} - \sqrt{k/n}e_{q+k})| \\ &= \sqrt{k(n-k)}/n|w_{p+k} - w_{q+k}| \leqq |w_{p+k} - w_{q+k}| . \end{aligned}$$

A similar computation shows that $|(Dh_k, g_k)| \leqq |w_{p+k} - w_{q+k}|$; therefore $\|(PD - DP)|M_k\| \leqq |w_{p+k} - w_{q+k}|$. □

LEMMA B. $\|PW - WP\| \leqq 2/\sqrt{n}$.

Proof. The basic idea of the proof is quite simple: each vector in the orthonormal set $\{g_0, \cdots, g_n, e_{p+n+1}, \cdots, e_{q-1}\}$ is mapped by W almost into the next one (and the last vector is mapped almost into the first one). It makes sense therefore that the span of the set, which is by definition the range of P, should be almost reducing. The computations are simplified if we note that $\|PW - WP\| \leqq \|(1-P)WP\| + \|PW(1-P)\|$; we shall show that each of the two summands on the right is bounded above by $1/\sqrt{n}$.

To estimate $\|(1-P)WP\|$, begin by computing $(1-P)Wf$ for each basis vector f in the orthonormal basis $\{g_0, \cdots, g_n, e_{p+n+1}, \cdots, e_{q-1}\}$ for the range of P. For each integer i with $p + n + 1 \leqq i \leqq q - 1$, the vector e_{i+1} is in the range of P (note that $e_q = g_0$); therefore $(1-P)We_i = (1-P)e_{i+1} = 0$. For each integer k with $0 \leqq k \leqq n-1$, the vector Wg_k is in $\bigvee\{e_{p+k+1}, e_{q+k+1}\} = \bigvee\{g_{k+1}, h_{k+1}\}$. Because the vector g_{k+1} is in the range of P, and because the vector h_{k+1} is in the range of $1 - P$, it follows that $(1-P)Wg_k = (Wg_k, h_{k+1})h_{k+1}$. Finally, $g_n = e_{p+n}$; therefore $(1-P)Wg_n = (1-P)e_{p+n+1} = 0$. The computations so far show that $(1-P)W$ annihilates all the vectors in the basis for the range of P except $g_0, \cdots, g_{n-1}$, each of which it maps into a multiple of the corresponding vector in the orthonormal set $\{h_1, \cdots, h_n\}$. The norm of $(1-P)WP$ is therefore simply the largest absolute value of the multiples; that is, $\|(1-P)WP\| = \max|(Wg_k, h_{k+1})|$, $(0 \leqq k \leqq n-1)$. Now,

$$|(Wg_k, h_{k+1})| = |\sqrt{k(n-k-1)} - \sqrt{(n-k)(k+1)}|/n \leqq 1/\sqrt{n} .$$

This proves that $\|(1-P)WP\| \leqq 1/\sqrt{n}$.

The estimation of $\| PW(1-P) \|$ is similar to that for $\| (1-P)WP \|$; therefore we give only a sketch. The vectors in the basis $\{\cdots, e_{p-2}, e_{p-1}, h_0, \cdots, h_n, e_{q+n+1}, e_{q+n+2}, \cdots\}$ for the range of $1-P$ are all annihilated by PW except $h_0, \cdots, h_{n-1}$, and $PWh_k = (Wh_k, g_{k+1})g_{k+1}$ for $0 \leqq k \leqq n-1$. Therefore $\| PW(1-P) \| \leqq \max |(Wh_k, g_{k+1})|$, $(0 \leqq k \leqq n-1)$, and the right side is again bounded by $1/\sqrt{n}$. $\square$

Proof of Theorem 2. Since the weight sequence $\{w_i\}_{-\infty}^{+\infty}$ is block-balanced, there exists for each positive integer n a pair of integers p and q such that $p+n<0<q$ and $|w_{p+k} - w_{q+k}| < 1/n$ for $0 \leqq k \leqq n$. Define P_n to be the projection P (depending on n, p, and q) defined just before Lemma A. We shall prove that the sequence $\{P_n\}_1^{+\infty}$ of projections of finite rank implements the quasidiagonality of $B = WD$.

Because $P_nB - BP_n = (P_nW - WP_n)D + W(P_nD - DP_n)$, it follows that

$$\| P_nB - BP_n \| \leqq \| P_nW - WP_n \| \, \| D \| + \| W \| \, \| P_nD - DP_n \| .$$

By the two lemmas, the right side is dominated by $2\| D \|/\sqrt{n} + \| W \|/n$, so that $\| P_nB - BP_n \| \to 0$.

It remains to show that $P_n \to 1$ strongly; it is sufficient to prove that $P_ne_i \to e_i$ as $n \to \infty$, for each basis vector e_i. Two cases must be considered: $i < 0$ and $i \geqq 0$. The proofs for the two cases differ in notation only; therefore we prove only the case $i \geqq 0$. Also, we need to consider only $n \geqq i$. If the q corresponding to n satisfies the inequality $q > i$, then $p+n+1 \leqq 0 \leqq i \leqq q-1$, so that e_i is in the range of P_n, and $P_ne_i = e_i$. If on the other hand, $q \leqq i$, let $k = i - q$, and note that $0 \leqq k \leqq n$. The basis vector $e_i = e_{q+k}$ is in $\bigvee \{e_{q+k}, e_{p+k}\} = \bigvee \{g_k, h_k\}$; consequently $P_ne_i = (e_{q+k}, g_k)g_k = \sqrt{(n-k)k}/ne_{p+k} + (n-k)/ne_{q+k}$. It follows that $P_ne_i - e_i = \sqrt{(n-k)k}/ne_{p+k} - k/ne_{q+k}$, so that $\| P_ne_i - e_i \|^2 = k/n$. The condition $0 < q$ implies that $k = i - q < i$, so that $\| P_ne_i - e_i \|^2 < i/n$. In summary, we have proved in either case ($q > i$ or $q \leqq i$) that $\| P_ne_i - e_i \|^2 < i/n$, provided $n \geqq i$. For each integer $i \geqq 0$, therefore, $P_ne_i \to e_i$ as $n \to \infty$. $\square$

COROLLARY. *If the weights of a two-way weighted shift are periodic, then it is quasidiagonal.*

Proof. If the weights are periodic, then they are block-balanced. $\square$

In case the weights are strictly positive, it is interesting to compare the corollary with a result of R. L. Kelley, which says that B is reducible if and only if the weights are periodic [7, Thm. 11,

p. 44]. The proof of Kelley's result produces projections that commute with B but that are (unavoidably) of infinite rank. The proof of the corollary produces projections that are of finite rank and that nearly commute with B, but that cannot actually commute with B.

2. Necessity. The theorems in § 1 produce two kinds of quasidiagonal two-way weighted shifts. In this section, we prove that they are the only ones. The key idea is that for each shift that does not belong to one of the two kinds, there is a nonquasitriangular operator in the C^*-algebra generated by the shift, so that by Theorem 3 the shift itself fails to be quasidiagonal.

Throughout this section, it is necessary to assume that the weights are nonnegative (Theorems 4 and 5 are false, otherwise), and it is convenient to assume that $\|B\| \leqq 1$ (so that $0 \leqq w_i \leqq 1$). No real generality is lost in either case, for the weighted shift with weights $\{w_i\}_{-\infty}^{+\infty}$ is unitarily equivalent to the weighted shift with weights $\{|w_i|\}_{-\infty}^{+\infty}$ [4, Problem 75], and quasidiagonality is unchanged by multiplication by scalars (this is obvious from the definition).

THEOREM 3. *If A is quasidiagonanl, then each operator in the C^*-algebra generated by A is also quasidiagonal.*

Proof. If the sequence $\{P_n\}$ of projections of finite rank implements the quasidiagonality of A, then $\{P_n\}$ also implements the quasidiagonality of A^*, because $\|P_nA^* - A^*P_n\| = \|P_nA - AP_n\|$. It therefore implements the quasidiagonality of finite products involving A and A^*, because $\|P_nST - STP_n\| \leqq \|P_nS - SP_n\|\,\|T\| + \|S\|\,\|P_nT - TP_n\|$; also it implements all polynomials (possibly noncommutative) in A and A^*, because

$$\|P_n(S+T) - (S+T)P_n\| \leqq \|P_nS - SP_n\| + \|P_nT - TP_n\| .$$

If, finally, $S_n \to S$ and each S_n is a polynomial in A and A^*, then by a diagonal argument it is possible to select a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ that implements the quasidiagonality of S; the inequality needed is

$$\begin{aligned}\|P_{n_k}S - SP_{n_k}\| \leqq \|P_{n_k}\|\,\|S - S_n\| + \|S_n - S\|\,\|P_{n_k}\| \\ + \|P_{n_k}S_n - S_nP_{n_k}\| .\end{aligned}$$ □

The next theorem provides the tool necessary for applying Theorem 3 in the context of weighted shifts. In its proof, we need an approximation theorem of the Stone-Weierstrass type. Let n be a fixed, nonnegative integer and let X be the Cartesian product of

$n+1$ copies of the closed unit interval $[0, 1]$, the component functions being indexed by the integers 0 to n (so that a typical point in X is $(x_0, x_1, \cdots, x_n)$ with $0 \leqq x_i \leqq 1$). Let $C^r(X)$ be the normed space of real-valued continuous functions on X with the sup norm. We need the fact that if the functions in a subset S of $C^r(X)$ vanish identically on a closed subset X_0 of X and separate points in the complement of X_0, then the algebra of functions generated by S is dense in the set C_0 of functions in $C^r(X)$ that vanish identically on X_0 [9, Thm. 7, p. 46].

THEOREM 4. *If u is a function in $C^r(X)$ that vanishes identically on the coordinate faces $x_j = 0$ $(0 \leqq j \leqq n)$, then the weighted shift B_u with weights $\{u(w_i, w_{i+1}, \cdots, w_{i+n})\}_{i=-\infty}^{+\infty}$ is in the C^*-algebra generated by the weighted shift B with weights $\{w_i\}_{-\infty}^{-\infty}$ $(0 \leqq w_i \leqq 1)$.*

Proof. Let $\mathscr{A}[B]$ be the C^*-algebra generated by B. If j is a nonnegative integer, then the operator D_j defined by the formula $D_j = \sqrt{B^{*(j+1)}B^{(j+1)}}$ is the diagonal operator with diagonal entries $\{w_i w_{i+1} \cdots w_{i+j}\}_{i=-\infty}^{+\infty}$ (the assumption $0 \leqq w_i$ has just been used), and D_j is in $\mathscr{A}[B]$ (because D_j is in $\mathscr{A}[D_j^2]$ by the Gelfand-Naĭmark theorem [3, Thm. 7, p. 876], and $\mathscr{A}[D_j^2] \subset \mathscr{A}[B]$). The product BD_j is the weighted shift with weights $\{w_i^2 w_{i+1} w_{i+2} \cdots w_{i+j}\}_{i=-\infty}^{+\infty}$, and it is in $\mathscr{A}[B]$. More generally, if $b_0, b_1, \cdots, b_n$ is a sequence of nonnegative integers, then the operator $BD_0^{b_0}D_1^{b_1}D_2^{b_2} \cdots D_n^{b_n}$ is the weighted shift whose ith weight is

$$w_i^{(1+b_0+b_1+\cdots+b_n)} w_{i+1}^{(b_1+b_2+\cdots+b_n)} w_{i+2}^{(b_2+b_3+\cdots+b_n)} \cdots w_{i+n}^{b_n},$$

and it is in $\mathscr{A}[B]$. Put another way, if $m_0, m_1, \cdots, m_n$ is a sequence of integers with $m_0 \geqq m_1 \geqq \cdots \geqq m_n \geqq 0$, and if u is the function in $C^r(X)$ defined by $u(x_0, \cdots, x_n) = x_0^{(m_0+1)} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, then the weighted shift B_u is in $\mathscr{A}[B]$.

Let C_0 be the subset of $C^r(X)$ consisting of the functions in $C^r(X)$ that vanish on the set X_0 of points in X one of whose coordinates is 0. We want to show that B_u is in $\mathscr{A}[B]$ whenever u is in C_0; the first paragraph proves that B_u is in $\mathscr{A}[B]$ whenever u is in the set S consisting of the functions in $C^r(X)$ that have the special form $u(x_0, \cdots, x_n) = x_0^{(m_0+1)} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where the m_i are integers such that $m_0 \geqq m_1 \geqq \cdots \geqq m_n \geqq 0$. Because the set S is closed under multiplication, each function in the algebra $\mathscr{S}$ generated by S is merely a linear combination of functions in S. Because $B_{u_1} + B_{u_2} = B_{u_1+u_2}$ and $B_{\alpha u} = \alpha B_u$ for each scalar α, the operator B_u is in $\mathscr{A}[B]$ whenever u is in $\mathscr{S}$. If $u_n \to u$ uniformly on X, then $B_{u_n} \to B_u$ uniformly; therefore B_u is in $\mathscr{A}[B]$ whenever u is in the uniform closure of $\mathscr{S}$. The set S separates points in $X - X_0$; therefore, by the remark preceding the theorem, the uniform closure of $\mathscr{S}$ is all of C_0. $\square$

The condition $0 \leqq w_i$ in Theorem 4 is necessary. To see this, suppose the weights of B are

$$\{\cdots, -1, 1, -1, (1), -1, -1, 1, -1, -1, \cdots\}$$

(note that the weight 1 alternates with a single -1 in the backward direction while it alternates with a pair of -1's in the forward direction). The operator B is quasidiagonal (because it is normal); therefore each operator in $\mathscr{A}[B]$ is quasidiagonal, by Theorem 3. If u is a continuous function of one variable such that $u(1) = 1$ and $u(-1) = 2$, then B_u has weights $\{\cdots, 2, 1, 2, (1), 2, 2, 1, 2, 2, \cdots\}$, and B_u fails to be quasidiagonal (as the next theorem shows); therefore B_u is not in $\mathscr{A}[B]$.

THEOREM 5. *If a two-way weighted shift with nonnegative weights is quasidiagonal, then either the weight sequence has* 0 *as a limit point in both directions, or it is block-balanced.*

Proof. Suppose that the weight sequence does not have 0 has a limit point in one direction and that it is not block-balanced. We must prove that B is not quasidiagonal. Since quasidiagonality is invariant under the adjoint operation, it is sufficient to prove the case where the weight sequence does not have 0 as a limit point in the forward direction. It is then possible to alter a finite number of the weights to produce a new weight sequence (which is still not block-balanced) that is bounded below in the forward direction by a positive number, say δ. Such a change amounts to a compact perturbation of B, and as such does not alter quasidiagonality.

The assumptions now are that $0 \leqq w_i \leqq 1$ for each integer i, that $0 < \delta \leqq w_i \leqq 1$ for $i \geqq 0$, and that there exists a positive number ε and a positive integer n such that to each pair of integers p and q with $p + n < 0 < q$ there corresponds an integer k $(0 \leqq k \leqq n)$ such that $|w_{p+k} - w_{q+k}| \geqq \varepsilon$. We must prove that B is not quasidiagonal. Let Y be the closure in the normed space X (defined just before Theorem 4) of the set of points $\{(w_i, w_{i+1}, \cdots, w_{i+n})\}_{i=0}^{+\infty}$ (note: in the forward direction only), and let Z be the closure of the set consisting first of the points in $\{(w_i, w_{i+1}, \cdots, w_{i+n})\}_{i=-\infty}^{n}$ (note: in the backward direction only) and second of the points in X at least one of whose components is 0. The assumption on ε and n implies that the set Y is at a distance at least ε from each point of the first kind in Z, and the assumption on δ implies that Y is at a distance at least δ from each point of the second kind in Z. The closed sets Y and Z are therefore disjoint. By Urysohn's lemma, there exists a continuous function u on X with $u[Z] = 0$ and $u[Y] = 1$. The function u vanishes on the face $x_0 = 0$ of Theorem 4 because the points in that face are contained in Z.

By Theorem 4, the shift B_u is in the C^*-algebra generated by B. If B were quasidiagonal, then by Theorem 3, B_u would also be quasidiagonal. Except for a compact perturbation, B_u is the weighted shift with weights $\{\cdots, 0, 0, 0, (1), 1, 1, \cdots\}$ (the weights with indices from $-n+1$ to -1 may be different). The quasidiagonality of B thus implies the quasidiagonality of $0 \oplus U$ (U is the unweighted unilateral shift). It is known, however, that $0 \oplus U$ fails even to be quasitriangular (proof: U is not quasitriangular [5, Thm. 3]; therefore neither is $0 \oplus U$ [2, Thm. 8]). The weighted shift B consequently fails to be quasidiagonal. □

3. **Applications.** In this section we give two applications that use the characterization of quasidiagonal weighted shifts developed in the previous two sections. The first application is a reformulation of the characterization in a special case, namely where the weighted shift has compact self-commutator (the self-commutator of A is $A^*A - AA^*$). The second example shows that quasidiagonality fails to be invariant for similarity.

THEOREM 6. *A two-way weighted shift with nonnegative weights that has compact self-commutator is quasidiagonal if and only if there exists a number that is both a forward limit point and a backward limit point of the weight sequence.*

L. G. Brown, R. G. Douglas, and P. A. Fillmore have proved that if an operator A with compact self-commutator is quasidiagonal, then it is a compact perturbation of a diagonal operator [1, Cor. 11.12]. Theorem 6 therefore also characterizes two-way weighted shifts that are compact perturbations of diagonal operators.

Proof. A diagonal operator with weights $\{d_i\}_{-\infty}^{+\infty}$ is compact if and only if the weights converge to 0 in both directions. Since $B^*B - BB^*$ is the diagonal operator with weights $\{w_i^2 - w_{i+1}^2\}_{-\infty}^{+\infty}$, it follows that $B^*B - BB^*$ is compact if and only if $\{w_i - w_{i+1}\}_{-\infty}^{+\infty}$ converges to 0 in both directions. In that case, the set of forward limit points is a closed interval R, and the set of backward limit points of $\{w_i\}_{-\infty}^{+\infty}$ is a closed interval L.

If R and L have a point in common, say x, then, because $\{w_i - w_{i+1}\}_{-\infty}^{+\infty}$ converges to 0 in both directions, there exist arbitrarily long blocks, on both sides of the center weight, such that the weights in the blocks are arbitrarily near to x, and therefore arbitrarily near to each other. The weight sequence is therefore block-balanced, so that B is quasidiagonal by Theorem 2.

If R and L are disjoint, then they are a positive distance apart. The condition that $\{w_i - w_{i+1}\}_{-\infty}^{+\infty}$ converges to 0 in both directions means that eventually the blocks to the right of the center weight are a positive distance (sup norm) from the blocks to the left of the center weight. The weight sequence is therefore not block-balanced. Neither does it have 0 as a limit point in both directions, since R and L are disjoint. By Theorem 5, the shift B fails to be quasidiagonal. $\square$

EXAMPLE. An operator that is similar to the (unweighted) bilateral shift W may fail to be quasidiagonal.

Proof. Let A be the two-way weighted shift with weights $\{\cdots, 1, 1, 1, (1/2), 2, 1/2, 2, \cdots\}$. If S is the diagonal operator with diagonal entries $\{\cdots, 1, 1, 1, (1), 2, 1, 2, \cdots\}$, then $S^{-1}AS = W$, so that A is similar to W. By Theorem 5, on the other hand, A fails to be quasidiagonal. $\square$

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