PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 2, 1982

RITT SCHEMES AND TORSION THEORY

ALEXANDRU BUIUM

There is a natural way to associate a torsion theory to any differential ring. Using this tool, one may prove that there is a duality between the category of reduced affine Ritt schemes and a full subcategory of the category of Ritt algebras. As a consequence, a brief investigation is made concerning morphisms of differential finite type and a differential version of Chevalley's constructibility theorem is proved for such morphisms.

1. Introduction. The category *Diff* has as its objects commutative rings A with unit together with m derivation operators $D_1, \cdots, D_m: A \to A$ which commute. A morphism $f: (A, D_1, \cdots, D_m) \to (\bar{A}, \bar{D}_1, \cdots, \bar{D}_m)$ in *Diff* is a ring homomorphism $f: A \to A$ with $\bar{D}_i f = fD_i$ for every $i = 1, \cdots, m$. Recall from [5, p. 110] that an LDR-space is pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf in *Diff* on X such that for each $P \in X$ the ring $\mathcal{O}_{X,P}$ is local and its maximal ideal is differential. A morphism of LDR-spaces $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (ψ, θ) where $\psi: X \to Y$ is continuous and $\theta: \mathcal{O}_Y \to \psi_* \mathcal{O}_X$ is a morphism of sheaves in *Diff* on Y such that for each $P \in X$, the morphism $\mathcal{O}_{Y,\psi(P)} \to \mathcal{O}_{X,P}$ is local. The category of LDR-spaces is denoted by LDR.

There exist two fundamental functors $\mathrm{Spec}_D: Diff \to LDR$ and $\Gamma_D: LDR \to Diff$ defined such as follows: for any differential ring A, the topological space $\mathrm{Spec}_D A$ consists of the prime differential ideals of A, the topology being induced by the natural inclusion $j: \mathrm{Spec}_D A \to \mathrm{Spec}\, A$ and the defining sheaf $\mathcal{O}_{\mathrm{Spec}_D A}$ being $\hat{A} = j^{-1}(\tilde{A})$, where $\tilde{A}$ is the defining sheaf of the scheme $\mathrm{Spec}\, A$, [3, p. 70]. Note that $\tilde{A}$ (and consequently $\hat{A}$) has a natural structure of sheaf in *Diff*. Indeed the derivations $D_1, \cdots, D_m$ on A canonically give derivations $D_{1,P}, \cdots, D_{m,P}$ on A_P for any $P \in \mathrm{Spec}\, A$; hence for any open set $U \subseteq \mathrm{Spec}\, A$, the ring $\Gamma(U, \tilde{A})$ becomes a differential ring with derivations $D_{1,U}, \cdots, D_{m,U}$ defined such as follows: for any $s \in \Gamma(U, A)$, $D_{i,U}(s)$ is the section defined by the family $\{D_{i,P}(s_P)\}_{P \in U}$. On the other hand for any LDR-space X, $\Gamma_D(X)$ will denote the differential ring of global sections $\Gamma(X, \mathcal{O}_X)$.

It was proved in [5] that at least in the case of a single derivation, the functors Spec_D and Γ_D give an adjunction between *Diff* and LDR^{op}. We shall prove in §2 that these functors give in fact an equivalence between sufficiently large subcategories.

For the remainder of this paper we shall suppose that all rings contain the field of rationals $\boldsymbol{Q}$. A differential ring which contains $\boldsymbol{Q}$ will be called a Ritt algebra. If A is a Ritt algebra, an LDR-space of the form $\mathrm{Spec}_D A$ will be called an affine Ritt scheme. A Ritt scheme will mean an LDR-space whice may be covered by affine Ritt schemes.

The equivalence proved in §2 makes possible an investigation of morphisms of differential finite type between affine Ritt schemes which will be made in §3. In §4 we make some remarks on the differential affine space.

The necessary information on rings and modules of quotients may be found in **[11]**. We will use this technique in the following context: given a commutative ring A and a subset X of $\mathrm{Spec}\, A$, one may associate to X an hereditary torsion class $\mathscr{T} = \{M \in \mathrm{Mod}\, A,\ M_P = 0 \text{ for any } P \in X\}$, a Gabriel topology $F = \{J \text{ ideal in } A,\ A/J \in \mathscr{T}\} = \{J \text{ ideal in } A,\ J \not\subseteq P \text{ for any } P \in X\}$ and a left exact radical $t\colon \mathrm{Mod}\, A \to \mathscr{T}$, $t(M) = \{x \in M,\ \mathrm{ann}\,(x) \in F\}$. For any $M \in \mathrm{Mod}\, A$ one defines the module of quotients $M_F = \varinjlim_{J \in F} \mathrm{Hom}\,(J, M/t(M))$. Then we have $\ker\,(\varphi_M\colon M \to M_F) = t(M)$, $\mathrm{coker}\,(\varphi_M) \in \mathscr{T}$ and $M_F \to (M_F)_F$ is an isomorphism. For any ideal I in A one defines the ideal $I^c = \{x \in A,\ I\colon x \in F\}$ and put $C_F(A) = \{I \text{ ideal in } A,\ I^c = I\}$. Now the set $F^e = \{J \text{ ideal in } A_F,\ J \cap A \in F\}$ is a Gabriel topology on A_F and there is a one-to-one correspondence between $C_F(A)$ and $C_{F^e}(A_F)$ given by $I \mapsto I_F$ and $J \mapsto J \cap A$. This correspondence induces a one-to-one correspondence between $\mathrm{Spec}\, A \cap C_F(A)$ and $\mathrm{Spec}\, A_F \cap C_{F^e}(A_F)$. Note that for any $P \in \mathrm{Spec}\, A$, we have $P \in F$ or $P \in C_F(A)$.

2. Duality given by Spec_D and Γ_D. For any Ritt algebra A let $\mathscr{T}_A$, F_A and t_A be the hereditary torsion class, the Gabriel topology and the radical associated to the subset $X = \mathrm{Spec}_D A$ of $\mathrm{Spec}\, A$ as in §1. Put $A_D = \Gamma_D\, \mathrm{Spec}_D A$.

PROPOSITION 2.1. *For any Ritt algebra A we have $t_A(A) = \ker\,(A \to A_D)$ and the canonical morphism $A \to A/t_A(A)$ induces an isomorphism of Ritt schemes* $\mathrm{Spec}_D A/t_A(A) \simeq \mathrm{Spec}_D A$

Proof. Our first statement follows directly from definitions. To prove the second statement, observe that for any $x \in t_A(A)$ and for any $P \in \mathrm{Spec}_D A$ we have $\mathrm{ann}\,(x) \not\subseteq P$, hence $t_A(A)$ is contained in every $P \in \mathrm{Spec}_D A$. We get that $\mathrm{Spec}_D A/t_A(A) \to \mathrm{Spec}_D A$ is a homeomorphism. It is sufficient to prove that the morphisms induced on the stalks are isomorphisms, i.e. that for any $P \in \mathrm{Spec}_D A$ we have $A_P \cong (A/t_A(A))_P$. But this isomorphism holds since $(t_A(A))_P = 0$ for any $P \in \mathrm{Spec}_D A$.

REMARK. If A is a Ritt domain with field of quotients K and U is an open subset of $X = \mathrm{Spec}_D A$, then the ring $\Gamma(U, \mathcal{O}_X)$ is equal to the intersection (taken in K) of all the local rings A_P as P runs through U.

COROLLARY 2.2. *Let X be an affine Ritt scheme. The following statements are equivalent*:

(1) *For any open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced (or integral).*

(2) *X is isomorphic to $\mathrm{spec}_D A$ where A is a reduced (or integral) Ritt algebra.*

Proof. (1) $\Rightarrow$ (2) If $X = \mathrm{Spec}_D B$, it follows by Proposition 2.1 that $A = B/t_B(B)$ is a subring of $B_D = \Gamma(X, \mathcal{O}_X)$, hence A is reduced (or integral). By Proposition 2.1 again, we get that X is isomorphic to $\mathrm{Spec}_D A$.

(2) $\Rightarrow$ (1) If A is reduced then every local ring A_P is reduced, hence every $\Gamma(U, \mathcal{O}_X)$ is reduced. If A is integral then $\Gamma(U, \mathcal{O}_X)$ is integral by the remark above.

A Ritt scheme will be called reduced (or integral) if $\Gamma(U, \mathcal{O}_X)$ is reduced (or integral) for any open subset U of X.

If A is a Ritt algebra then $\mathrm{Spec}_D A$ is quasi-compact by [6]. If J is an ideal in A, then $r(J)$, $[J]$ and $\{J\}$ will denote the radical ideal, then differential ideal and the radical differential ideal respectively, which are generated by J. By [8, p. 13], $\{J\}$ is the intersection of all prime differential ideals which contain J and by [4, Lemma 1.8] we have $\{J\} = r([J])$.

For any A-module M, put $\hat{M} = j^{-1}(\tilde{M})$ where j is the natural inclusion $\mathrm{Spec}_D A \to \mathrm{Spec}\, A$ and $\tilde{M}$ is the sheaf on $\mathrm{Spec}\, A$ defined by M [3, p. 110]. The stalk of $\hat{M}$ at $P \in \mathrm{Spec}_D A$ is $M_P = M \otimes_A A_P$. There exists a natural morphism of A-modules $\theta_M: M \to M_D = \Gamma(\mathrm{Spec}_D A, \hat{M})$. It is apparent that $\ker(\varphi_M: M \to M_F) = \ker(\theta_M: M \to M_D) = t(M)$ where $F = F_A$ and $t = t_A$.

LEMMA 2.3. *For every A-module M there exists a natural injective morphism of A-modules $\psi_M: M_F \to M_D$ such that $\psi_M \varphi_M = \theta_M$.*

Proof. Take $x \in M_F$. Since coker $(\varphi_M) \in \mathcal{T}_A$ it follows that for every $P \in \mathrm{Spec}_D A$ there exist elements $s_P \in A \backslash P$ and $x_P \in M$ such that $s_P x = \varphi_M(x_P)$. Since $\varphi_M(s_P x_Q - s_Q x_P) = 0$ for every $P, Q \in \mathrm{Spec}_D A$ it follows that $x_P/s_P = x_Q/s_Q$ in every M_R with $R \in \mathrm{Spec}_D A$, hence the elements $x_P/s_P \in M_{s_P}$ stick together and define a global section $s \in M_D$. It is apparent that s depends only on x and so we put $\psi_M(x) = s$.

ψ_M is injective because the condition $s = 0$ implies that for any $P \in \mathrm{Spec}_D A$ there exists $u_P \in A \backslash P$ such that $u_P x_P = 0$. It follows that $u_P s_P x = 0$ for any $P \in \mathrm{Spec}_D A$, hence $x \in t(M_F) = 0$.

For any A-module M, $\mathrm{Ass}(M)$ will denote the set of all primes P in A for which there exists $x \in M$ such that P is minimal among the prime ideals containing $\mathrm{ann}(x)$.

LEMMA 2.4. *Suppose that* $\mathrm{Ass}(M) \subseteqq \mathrm{Spec}_D A$. *Then* $\varphi_M: M \to M_F$ *is injective and* $\psi_M: M_F \to M_D$ *is bijective.*

Proof. Since $\mathrm{Ass}(M) \subseteqq \mathrm{Spec}_D(A)$ it follows that for every $x \in M$ we have $\{\mathrm{ann}(x)\} = r(\mathrm{ann}(x))$. For the first assertion of the lemma it is sufficient to prove that $t(M) = 0$. If $x \in t(M)$, for every $P \in \mathrm{Spec}_D A$ there exists $s_P \in A \backslash P$ such that $s_P \in \mathrm{ann}(x)$. Since the differential ideal generated by these s_P as P runs through $\mathrm{Spec}_D A$ is equal to A, it follows that $1 \in \{\mathrm{ann}(x)\} = r(\mathrm{ann}(x))$, hence $x = 0$. To prove the second assertion it is sufficient to show that ψ_M is surjective. Take $s \in M_D$. By quasi-compacity of $\mathrm{Spec}_D A$ there exist $f_1, \cdots, f_k \in A$ and $x_1, \cdots, x_k \in M$ such that $\{(f_1, \cdots, f_k)\} = A$ and the restriction of s at $D(f_i) = \{P \in \mathrm{Spec}_D A,\ f_i \notin P\}$ is given by $x_i/f_i \in M_{f_i}$. Since $x_i/f_i = x_j/f_j$ in any M_P with $P \in D(f_i f_j)$ it follows that for any such P there exists $s_{ijP} \in A \backslash P$ with $s_{ijP} \in \mathrm{ann}(f_i x_j - f_j x_i)$. Obviously, for a fixed pair (i, j) the element $f_i f_j$ is contained in the radical differential ideal generated by all the s_{ijP} as P runs through $D(f_i f_j)$, hence $f_i f_j \in \{\mathrm{ann}(f_i x_j - f_j x_i)\} = r(\mathrm{ann}(f_i x_j - f_j x_i))$. So there exists a common N such that $(f_i f_j)^N (f_i x_j - f_j x_i) = 0$ for all i and j. Replacing x_i/f_i by $x_i f_i^N / f_i^{N+1}$ we may suppose $N = 0$. Consider the morphisms of A-modules $u: A^k \to J = f_1 A + \cdots + f_k A$ and $v: A^k \to M$ sending the elements of a basis of A^k into $f_1, \cdots, f_k$ and $x_1, \cdots, x_k$ respectively. Notice that $v(\ker(u)) \subseteqq t(M)$. Indeed if we have $a_1, \cdots, a_k \in A$ such that $\sum_{i=1}^k a_i f_i = 0$ we get that

$$f_j\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{i=1}^{k} a_i f_i x_j = 0$$

for all j, hence $\sum_{i=1}^k a_i x_i \in t(M)$. So v induces a morphism of A-modules $\tilde{v} \in \mathrm{Hom}(J, M/t(M))$. Since $J \in F$ we may consider the image of $\tilde{v}$ in $M_F = \varinjlim I \in F \,\mathrm{Hom}(I, M/t(M))$ and denote it by x. It is apparent that $\psi_M(x) = s$.

LEMMA 2.5. *Suppose A satisfies one of the conditions*:

(1) *A is reduced*

(2) *A is Noetherian*

Then $\mathrm{Ass}(A) \subseteqq \mathrm{Spec}_D A$.

Proof. Suppose first that A is reduced and take $P \in \mathrm{Ass}(A)$. Since $PA_P \in \mathrm{Ass}(A_P)$ we may suppose that A is P-local. Since P is minimal among the primes which contain $\mathrm{ann}(x)$ for a certain $x \in A$, it follows that $P = r(\mathrm{ann}(x))$. On the other hand since the ideal (0) is radical and differential, so is $(0):(x)$ [4, Lemma 1.4] hence P is differential.

If A is supposed to be Noetherian, our statement follows for instance from [10].

We say that a Ritt algebra A is closed if the morphism $\theta_A: A \to A_D$ is an isomorphism. Every local Ritt algebra whose maximal ideal is differential is closed by [7, Proposition 3.3]. The following result shows that any ring of global sections of a reduced affine Ritt scheme is closed. In §4 we will also show that an algebra of differential polynomials over a closed Ritt domain is closed.

THEOREM 2.6. *Let A be a Ritt algebra such that* $\mathrm{Ass}(A) \subseteqq \mathrm{Spec}_D A$. *Then the morphism $\theta_A: A \to A_D$ is injective and induces an isomorphism of Ritt schemes* $\mathrm{Spec}_D A_D \to \mathrm{Spec}_D A$. *Consequently, A_D is closed.*

Proof. By Lemma 2.3, $\psi_A: A_F \to A_D$ is an isomorphism, hence it is sufficient to prove that $\mathrm{Spec}_D \varphi_A$ is an isomorphism of Ritt schemes. Let us prove first that $\mathrm{Spec}_D \varphi_A$ is a homeomorphism between the underlying topologycal spaces. Since $\mathrm{Spec}_D A \cap F = \varnothing$ we get that $\mathrm{Spec}_D A \subseteqq C_F(A)$. Applying this remark to the ring $B = A_F$ we get that $\mathrm{Spec}_D B \subseteqq C_{F_B}(B)$. On the other hand we see that $F^e \subseteqq F_B$. Indeed if $J \in F^e$ then J cannot be contained in any $P \in \mathrm{Spec}_D B$ because if we had $J \subseteqq P$ for such a P, we would get $J \cap A \subseteqq P \cap A \in \mathrm{Spec}_D A$ which is a contradiction. Hence $C_{F_B}(B) \subseteqq C_{F^e}(B)$ and finally we deduce that $\mathrm{Spec}_D B \subseteqq C_{F^e}(B)$. Let us also observe that for any $P \in \mathrm{Spec}_D A$ we have $P_F \in \mathrm{Spec}_D B$. Indeed there exists a canonical morphism of A-algebras $A_F \xrightarrow{\beta} A_P$ and since $PA_P \in \mathrm{Spec}_D A_P$ it follow that $\bar{P} = PA_P \cap A_F \in \mathrm{Spec}_D A_F \subseteqq C_{F^e}(A_F)$. Since $\bar{P} \cap A = P_F \cap A = P$, by the one-to-one correspondence between $C_F(A)$ and $C_{F^e}(A_F)$ we get $P_F = \bar{P}$, hence $P_F \in \mathrm{Spec}_D A_F$. We may conclude that the one-to-one correspondence between $\mathrm{Spec}\, A \cap C_F(A)$ and $\mathrm{Spec}\, A_F \cap C_{F^e}(A_F)$ induces one-to-one correspondence between $\mathrm{Spec}_D A$ and $\mathrm{Spec}_D A_F$. One knows that $O \mapsto O \cap A$ is continuous. To prove that $P \overset{h}{\mapsto} P_F$ is continuous, take $y \in A_F$ and calculate $h^{-1}(D(y))$ where $D(y)$ is the principal open subset of $\mathrm{Spec}_D A_F$ defined by y. There exist $f_1, \cdots, f_k \in A$ and $y_1, \cdots, y_k \in A$ such that $\{(f_1, \cdots, f_k)\} = A$ and $\varphi_A(f_i)y = \varphi_A(y_i)$ for every i. Then we have

$$h^{-1}(D(y)) = \bigcup_i h^{-1}(D(\varphi_A(f_i)y)) = \bigcup_i D(y_i)$$

which is an open set in $\mathrm{Spec}_D A$.

Now we only have to prove that $\mathrm{Spec}_D \varphi_A$ gives an isomorphism on each stalk, i.e. that for every $P \in \mathrm{Spec}_D A$ we have $A_P \cong (A_F)_{P_F}$. We have the commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A_F & \xrightarrow{\beta} & A_P \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \Vert \\ A_P & \xrightarrow{\bar{\alpha}} & (A_F)_{P_F} & \xrightarrow{\bar{\beta}} & (A_P)_{PA_P} \end{array}$$

where $\alpha = \varphi_A$ and $\bar{\beta}\bar{\alpha}$ is the identity of A_P. It is sufficient to prove that $\bar{\beta}$ is injective. Suppose we have $y \in (A_F)_{P_F}$ with $\bar{\beta}(y) = 0$. After multiplying y with a unit in $(A_F)_{P_F}$ we may suppose $y = \gamma(x)$ with $x \in A_F$. Since $\mathrm{coker}\,(\alpha) \in \mathscr{T}_A$ it follows that there exist $s \in A \backslash P$ and $a \in A$ such that $\alpha(s)x = \alpha(a)$. Applying $\bar{\beta}\gamma$ to this equality we get $\delta(a) = 0$, hence there exists $u \in A \backslash P$ such that $ua = 0$. We get that $\alpha(us)x = 0$ and since $\alpha(us) \notin P_F$ we get $y = 0$.

COROLLARY 2.7. *The functors* Spec_D *and* Γ_D *give a duality between the category RA of reduced affine Ritt schemes and the category RC of reduced closed Ritt algebras. Consequently, the category RC is a coreflective subcategory of the category R of reduced Ritt algebras.*

Proof. The Corollary is a consequence of Lemma 2.5 and Theorem 2.6.

REMARK. Our Theorem 2.6 may be stated in more general terms. If in the definition of the category *LDR* we forget the condition "maximal ideals in the stalks of $\mathscr{O}$ are differential", we get, by [5], the category of differential local ringed spaces which is denoted by *Diff* loc. There are two standard functors Spec: *Diff* $\to$ *Diff* loc and Γ: *Diff* loc $\to$ *Diff*, where Spec A, as topological space, is the set of all prime ideals of A with the Zariski topology, A being any differential ring. Suppose we have another functor S: *Diff* $\to$ *Diff* loc which satisfies the follows:

1) There exists a functorial morphism $j\colon S \to \mathrm{Spec}$ such that for any differential ring A, the morphism $j_A\colon S(A) \to \mathrm{Spec}\,A$ is an inclusion of sets, the topology and the defining sheaf on $S(A)$ being obtained by inverse image from the structures of Spec A.

2) For any differential ring A and for any $P \in S(A)$ we have $PA_P \in S(A_P)$.

Notice that the functor Spec_D in arbitrary characteristic satisfies these axioms. The functor Spec_q considered in [7] satisfies them too. It is easy to see that our method leads in fact to the following result:

THEOREM 2.8. *If A is a differential ring such that $S(A)$ is quasi-compact and* $\mathrm{Ass}\,(A) \subseteq S(A)$, *then the morphism $A \to \Gamma S(A)$ is injective and induces an isomorphism $S\Gamma S(A) \to S(A)$.*

3. Morphisms of differential finite type. We say that a morphism $f: X \to Y$ of affine Ritt schemes is of differential finite type if there exists a morphism of Ritt algebras $u: A \to B$ with B finitely generated over A as a differential algebra [8, p. 59] and there exist isomorphisms $X \cong \mathrm{Spec}_D B$, $Y \cong \mathrm{Spec}_D A$ such that $\mathrm{Spec}_D u$ induces f via these isomorphisms. If $b_1, \cdots, b_s$ generate B as an A-differential algebra we use the notation $B = A\{b_1, \cdots, b_s\}$. A morphism of Ritt schemes $f: X \to Y$ is called dominant if $f(X)$ is a dense subset in Y. One may easily check that given a morphism of Ritt algebras $u: A \to B$, then $\mathrm{Spec}_D u: \mathrm{Spec}_D B \to \mathrm{Spec}_D A$ is dominant if and only if $\ker(u) \subseteq \mathrm{nil}\,(A)$.

LEMMA 3.1. *If $u: A \to B$ is an injective morphism of reduced Ritt algebras, then the morphism $u_D: A_D \to B_D$ is also injective.*

Proof. Since u is injective, it follows that the morphism $\mathrm{Spec}_D B \to \mathrm{Spec}_D A$ is dominant. By Theorem 2.6 we get that the morphism $\mathrm{Spec}_D B_D \to \mathrm{Spec}_D A_D$ is dominant, hence $\ker(u_D) \subseteq \mathrm{nil}\,(A_D) = 0$

LEMMA 3.2. *Let $f: X \to Y$ be a morphism of reduced affine Ritt schemes. Put $A = \Gamma(Y, \mathcal{O}_Y)$ and $B = \Gamma(X, \mathcal{O}_X)$. Then f is of differential finite type if and only if there exists a differential sub-A-Algebra C of B such that C is finitely generated over A as a differential algebra and $C_D = B$.*

Proof. By Corollary 2.2 we have $Y \cong \mathrm{Spec}_D A_1$ where A_1 is reduced. By Theorem 2.6 we get that $\mathrm{Spec}_D A_1 \cong \mathrm{Spec}_D (A_1)_D = \mathrm{Spec}_D A$, hence $Y \cong \mathrm{Spec}_D A$. In the same way we get $X \cong \mathrm{Spec}_D B$. If we suppose there exists an algebra C as in the statement of our lemma, then by Theorem 2.6 we get that $\mathrm{Spec}_D C \cong \mathrm{Spec}_D B$ and so f is of differential finite type. Conversely, suppose that f is of differential finite type. Let $u: \bar{A} \to \bar{B}$ be a morphism of Ritt algebras such that $\bar{B}$ is finitely generated over $\bar{A}$ as a differential algebra and suppose there exist isomorphisms $Y \cong \mathrm{Spec}_D \bar{A}$, $X \cong \mathrm{Spec}_D \bar{B}$

such that f is given by $\mathrm{Spec}_D\, u$ via these isomorphisms. Since $t_{\bar{A}}(\bar{A}) \subseteqq t_{\bar{B}}(\bar{B})$, by Proposition 2.1 we may replace $\bar{A}$ and $\bar{B}$ by $\bar{A}/t_{\bar{A}}(\bar{A})$ and $\bar{B}/t_{\bar{B}}(\bar{B})$ which are reduced being subrings in $\bar{A}_D = \mathrm{A}$ and $\bar{B}_D = B$ respectively. Hence we may suppose that $\bar{A}$ and $\bar{B}$ are reduced. If u_D denotes the morphism $A \to B$, put $C = u_D(A)[\bar{B}] \subseteqq B$. Applying Lemma 3.1 to $\bar{B} \subseteqq C \subseteqq B$ we get $B = \bar{B}_D \subseteqq C_D \subseteqq B$, hence $C_D = B$. Since C is finitely generated over A as a differential algebra, the lemma is proved.

THEOREM 3.3. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of differential finite type between reduced affine Ritt schemes, then gf is also of differential finite type.*

Proof. Put $A = \Gamma(Z, \mathscr{O}_Z)$, $B = \Gamma(Y, \mathscr{O}_Y)$, $C = \Gamma(X, \mathscr{O}_X)$. By Lemma 3.2 there exist morphisms $A \xrightarrow{a} E \subseteqq B = E_D$ and $B \xrightarrow{b} G \subseteqq C = G_D$ where E and G are finitely generated over A and B respectively, as differential algebras. Put $T = b(E)\{x_1, \cdots, x_k\} \subseteqq C$, where $x_1, \cdots, x_k$ generate G over B as a differential algebra. Applying the functor $\Gamma_D\,\mathrm{Spec}_D$ to $E \to T \subseteqq C$ we get by Lemma 3.1 $B = E_D \to T_D \subseteqq C_D = C$, hence T_D contains the ring $b(B)$ and the elements $x_1, \cdots, x_k$. Consequently T_D contains G and so applying again Lemma 3.1 to $G \subseteqq T_D \subseteqq C$ we get $C = G_D \subseteqq T_D \subseteqq C$ hence $T_D = C$. Since T is finitely generated over A as a differential algebra, we may apply Lemma 3.2 and we get that gf is of differential finite type.

Now we prove the following differential version of Chevalley's constructibility theorem:

THEOREM 3.4 *Let $X \xrightarrow{f} Y$ be a morphism of differential finite type between ordinary affine Ritt schemes ("ordinary" means there is a single derivation). Suppose that Y has a Noetherian underlying topological space. Then f is constructible.*

Proof. Suppose that $f = \mathrm{Spec}_D\, u$ where u is a morphism of Ritt algebras $A \to B$ such that B is finitely generated over A as a differential algebra.

Suppose for the beginning that B is finitely generated over A as an algebra (in the nondifferential sense!). It is sufficient to prove that $f(\mathrm{Spec}_D\, B)$ is constructible. Since $\mathrm{Spec}_D\, A$ is a Noetherian topological space, it is sufficient, applying classical criterion [9, 6. C], to prove that whenever the morphism $\mathrm{Spec}_D\,(B/PB) \xrightarrow{g} \mathrm{Spec}_D\,(A/P)$ is dominant for a certain $P \in \mathrm{Spec}_D\, A$, it follows that the image of g contains a nonempty open subset in $\mathrm{Spec}_D\,(A/P)$. But if g is dominant, we get that $A/P \to B/PB$ is injective. So we may suppose that A is a domain and $A \subseteq B$, all we have to show being that $f(\operatorname{Spec}_D B)$ contains a nonempty open subset in $\operatorname{Spec}_D A$. But $f(\operatorname{Spec} B)$ contains a nonempty open subset in $\operatorname{Spec} A$ by [9, proof of 6. E] and our statement follows from the general formula

$$f(\operatorname{Spec}_D B) = f(\operatorname{Spec} B) \cap \operatorname{Spec}_D A$$

and from the fact that $\operatorname{Spec}_D A$ is dense in $\operatorname{Spec} A$.

Now come back to the general case and suppose that B is finitely generated over A as a differential algebra. It is sufficient to prove that $f(\operatorname{Spec}_D B)$ is constructible. Applying [9, 6. C] again, we reduce ourselves to the case A domain, $A \subseteq B$ and we have to prove that the image of f contains a nonempty open set. Suppose $B = A\{y_1, \cdots, y_n\}$. Let $y_1, \cdots y_N$ be a maximal family of differentially algebraically independent elements over A[8, p. 69] and put $C = A\{y_1, \cdots, y_N\}$. Since $\operatorname{Spec}_D A$ is a Noetherian topological space and B is finitely generated over A as a differential algebra, it follows from [8, Theorem 1, p. 126] that $\operatorname{Spec}_D B$ is also Noetherian and hence by [8, Theorem 1, p. 14] every radical differential ideal in B is a finite intersection of prime differential ideals. Consequently, $\operatorname{nil}(B) = P_1 \cap \cdots \cap P_r$, $P_i \in \operatorname{Spec}_D B$ for all i and so we get $(0) = \operatorname{nil}(B) \cap C = (P_1 \cap C) \cap \cdots (P_r \cap C)$. Hence there exists an index i such that the morphism $C \to B/P_i$ is injective. Put $z_j = y_j \bmod P_i$ for all $j \geqq N+1$. For any such j take a differential polynomial [8, p. 70] $F_j \in C\{Y\}$, $F_j \neq 0$, $F_j(z_j) = 0$. Suppose that we have chosen each F_j of minimum order n_j and of minimum degree in $Y^{(n_j)}$. Consider $S_j = \partial F_j / \partial Y^{(n_j)}$ the separant of F_j [8, p. 75]. We have $S_j \neq 0$ (because of the characteristic) and $S_j(z_j) \neq 0$ by the minimality of F_j. Put $S = \prod_{j=N+1}^{n} S_j(z_j)$ which is a nonzero element in $E = B/P_i$. We claim that $E[1/S]$ is finitely generated as a C-algebra (in the nondifferential sense). Indeed, if

$$F_j = \sum_k G_{kj}(Y^{(n_j)})^k\,, \qquad G_{kj} \in C[Y, Y', \cdots, Y^{(n_j-1)}]$$

we get

$$0 = (F_j(z_j))' = \sum_k (G_{kj}(z_j))'(z_j^{(n_j)})^k + S_j(z_j) z_j^{(n_j+1)}$$

We get then by induction that for any $q \geqq 0$

$$z_j^{(q)} \in C[z_{N+1}, \cdots, z_{N+1}^{(M)}, \cdots, z_n, \cdots, z_n^{(M)}, 1/S]\,, \qquad M = \max_j n_j$$

By the first part of our proof, the morphism

$$\operatorname{Spec}_D(E[1/S]) \longrightarrow \operatorname{Spec}_D C$$

is constructible and since it is dominant we get that its image contains a principal open set $D(H) \subseteq \operatorname{Spec}_D C$ with $H \neq 0$. Now if

h is a nonzero coefficient of H it follows that $D(h) \subseteqq \operatorname{Spec}_D A$ is contained in $f(\operatorname{Spec}_D((B/P_i)[1/S])) \subseteqq f(\operatorname{Spec}_D B)$ and the theorem is proved.

4. Remarks on the differential affine space. For any Ritt algebra A, let $B = A\{Y_1, \cdots, Y_n\}$ be the ring of differential polynomials over A. The Ritt scheme $A_A^n = \operatorname{Spec}_D B$ will be called the differential n-affine space over A. For all $a = (a_1, \cdots, a_m) \in N^m$ and for all $F \in B$ we shall write $F^{(a)}$ instead of $D_1^{a_1} \cdots D_m^{a_m} F$. Order the set of all indeterminates $Y_i^{(a)}$ lexicographically [8, p. 75]. Then for any $F \in B$, the leader u_F of F denotes the highest derivative $Y_i^{(a)}$ present in F[8, p. 75].

LEMMA 4.1. *Let A be Ritt domain, $B = A\{Y_1, \cdots, Y_n\}$ and $0 \neq F \in B$. Then any element $x \in \Gamma(D(F), \hat{B})$ may be written as $x = H/G$, $H, G \in B$ and $u_G \leqq u_F$.*

Proof. Let K be the field of quotients of A. There exist $F_1, \cdots, F_r, G_1, \cdots, G_r \in B$ such that $F \in \{(G_1, \cdots, G_r)\}$ and $x = F_i/G_i$ for all $i = 1, \cdots, r$. Let $H, G \in B$ be such that $x = H/G$ and H, G have no common prime divisor in the ring $K\{Y_1, \cdots, Y_n\}$ which is factorial. Since $HG_i = F_iG$ for all i, it follows that G divides G_i in $K\{Y_1, \cdots, Y_n\}$ and so there exist $g_1, \cdots, g_r \in A$ such that $g_iG_i \in (G)B$ for all i. If $g = g_1g_2 \cdots g_r$, we have $\{(G)\} \supseteqq \{(gG_1, \cdots, gG_r)\} = \{(g)\} \cap \{(G_1, \cdots, G_r)\} \ni gF$. Now take a decomposition of G into prime factors $G = E_1^{i_1} \cdots E_k^{i_k}$ in the ring $K\{Y_1, \cdots, Y_n\}$. Since $gF \in \{E_i\}$, we cannot have $u_{gF} < u_{E_i}$ because if we had such an inequality, gF would be reduced with respect to E_i[8, p. 77] and hence gF would be divisible by E_i[8, p. 155] which is a contradiction. Hence $u_F = u_{gF} \geqq u_{E_i}$, for all $i = 1, \cdots, r$ and since $u_G = \max\{u_{E_i}, 1 \leqq i \leqq r\}$ we get $u_F \geqq u_G$.

THEOREM 4.2. *Let A be a Ritt domain. There is a natural isomorphism*

$$(A\{Y_1, \cdots, Y_n\})_D \cong A_D\{Y_1, \cdots, Y_n\}$$

Consequently, if A is closed, so is $A\{Y_1, \cdots, Y_n\}$.

Proof. Both rings are subrings in field of quotients of $B = A\{Y_1, \cdots, Y_n\}$. Obviously, $A_D\{Y_1, \cdots, Y_n\} \subseteqq B_D$. Conversely, if $x \in B_D = \Gamma(\operatorname{Spec}_D B, \hat{B})$, Lemma 4.1 x may be written as F/g with $F \in B$ and $g \in A$. Let us prove that $F/g \in A_D\{Y_1, \cdots, Y_n\}$ by induction on the number of monomials in F. Take $p \in \operatorname{Spec}_D A$ and put $P = pB + [Y_1, \cdots, Y_n] \in \operatorname{Spec}_D B$. Since $F/g \in B_P$ we get that there exist

$W, H \in B$ with $W(0, \cdots, 0) = w \notin p$ and $FW = gH$. Let $fM = f\prod_{i,a}(Y_i^{(a)})^{k_{ia}}$ be a monomial of minimum degree in F where $f \in A$. Identifying the coefficients of M we get $fw = gh$ with $h \in A$, hence $f/g \in A_p$. Since p runs through $\mathrm{Spec}_D A$ it follows that $f/g \in A_D$. Hence $fM/g \in A_D\{Y_1, \cdots, Y_n\}$. Applying the induction hypothesis to $(F - fM)/g$ we get that $(F - fM)/g \in A_D\{Y_1, \cdots, Y_n\}$ and so $F/g \in A_D\{Y_1, \cdots, Y_n\}$.

Let K be a field of characteristic zero. A consequence of Theorem 4.2 is the fact that the algebra $K\{Y_1, \cdots, Y_n\}$ is closed. It is natural to look for its factorizations which remain closed. Suppose the derivation operators are independent on K[8, p. 95] and that K has an uncountable field of constants.

PROPOSITION 4.3. *Let P be a differential prime ideal in $K\{Y_1, \cdots, Y_n\}$ such that the algebra $K\{Y_1, \cdots, Y_n\}/P = A$ is cloesd. If the field of quotients $Q(A)$ is differentially algebraic over K, then A is a field, algebraic extension of K.*

Proof. Suppose there exists $y \in A$ which is transcendental over K. Since y is differentially algebraic over K, there exists a differential polynomial $0 \neq F \in K\{Z\}$ such that $F(y) = 0$. Since the derivation operators are independent on K, there exists by [8, p. 99] $v \in K$ such that $F(v) \neq 0$. By Taylor's formula,

$$F(v + T) = F(v) + \sum_a (\partial F/\partial Z^{(a)})(v)T^{(a)} + \cdots$$

Put

$$G(T) = F(v) + (\partial F/\partial Z)(v)T + \cdots$$

Since G is a nonzero polynomial in $K[T]$, it has only a finite set of roots (perhaps non) in the field of constants K_0 of K. Consequently, there is an uncountable set $\Omega \subsetneqq K_0$ such that for every $c \in \Omega$ we have $F(v + c) = G(c) \neq 0$. Put $z = y - v$ which is also transcendental over K. Since $F((z - c) + (v + c)) = 0$ we get that for every $c \in \Omega$ the differential ideal generated by $z - c$ in A is the whole ring A. Consequently, for every $c \in \Omega$ we have $1/(z - c) \in A_D = A$. But the family $\{1/(z - c), c \in \Omega\}$ is uncountable and liniarly independent over K, which contradicts the obvious fact that A is generated as a vector space by a countable set. Consequently A is algebraic over K, hence A is a field.

COROLLARY 4.4. *If P is a nonzero differential prime ideal prime ideal in $K\{Y\}$, then $K\{Y\}/P$ cannot be closed unless it is a field.*

For any topological space X, $H^i(X,\ )$ will denote the derived functors of $\Gamma(X,\ ): Ab(X) \to Ab$, [3, p. 207]. The following result shows that there is a great difference from the cohomological point of view between schemes and Ritt schemes.

THEOREM 4.5. *Let A be a Ritt domain, $n \geqq 1$ and A_A^n the differential n-affine space over A, $\mathcal{O}$ being its defining sheaf. Then we have*:

$$H^1(U, \mathcal{O}) \neq 0$$

for any nonempty open subset U of A_A^n.

Proof. Suppose $H^1(U, \mathcal{O}) = 0$ where $U = D(I)$, I being a nonzero ideal in $B = A\{Y_1, \cdots, Y_n\}$. Replacing $A\{Y_1, \cdots, Y_{n-1}\}$ by A we may suppose that $n = 1$ and put $Y = Y_1$. Choose $F \in I$, $F \notin A$, $u_F = Y^{(b)}$. Let us also take $a \in N^m$, $a > b$ in the lexicographic order and take $c \in N^m$, $c \neq (0, \cdots, 0)$. Put $y = Y^{(a)} - Y^{(a+c)}$ and consider the exact sequence of B-modules

$$0 \longrightarrow B \xrightarrow{w} B \longrightarrow B/yB = M \longrightarrow 0$$

where w is the multiplication by y. This sequence induces an exact sequence of $\hat{B} = \mathcal{O}$-modules

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \hat{M} \longrightarrow 0$$

We get an exact sequence

$$\Gamma(U, \mathcal{O}) \xrightarrow{p} \Gamma(U, \hat{M}) \longrightarrow H^1(U, \mathcal{O}) = 0$$

Put $F_1 = Y^{(a)} - 1$ and $F_2 = Y^{(a+c)} - 1$. Since $F_1^{(c)} - F_2 = 1$ we get that $D(F_1) \cup D(F_2) = \operatorname{Spec}_D B$. Put

$$s_1 = \bar{1}/F_1 \in M_{F_1} \qquad s_2 = \bar{1}/F_2 \in M_{F_2}$$

where for all $x \in B$, $\bar{x}$ is the image of x in M. Since s_1 and s_2 stick together, we get a section $s \in \Gamma(U, \hat{M})$. Since p is surjective there exists $t \in \Gamma(U, \mathcal{O})$ such that $p(t) = s$. By Lemma 4.1, we may write $t = W/H$ with $W, H \in B$ and $u_H \leqq u_F$. Since W/H and $1/F_1$ have the same image in any M_P with $P \in D(F_1) \cap U = D(F_1 I)$, we get that for any such P, there exist $T_P \in B \backslash P$ and $G_P \in B$ such that

$$T_P((Y^{(a)} - 1)W - H) = G_P(Y^{(a)} - Y^{(a+c)})$$

But $Y^{(a)} - Y^{(a+c)}$ cannot divide the polynomial $E = (Y^{(a)} - 1)W - H$ because if it did, making in E the substitution $Y^{(a)} = Y^{(a+c)} = 1$ we would get $H = 0$ (since H does not change under this substitution).

Consequently, $Y^{(a)} - Y^{(a+c)}$ divides T_P and so $T_P \in [Y^{(a)}]$ for any $P \in D(F_1 I)$. Let J be the radical differential ideal generated by all T_P as P runs through $D(F_1 I)$. Obviously we have $F_1 I \subseteq J$ and so

$$F_1 F \in [Y^{(a)}]$$

On the other hand $F_1 = Y^{(a)} - 1 \notin [Y^{(a)}]$ and $F \notin [Y^{(a)}]$ because $u_F = Y^{(b)}$ and $b < a$. We have obtained a contradiction, because the ideal $[Y^{(a)}]$ is prime.

References

1. P. Blum, *Complete models of differential fields*, Trans. Amer. Math. Soc., **137** (1969), 309–325.
2. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J., **9** (1957), 119–221.
3. R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
4. I. Kaplanski, *An introduction to differential algebra*, Hermann, Paris, 1957.
5. W. Keigher, *Adjunction and comonads in differential algebra*, Pacific J. Math., **59** (1975), 99–112.
6. ———, *Prime Differential Ideals in Differential Rings*, Contributions to algebra: A Collection of Papers Dedicated to Ellis Kolchin; Bass, Cassidy, Kovacic (eds.), Academic Press, New York, 1977.
7. ———, *On the quasi-affine scheme of a differential ring*, to appear.
8. E. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.
9. H. Matsumura, *Commutative Algebra*, W. A. Benjamin Co., New York, 1970.
10. N. Radu, *Sur la décomposition primaire differentiels*, Rév. Roumaine Math. Pures Appl., Vol. **16**, **9** (1971).
11. B. Stenström, *Rings and modules of quotients*, Lecture Notes in Mathematics, Vol. 237, Springer Verlag, Berlin, 1971.

Received April 3, 1980 and in revised form December 9, 1980.

UNIVERSITY OF BUCHAREST
BUCHAREST, ROMANIA