PACIFIC JOURNAL OF MATHEMATICS
Vol. 99, No. 1, 1982

THE CONSTRUCTION OF CERTAIN BMO FUNCTIONS AND THE CORONA PROBLEM

AKIHITO UCHIYAMA

In Euclidean space R^d, let I denote any cube with sides parallel to the axes and write $|I|$ for the measure of I. A real valued locally integrable function $f(x)$ on R^d has bounded mean oscillation, $f \in \mathrm{BMO}$, if

$$\sup_I \inf_{c \in R} \int_I |f(x) - c|\,dx/|I| = \|f\|_{\mathrm{BMO}} < \infty .$$

Our result is the following.

THEOREM 1. Let $\lambda > 1$. Let $E_1, \cdots, E_N \subset R^d$ be measurable sets such that

$$\min_{1 \le j \le N} |I \cap E_j|/|I| < 2^{-2d\lambda} \tag{1.1}$$

for any I. Then, there exist functions $\{f_j(x)\}_{j=1}^N$ such that

$$\sum_{j=1}^N f_j(x) \equiv 1 , \tag{1.2}$$

$$0 \leqq f_j(x) \leqq 1 , \quad 1 \leqq j \leqq N , \tag{1.3}$$

$$f_j(x) = 0 \quad \text{a.e.} \quad \textbf{on} \quad E_j , \quad 1 \leqq j \leqq N , \tag{1.4}$$

$$\|f_j\|_{\mathrm{BMO}} \leqq c_1(d, N)/\lambda , \quad 1 \leqq j \leqq N . \tag{1.5}$$

Converely, if there exist $\{f_j(x)\}_{j=1}^N$ that satisfy (1.2)-(1.4) and

$$\|f_j\|_{\mathrm{BMO}} \leqq c_2(d, N)/\lambda , \quad 1 \leqq j \leqq N , \tag{1.6}$$

then (1.1) holds.

In particular, if $N = 2$, then the following holds.

COROLLARY 1. *Let $\lambda > 1$. Let $A, B \subset R^d$ be measurable sets such that*

$$\min(|I \cap A|/|I|, |I \cap B|/|I|) < 2^{-2d\lambda} \tag{*}$$

for any I. Then, there exists a function $f(x)$ such that

$$f(x) = 1 \quad \text{a.e.} \quad on \quad A , \tag{1.7}$$

$$f(x) = 0 \quad \text{a.e.} \quad on \quad B , \tag{1.8}$$

$$\|f\|_{\mathrm{BMO}} \leqq c_1(d, 2)/\lambda .$$

Conversely, if there exists $f(x)$ that satisfy (1.7)-(1.8) and

$$\|f\|_{\mathrm{BMO}} \leqq c_2(d, 2)/\lambda ,$$

then (*) *holds.*

Corollary 1 is implicit in Garnett-Jones [10] and is the essential part of their proof. [See also Jones [13].] Thus, Theorem 1 is *an* extension of [10]. In § 3, we give the proof of Theorem 1.

Recently, Jones [14] showed that their paper [10] is closely related to the corona problem. Using [10], he gave an estimate for corona solutions. In §§ 4 and 5, we refine Jones' result by using Theorem 1 instead of [10].

I would like to thank Professor P. W. Jonse for sending his papers [13]-[16]. I would like to thank Professor M. Kaneko who suggested me the condtition (*) and Professor K. Yabuta who gave me a valuable information. I would like to thank referee for his helpful suggestions and for finding some errors.

A comment on notation: The letter C will denote the various constants which depend only on d and N. The latters h, i, j, k, m, n and p will denote integers.

2. Preliminaries. First, we prepare some notations and lemmas.

For a cube I, I^* denotes the cube having the same center as I and $\ell(I^*) = 3\ell(I)$, where $\ell(I)$ denotes the side length of I.

We say that $a(x) \in C(R^d)$ is adapted to a cube I if

$$\operatorname{supp} a \subset I^*$$

and

$$|a(x) - a(y)| \leqq |x - y|/\ell(I) .$$

Let q be a large integer, depending only on d and N, such that

$$1 + N3^{2d}q \leqq 2^q . \tag{2.1}$$

In the following, q will be fixed.

A dyadic cube is a cube of the form

$$[k_1 2^{-h}, (k_1 + 1)2^{-h}) \times \cdots \times [k_d 2^{-h}, (k_d + 1)2^{-h})$$

where h and k_j $(1 \leqq j \leqq d)$ are integers. Let D_h denote the set of all dyadic cubes with side length 2^{-hq}.

For each I, set

$$g_j(I) = \log_2(|I|/|I \cap E_j|) , \quad 1 \leqq j \leqq N ,$$

where $\log(|I|/0)$ means ∞.

LEMMA 2.1. *If* $I \subset J$ *and* $2^{kd}|I| = |J|$, *then*

$$g_j(I) \geqq g_j(J) - kd .$$

Proof.

$$\begin{aligned} g_j(I) &= \log_2(|I|/|I\cap E_j|) = \log_2(|J|2^{-kd}/|I\cap E_j|) \\ &= \log_2(|J|/|I\cap E_j|) - kd \geqq \log_2(|J|/|J\cap E_j|) - kd \\ &= g_j(J) - kd\ . \end{aligned}$$

LEMMA A [*See Fefferman-Stein* [7]]. *If* $f\in \mathrm{BMO}(R^d)$, *then*

$$|(f)_I - (f)_{I^*}| \leqq 2(1+3^d)\|f\|_{\mathrm{BMO}}\ ,$$

where $(f)_I = \int_I f(y)dy/|I|$.

Proof. Note that

$$\begin{aligned} \int_I |f(y)-(f)_I|dy/|I| &\leqq \int_I |f(y)-c|dy/|I| + |c-(f)_I| \\ &\leqq 2\int_I |f(y)-c|dy/|I| \quad \text{for any} \quad c\in R\ . \end{aligned}$$

Thus, $\int_I |f(y)-(f)_I|dy/|I| \leqq 2\|f\|_{\mathrm{BMO}}$. So,

$$\begin{aligned} |(f)_I - (f)_{I^*}| &\leqq \int_I |f(y)-(f)_I|dy/|I| + \int_I |f(y)-(f)_{I^*}|dy/|I| \\ &\leqq 2\|f\|_{\mathrm{BMO}} + 3^d\int_{I^*} |f(y)-(f)_{I^*}|dy/|I^*| \\ &\leqq 2(1+3^d)\|f\|_{\mathrm{BMO}}\ . \end{aligned}$$

LEMMA B [*See Coifman-Weiss* [6]].

$$\|f\|_{\mathrm{BMO}} = \sup\left\{\left|\int_{R^d} f(y)h(y)\,dy\right| : \textit{there exists a cube } I \textit{ such that } \operatorname{supp} h\subset I,\ \|h\|_\infty \leqq |I|^{-1},\ \int_I h(y)dy = 0\right\}\ .$$

REMARK 2.1. The function $h(x)$ satisfying the above conditions is called "1-atom".

Lemma B follows immediately from the argument of dual spaces. We omit the proof.

LEMMA C [*John-Nirenberg* [12]]. *If* $f\in \mathrm{BMO}(R^d)$, *then*

$$|\{x\in I: |f(x)-(f)_I| > \lambda\}/|I| \leqq c_3(d)2^{-c_4(d)\lambda/\|f\|_{\mathrm{BMO}}}$$

for any cube I *and any* $\lambda > 0$.

For the proof of Lemma C, see [12].

3. Proof of Theorem 1. The converse part of Theorem 1 is an immediate consequence of Lemma C.

Let I be any cube. By (1.2), there exists $j_0 \in \{1, \cdots, N\}$ such that

$$(f_{j_0})_I \geqq 1/N .$$

Thus,

$$\begin{aligned} |I \cap E_{j_0}|/|I| &\leqq |\{x \in I: |f_{j_0}(x) - (f_{j_0})_I| \geqq 1/N\}|/|I| \quad \text{by (1.4)} \\ &\leqq c_3(d) 2^{-c_4(d)/(Nc_2(d,N)/\lambda)} \quad \text{by (1.6) and Lemma C} \\ &\leqq 2^{-2d\lambda} \quad \text{by} \quad \lambda > 1 \end{aligned}$$

if $c_2(d, N)$ is sufficiently small. This concludes the proof of the converse part of Theorem 1.

The difficult part of our proof is the construction of $f_1, \cdots, f_N$. The idea of the following construction is essentially due to P. W. Jones [13]. [See also L. Carleson [3].]

By (1.1),

$$\left| \bigcap_{j=1}^{N} E_j \right| = 0 .$$

Thus, if λ is not so large, then

$$f_j = \chi_{E_j^c} \Big/ \sum_{k=1}^{N} \chi_{E_k^c} , \quad 1 \leqq j \leqq N ,$$

satisfy the desired properties, where χ_E denote the characteristic function of a measurable set E. So we may assume that λ is large enough.

First, we assume

(3.1) $$E_1, \cdots, E_N \subset [0, 1) \times \cdots \times [0, 1) = I_0 .$$

We will inductively construct the sequences of BMO functions $\{\mathscr{f}_{j,h}\}_{h=1}^{\infty}$ $(1 \leqq j \leqq N)$ such that

(1.2)′ $$\sum_{j=1}^{N} \mathscr{f}_{j,h}(x) \equiv \lambda ,$$

(1.3)′ $$0 \leqq \mathscr{f}_{j,h}(x) \leqq \lambda ,$$

(1.4)′ $$\mathscr{f}_{j,h}(x) \leqq g_j(I)/d \quad \text{on} \quad I \quad \text{if} \quad I \in D_h ,$$

(1.5)′ $$\|\mathscr{f}_{j,h}\|_{\mathrm{BMO}} \leqq c_1(d, N) .$$

If the above $\{\mathscr{f}_{j,h}\}$ have been built, then there exists a sequence

$$1 \leqq h_1 < h_2 < h_3 < \cdots$$

such that $\{\mathscr{f}_{j,h_k}\}_{k=1}^{\infty}$ $(1 \leqq j \leqq N)$ converge weakly* in L^∞ since $\|\mathscr{f}_{j,h}\|_\infty \leqq \lambda$

by (1.3)$'$. Set

$$f_j = w^*\text{-}\lim_{k\to\infty} \mathscr{f}_{j,h_k}/\lambda\ ,$$

Then, (1.2) and (1.3) follow from (1.2)$'$ and (1.3)$'$. Let $h(x)$ be any 1-atom. Then,

$$\begin{aligned}\left|\int f_j(y)h(y)dy\right| &= \left|\lim_{k\to\infty}\int \mathscr{f}_{j,h_k}(y)h(y)dy/\lambda\right| \\ &\leqq \limsup_{k\to\infty} \|\mathscr{f}_{j,h_k}\|_{\mathrm{BMO}}/\lambda \quad \text{by Lemma B} \\ &\leqq c_1(d, N)/\lambda \quad \text{by } (1.5)'\ .\end{aligned}$$

Thus, (1.5) follows from Lemma B. Since

$$\lim_{I\ni x,\, |I|\to 0} g_j(I) = 0$$

for almost every $x \in E_j$ by Lebesgue's theorem,

$$\lim_{h\to\infty} \mathscr{f}_{j,h}(x) = 0 \quad \text{a.e. on} \quad E_j$$

by (1.4)$'$. Thus, (1.4) follows. Hence, $f_1, \cdots, f_N$ are the desired functions.

It is fairly easy to remove the restriction (3.1). By the same argument as above, for any positive integer p, we can construct $f_{j,p}$, $1 \leqq j \leqq N$, such that

$$\begin{gathered}\sum_{j=1}^{N} f_{j,p}(x) \equiv 1\ , \\ 0 \leqq f_{j,p}(x) \leqq 1\ , \\ f_{j,p}(x) = 0 \quad \text{on} \quad E_j \cap \{(x_1, \cdots, x_d)\colon |x_n| \leqq p,\ 1 \leqq n \leqq d\}\ , \\ \|f_{j,p}\|_{\mathrm{BMO}} \leqq c_1(d, N)/\lambda\ .\end{gathered}$$

There exists a sequence

$$1 \leqq p_1 < p_2 < \cdots$$

such that $\{f_{j,p_k}\}_{k=1}^{\infty}$ $(1 \leqq j \leqq N)$ converge weakly* in L^∞. Then,

$$f_j = w^*\text{-}\lim_{k\to\infty} f_{j,p_k}\ , \quad 1 \leqq j \leqq N\ ,$$

are the desired functions.

Thus, all we have to show is the construction of $\{\mathscr{f}_{j,h}\}$ that satisfy (1.2)$'$–(1.5)$'$. In Lemma 3.1, we will construct $\{\mathscr{f}_{j,h}\}$ and show that they satisfy (1.2)$'$–(1.4)$'$. In Lemma 3.3, we will show that they satisfy (1.5)$'$.

Lemma 3.1. *If $E_1, \cdots, E_N$ satisfy* (1.1) *and* (3.1), *then there exist $\{\mathscr{f}_{j,h}(x)\}$ and $A_{j,h} \subset D_h$, where $1 \leqq j \leqq N$ and $1 \leqq h$, having the prop-*

erties (1.2)′-(1.4)′ *and*

$$|f_{j,h}(x) - f_{j,h}(y)| \leqq 2^{(h+1)q}|x-y| , \tag{3.2}$$

$$A_{j,k} = \{I \in D_h : \sup_{x \in I} f_{j,h-1}(x) > g_j(I)/d\} , \tag{3.3}$$

$$f_{j,h}(x) \geqq f_{j,h-1}(x) - 3^d q , \tag{3.4}$$

$$f_{j,h}(x) \geqq f_{j,h-1}(x) \quad on \quad \Big(\bigcup_{I \in A_{j,h}} I^*\Big)^c . \tag{3.5}$$

Proof. By (1.1), for any I

$$\max_{1 \leq j \leq N} g_j(I) \geqq 2d\lambda .$$

Set

$$s(I) = \min\{j : 1 \leqq j \leqq N,\ g_j(I^*) \geqq 2d\lambda\} .$$

We may assume $s(I_0) = 1$. Set

$$f_{1,0}(x) \equiv \lambda ,$$
$$f_{j,0}(x) \equiv 0 , \quad 2 \leqq j \leqq N .$$

Then, $\{f_{j,0}\}$ satisfy (1.2)′-(1.4)′ and (3.2). Assume that $A_{j,h}$ $(1 \leqq j \leqq N, 1 \leqq h \leqq k-1)$ and $f_{j,h}$ $(1 \leqq j \leqq N,\ 0 \leqq h \leqq k-1)$ have been defined so that they satisfy (1.2)′-(1.4)′ and (3.2)-(3.5).

Define $A_{j,k}$ by (3.3). By modifying $f_{j,k-1}$, we will build $f_{j,k}$.

Let $b_I(x)$ be adapted to I, $0 \leqq b_I(x) \leqq 1$ and

$$b_I(x) = 1 \quad \text{on} \quad I . \tag{3.6}$$

Let $A_{j,k} = \{I_m\}_{m=1,\cdots,p}$. Set

$$a_{I_1}(x) = \min(qb_{I_1}(x), f_{j,k-1}(x))$$
$$a_{I_m}(x) = \min\Big(qb_{I_m}(x), f_{j,k-1}(x) - \sum_{n=1}^{m-1} a_{I_n}(x)\Big)$$
$$= \min\Big(qb_{I_m}(x), \max\Big(f_{j,k-1}(x) - \sum_{n=1}^{m-1} qb_{I_n}(x), 0\Big)\Big)$$
$$\text{for} \quad m = 2, \cdots, N .$$

Since the supports of $\{b_{I_m}\}$ overlap at most 3^d times, $3^{-d}q^{-1}a_{I_m}$ are adapted to I_m. Set

$$\tilde{f}_{j,k}(x) = f_{j,k-1}(x) - \sum_{I \in A_{j,k}} a_I(x) = f_{j,k-1}(x) - v_{j,k}(x) .$$

Since

$$\tilde{f}_{j,k}(x) = \max\Big(f_{j,k-1}(x) - \sum_{I \in A_{j,k}} qb_I(x), 0\Big) ,$$

we get

$$\max(f_{j,k-1}(x) - 3^d q, 0) \leqq \tilde{f}_{j,k}(x) \leqq f_{j,k-1}(x) ,$$
$$f_{j,k-1}(x) = \tilde{f}_{j,k}(x) \quad \text{on} \quad \Big(\bigcup_{I \in A_{j,k}} I^*\Big)^c .$$

Thus, $\{\tilde{f}_{j,k}\}_{j=1}^N$ satisfy (1.3)′, (3.4) and (3.5).

If $I \in A_{j,k}$ and $x \in I$, then

$$\begin{aligned} \tilde{f}_{j,k}(x) &\leqq \max(f_{j,k-1}(x) - q, 0) \quad \text{by (3.6)} \\ &\leqq \max(g_j(J)/d - q, 0) , \quad \text{where} \quad J \in D_{k-1} \quad \text{and} \quad J \supset I , \\ &\leqq g_j(I)/d \quad \text{by Lemma 2.1} . \end{aligned}$$

If $I \in D_k \backslash A_{j,k}$ and $x \in I$, then

$$\tilde{f}_{j,k}(x) \leqq f_{j,k-1}(x) \leqq g_j(I)/d$$

by the definition of $A_{j,k}$. So, $\{\tilde{f}_{j,k}\}_{j=1}^N$ satisfy (1.4)′. But, they don't satisfy (1.2)′. So, we have to modify $\{\tilde{f}_{j,k}\}$ further.

Set

$$\begin{aligned} f_{j,k}(x) &= \tilde{f}_{j,k}(x) + \sum_{I \in \bigcup_{m=1}^N A_{m,k}, s(I)=j} a_I(x) \\ &= \tilde{f}_{j,k}(x) + w_{j,k}(x) . \end{aligned} \tag{3.7}$$

Since

$$-\sum_{j=1}^N v_{j,k}(x) + \sum_{j=1}^N w_{j,k}(x) \equiv 0 ,$$

$\{f_{j,k}\}_{j=1}^N$ satisfy (1.2)′. (1.3)′, (3.4) and (3.5) are clear since $a_I(x) \geqq 0$.

If $I \in D_k$ and $w_{j,k}(x) \equiv 0$ on I, then

$$f_{j,k}(x) = \tilde{f}_{j,k}(x) \leqq g_j(I)/d \quad \text{on} \quad I$$

since $\tilde{f}_{j,k}$ satisfies (1.4)′. If $I \in D_k$ and $w_{j,k}(x) \not\equiv 0$ on I, then, by the definition of $w_{j,k}$ in (3.7), there exists $J \in D_k$ such that

$$J^* \supset I \qquad \text{and} \qquad g_j(J^*) \geqq 2d\lambda .$$

By Lemma 2.1,

$$g_j(I) \geqq g_j(J^*) - (\log_2 3)d \geqq \lambda d$$

since λ is large. So, by (1.3)′

$$f_{j,k}(x) \leqq \lambda \leqq g_j(I)/d$$

and (1.4)′ holds.

Lastly, we show (3.2). If $x, y \in J$ and $J \in D_k$, then

$$\begin{aligned} &|(-v_{j,k}(x) + w_{j,k}(x)) - (-v_{j,k}(y) + w_{j,k}(y))| \\ &\qquad \leqq \sum_{I \in \bigcup_{m=1}^N A_{m,k}} |a_I(x) - a_I(y)| . \end{aligned} \tag{3.8}$$

Since the supports of $\{a_I\}_{I\in\cup_{m=1}^N A_{m,k}}$ overlap at most $N3^d$ times, (3.8) is dominated by

$$N3^d\cdot 3^d\cdot q\cdot|x-y|\cdot 2^{kq} .$$

So,

$$\begin{aligned}|\ell_{j,k}(x)-\ell_{j,k}(y)| &\leqq |\ell_{j,k-1}(x)-\ell_{j,k-1}(y)|+N3^{2d}2^{kq}q|x-y| \\ &\leqq \{1+N3^{2d}q\}2^{kq}|x-y| \\ &\leqq 2^{(k+1)q}|x-y| \quad \text{by (2.1)} .\end{aligned}$$

This concludes the proof of Lemma 3.1. □

LEMMA 3.2. *$\ell_{j,h}(x)\leqq g_j(I)/d-hq-\log_2(\ell(I))+3\cdot 2^q d^{1/2}+2$ on I for any I such that $\ell(I)\leqq 3\cdot 2^{-hq}$.*

Proof. There exist at most 4^d dyadic cubes $J_1,\cdots,J_{k(I)}\in D_h$, $k(I)\leqq 4^d$, such that

$$J_i\cap I\neq\varnothing .$$

Let

$$r=\min_{1\le i\le k(I)} g_j(J_i) .$$

Then, by (1.4)′

$$\inf_{x\in I}\ell_{j,h}(x)\leqq r/d .$$

So, by (3.2)

$$\ell_{j,h}(x)\leqq r/d+3\cdot 2^q d^{1/2} \quad \text{on} \quad I . \tag{3.9}$$

On the other hand,

$$\begin{aligned} g_j(I) &= \log_2(|I|/|I\cap E_j|) \\ &\geqq \log_2(|I|/\sum_{1\le i\le k(I)}|J_i\cap E_j|) \\ &\geqq \log_2(|I|/(4^d\max_{1\le i\le k(I)}|J_i\cap E_j|)) \\ &= r+\log_2(|I|/2^{-hqd})-2d .\end{aligned} \tag{3.10}$$

Thus, the desired result follows from (3.9) and (3.10). □

LEMMA 3.3. *$\|\ell_{j,h}\|_{\mathrm{BMO}}\leqq c_1(d,N)$.*

Proof. Let I be any cube. If $\ell(I)\leqq 2^{-hq}$, then by (3.2)

$$\inf_{c\in R}\int_I|\ell_{j,h}(y)-c|dy/|I|\leqq 2^q d^{1/2} \tag{3.11}$$

If $0\leqq n<h$ and $2^{-(n+1)q}<\ell(I)\leqq 2^{-nq}$, put

$$\beta_j = \int_I f_{j,n}(y)dy/|I| .$$

Note that by Lemma 3.2

$$\beta_j \leqq g_j(I^*)/d + q + 3\cdot 2^q d^{1/2} + 2 . \tag{3.12}$$

We will show

$$\int_I |f_{j,h}(y) - \beta_j|dy/|I| \leqq C . \tag{3.13}$$

Put

$$\begin{aligned} &\{x \in I : |f_{j,h}(x) - \beta_j| > \alpha\} \\ &\quad = \{x \in I : f_{j,h}(x) < \beta_j - \alpha\} \cup \{x \in I : f_{j,h}(x) > \beta_j + \alpha\} \\ &\quad = G(I, j, \alpha) \cup H(I, j, \alpha) . \end{aligned} \tag{3.14}$$

First, we estimate $|G(I, j, \alpha)|$. Let $\alpha > d^{1/2}2^q$. Note that $f_{j,n}(x) > \beta_j - d^{1/2}2^q$ on I by (3.2). So, if $x \in G(I, j, \alpha)$, then, by (3.5), there exists $J \in A_{j,k}$, $n < k \leqq h$, such that

$$x \in J^* ,$$
$$f_{j,k}(x) < \beta_j - \alpha .$$

So,

$$f_{j,k-1}(x) < \beta_j - \alpha + 3^d q \quad \text{by (3.4)}$$

and

$$f_{j,k-1}(y) < \beta_j - \alpha + 3^d q + 2d^{1/2} \quad \text{on} \quad J \text{ by (3.2)} .$$

Thus,

$$g_j(J)/d < \beta_j - \alpha + 3^d q + 2d^{1/2} \quad \text{by (3.3)} .$$

Noticing the above fact, we can take disjoint dyadic cubes $\{J_m\} \subset \bigcup_{n<k\leqq h} A_{j,k}$ such that

$$\begin{aligned} &J_m \subset I^* , \\ &G(I, j, \alpha) \subset \bigcup_m J_m^* , \\ &g_j(J_m)/d < \beta_j - \alpha + 3^d q + 2d^{1/2} . \end{aligned} \tag{3.15}$$

Thus,

$$\begin{aligned} |G(I, j, \alpha)| &\leqq 3^d \sum_m |J_m| = 3^d \sum |J_m \cap E_j| 2^{g_j(J_m)} \\ &\leqq C2^{\beta_j d - \alpha d} \sum |J_m \cap E_j| \quad \text{by (3.15)} \\ &\leqq C2^{g_j(I^*) - \alpha d} \sum |J_m \cap E_j| \quad \text{by (3.12)} \\ &\leqq C2^{g_j(I^*) - \alpha d} |I^* \cap E_j| \leqq C|I|2^{-\alpha d} . \end{aligned} \tag{3.16}$$

Next, we estimate $|H(I, j, \alpha)|$. Let $\alpha > (N-1)d^{1/2}2^q$. Note that $\sum_{m=1}^{N} \beta_m = \lambda$ by (1.2)$'$. So, if $x \in H(I, j, \alpha)$, then

$$\begin{aligned}\sum_{1 \le m \le N, m \ne j} f_{m,h}(x) &= \lambda - f_{j,h}(x) \\ &= \sum_{m=1}^{N} \beta_m - f_{j,h}(x) = \Big(\sum_{1 \le m \le N, m \ne j} \beta_m\Big) - (f_{j,h}(x) - \beta_j) \\ &\leqq \Big(\sum_{1 \le m \le N, m \ne j} \beta_m\Big) - \alpha .\end{aligned}$$

Thus,

$$\sum_{1 \le m \le N, m \ne j} (\beta_m - f_{m,h}(x)) \geqq \alpha .$$

So,

$$x \in \bigcup_{1 \le m \le N, m \ne j} G(I, m, \alpha/(N-1)) ,$$

Thus,

$$H(I, j, \alpha) \subset \bigcup_{1 \le m \le N, m \ne j} G(I, m, \alpha/(N-1)) .$$

By (3.16),

$$|H(I, j, \alpha)| \leqq (N-1)C|I|2^{-\alpha d/(N-1)} . \tag{3.17}$$

Thus, if $1 \geqq \ell(I) \geqq 2^{-hq}$, then (3.13) follows from (3.16), (3.17) and (3.14).

If $\ell(I) > 1$, put

$$\begin{aligned}&\beta_1 = \lambda \\ &\beta_j = 0 , \quad 2 \leqq j \leqq N .\end{aligned}$$

Then, (3.13) follows from the same argument. Thus, Lemma 3.3 follows from (3.11) and (3.13). □

4. A refinement of Jones' paper "Estimates for the corona problem". Let H^∞ denote the Banach algebra of bounded analytic functions defined on $R^2_+ = \{z = (x, y) : x \in R^1, y > 0\}$, endowed with the usual sup norm. The corona problem is as follows. We are given a finite number of functions $F_1, F_2, \cdots, F_N \in H^\infty$ which satisfy

$$\inf_{z=(x,y) \in R^2_+} \sup_{1 \le j \le N} |F_j(z)| > 0 .$$

We then must produce $G_1, G_2, \cdots, G_N \in H^\infty$ such that

$$\sum_{j=1}^{N} F_j(z)G_j(z) \equiv 1 .$$

The functions G_j are called corona solutions. As is well known, the corona problem was solved affirmatively by L. Carleson [1]. [See also [2], [11], [8] and [18].]

Recently, Jones [14] gave an estimate for the corona solutions.

THEOREM A. *Let* $0 < \varepsilon < c_6(N)$. *Suppose* $F_1, \cdots, F_N \in H^\infty$ *satisfy*

$$\|F_j\|_\infty \leqq 1\,, \quad 1 \leqq j \leqq N\,,$$
$$\max_{1\le j\le N} |F_j(z)| > 1 - \varepsilon \quad \text{for any} \quad z \in R^2_+\,. \tag{4.1}$$

Then, there are corona solutions $G_1, \cdots, G_N \in H^\infty$ *satisfying*

$$\|G_j\|_\infty \leqq 1 + A(N, \varepsilon)\,, \quad 1 \leqq j \leqq N\,,$$
$$\sum_{j=1}^N |F_j(z)G_j(z)| \leqq 1 + A(N, \varepsilon) \quad \text{for any} \quad z \in R^2_+\,,$$
$$\sum_{j=1}^N |\mathrm{Im}\,(F_j(z)G_j(z))| \leqq A(N, \varepsilon) \quad \text{for any} \quad z \in R^2_+\,,$$

where

$$A(N, \varepsilon) = c_7(N)(\log^{(N-1)}(1/\varepsilon))^{-1} \tag{4.2}$$
$$\log^{(k+1)} t = \log\,(\log^{(k)} t)\,.$$

As is pointed out in [14], (4.2) is the best order possible when $N = 2$. In this section, as an application of Theorem 1, we show

THEOREM 2. *In Theorem* A, *we can replace* (4.2) *by*

$$A(N, \varepsilon) = c_8(N)(\log\,(1/\varepsilon))^{-1}\,. \tag{4.3}$$

REMARK 4.1. (4.3) is the best order possible when N is fixed.

In [14], Jones showed two kinds of proofs. In this note, we show Theorem 2 by refining the second proof of [14].

As is shown in [14], though it is not explicitly stated, for the proof of Theorem 2, it suffices to show

THEOREM 3. *Let* $F_1, \cdots, F_N$ *and* ε *be as in Theorem* A. *Then, there exist* $f_1, \cdots, f_N \in \mathrm{BMO}\,(R^1)$ *satisfying*

$$\sum_{j=1}^N f_j(x) \equiv 1\,, \tag{4.4}$$

$$0 \leqq f_j(x) \leqq 1\,, \quad 1 \leqq j \leqq N\,, \tag{4.5}$$

$$\int P_y(x-t)f_j(t)dt < 1/(2N) \quad \text{if} \quad |F_j(x, y)| < 1 - \varepsilon^{1/3}\,, \tag{4.6}$$

$$\|f_j\|_{\mathrm{BMO}} \leqq c_9(N)(\log\,(1/\varepsilon))^{-1}\,, \quad 1 \leqq j \leqq N\,, \tag{4.7}$$

where

$$P_y(x) = y/(\pi(x^2 + y^2))$$

that is the Poisson kernel.

The proof of the fact that Theorem 3 implies Theorem 2 is complicated. We omit it in this note. Roughly speaking, it is through "Carleson measure" that H^∞ relates to BMO(R^1). For the definition of "Carleson measure" and for detailed discussion about the relation between Theorem 2 and Theorem 3, that is the relation among H^∞, BMO(R^1) and "Carleson measure", see [14].

In the following, we prove Theorem 3.

For an interval $I\subset R^1$, let

$$T(I)=\{z=(x, y): x\in I, |I|/2<y<|I|\} ,$$
$$F_j(I)=\inf_{z\in T(I)}|F_j(z)|, 1\leqq j\leqq N .$$

All we need is the following

THEOREM 4. *Let $F_1, \cdots, F_N$ and ε be as in Theorem* A. *Then, there exist measurable sets $E_1, \cdots, E_N\subset R^1$ such that*

$$\text{(C.1)}\qquad \min_{1\leq j\leq N}|I\cap E_j|/|I|<\varepsilon^{1/26}\quad \textit{for any interval}\quad I ,$$

$$\text{(C.2)}\qquad |I\cap E_j|/|I|>1-\varepsilon^{1/101}\quad \textit{if}$$

$$\text{(4.8)}\qquad F_j(I)<1-\varepsilon^{1/3} .$$

Jones showed Theorem 4 for the case $N=2$. Since our proof is very complicated, we postpone it to § 5.

It is fairly easy to show that Theorem 3 follows from Theorem 4 and Theorem 1. This idea is also due to [14]. First, by Theorem 4, we get $E_1, \cdots, E_N$ satisfying (C.1) and (C.2). Next, we apply Theorem 1 to these $E_1, \cdots, E_N$ and $\lambda=-(\log_2\varepsilon)/(52d)$. Then, we get $f_1, \cdots, f_N$ satisfying (1.2)-(1.5). (4.4), (4.5) and (4.7) follow from (1.2), (1.3) and (1.5). So, it suffices to show (4.6).

Let $(x, y)\in R_+^2$ and $1\leqq j\leqq N$ be such that

$$|F_j(x, y)|<1-\varepsilon^{1/3} .$$

Put

$$I=(x-y, x+y) .$$

Then,

$$F_j(I)<1-\varepsilon^{1/3} .$$

So, by (C.2) and (1.4),

$$\text{(4.9)}\qquad \int_I f_j(t)dt/|I|<\varepsilon^{1/101} .$$

On the other hand, by Lemma A and (4.7),

$$\text{(4.10)}\qquad \left|\int_{x-2^k y}^{x+2^k y} f_j(t)dt/2^{k+1}y-\int_{x-2^{k-1}y}^{x+2^{k-1}y} f_j(t)dt/2^k y\right|<8c_9(N)(\log(1/\varepsilon))^{-1}$$

for $k = 1, 2, \cdots$. So, by (4.9) and (4.10),

$$\begin{aligned}\int P_y(x-t)f_j(t)dt &\leqq C\sum_{k=0}^{\infty}\int_{x-2^k y}^{x+2^k y} f_j(t)dt\, 2^{-2k}y^{-1} \\ &\leqq C\sum_{k=0}^{\infty} 2^{-k}\{k(\log(1/\varepsilon))^{-1} + \varepsilon^{1/101}\} \\ &\leqq C(\log(1/\varepsilon))^{-1} \\ &\leqq 1/2N \quad \text{if} \quad c_6(N) \text{ is small enough}.\end{aligned}$$

Thus, (4.6) follows.

5. Proof of Theorem 4. First, we prepare some definitions and lemmas.

DEFINITION. For an interval I, a function $F(x, y)$ defined on R^2_+ and a positive number a, let

$$\begin{aligned}\Gamma(x, a) &= \{(u, v): |x-u| < 2v,\ 0 < v \leqq a\}, \\ F^{*a}(x) &= \inf_{(u,v)\in\Gamma(x,a)} |F(u, v)|, \\ R(I, F, \delta) &= \{x \in I: F^{*|I|}(x) < 1-\delta\}.\end{aligned}$$

For a measurable set E and $x \in R$, let

$$M_E(x) = \sup_{I \ni x} |I \cap E|/|I|.$$

LEMMA 5.1. *Let $F(x, y)$ be as above. Let $\delta > 0$. Let I and J be intervals such that*

$$I \subset J \quad \textit{and} \quad F(I) = \inf_{z\in T(I)} |F(z)| < 1-\delta.$$

Then, $I \subset R(J, F, \delta)$.

Since $\Gamma(x, |J|) \supset T(I)$ for any $x \in I$, this follows very easily. See Fig. 1.

LEMMA D [Jones [14]. See also [4] and [17]]. *Let $0 < \varepsilon < c_{10}$. Let $F(x, y)$ be a complex valued function, harmonic over R^2_+ and satisfying*

$$\|F\|_\infty \leqq 1.$$

Let I be an interval such that

$$\sup_{z\in T(I)} |F(z)| > 1-\varepsilon.$$

Then,

$$|R(I, F, \varepsilon^{1/3})| \leqq \varepsilon^{1/4}|I|.$$

For the proof of Lemma D, see [14].

Our fist claim is the construction of the measurable sets $\mathscr{E}_1, \cdots, \mathscr{E}_N \subset R^1$ such that

$$\text{(C.1)}' \qquad \max_{1\leq j\leq N} |I \cap \mathscr{E}_j|/|I| \geqq 1 - \varepsilon^{1/25} \quad \text{if} \quad I \subset I_1 = (-1, 1)\,,$$

$$\text{(C.2)}' \qquad |I \cap \mathscr{E}_j|/|I| \leqq \varepsilon^{1/100} \quad \text{if} \quad I \subset I_1 \quad \text{and if (4.8)}\,.$$

Note that if these $\mathscr{E}_1, \cdots, \mathscr{E}_N$ have been constructed, then

$$E_j^1 = (\mathscr{E}_j)^c\,, \quad 1 \leqq j \leqq N\,, \tag{5.1}$$

satisfy

$$\text{(C.1)}'' \qquad \min_{1\leq j\leq N} |I \cap E_j^1|/|I| < \varepsilon^{1/25} \quad \text{if} \quad I \subset I_1\,,$$

$$\text{(C.2)}'' \qquad |I \cap E_j^1|/|I| > 1 - \varepsilon^{1/100} \quad \text{if} \quad I \subset I_1 \quad \text{and if (4.8)}\,.$$

In particular, $E_1^1, \cdots, E_N^1$ satisfy (C.1) and (C.2) if $I \subset I_1$.

Now, we show the first step of this construction. See Fig. 2. By (4.1), there exists $p(1) \in \{1, \cdots, N\}$ such that

$$\sup_{z\in T(I_1)} |F_{p(1)}(z)| > 1 - \varepsilon\,.$$

Set

$$R = R(I_1, F_{p(1)}, \varepsilon^{1/3})\,,$$
$$\mathscr{E}(1) = I_1 \backslash R\,.$$

Set

$$\begin{aligned} &\mathscr{E}_{p(1),1} = \mathscr{E}(1)\,, \\ &\mathscr{E}_{j,1} = \varnothing \quad \text{if} \quad j \neq p(1) \quad \text{and} \quad 1 \leqq j \leqq N\,. \end{aligned} \tag{5.2}$$

By Lemma D,

$$|R| \leqq \varepsilon^{1/4}|I_1|\,. \tag{5.3}$$

Set

$$G = \{x \in I_1 \colon M_R(x) > \varepsilon^{1/25}\}\,.$$

By the Hardy-Littlewood maximal theorem and (5.3),

$$|G| \leqq C\varepsilon^{-1/25}|R| \leqq \varepsilon^{1/25}|I_1|\,.$$

If $I \subset I_1$ and $I \not\subset G$, then

$$|I \cap R|/|I| \leqq \varepsilon^{1/25}$$

by the definition of G. So,

$$|I \cap \mathscr{E}_{p(1),1}|/|I| > 1 - \varepsilon^{1/25}\,. \tag{5.4}$$

If $I \subset I_1$ and if $F_{p(1)}(I) < 1 - \varepsilon^{1/3}$, then $I \subset R$ by Lemma 5.1. So,

$$I \cap \mathscr{E}_{p(1),1} = \varnothing . \tag{5.5}$$

Thus, by (5.4) and (5.5), $\mathscr{E}_{1,1}, \cdots, \mathscr{E}_{N,1}$ satisfy (C.1)′ and (C.2)′ under an additional condition $I \not\subset G$. This concludes the first step.

In the second step, we make each $\mathscr{E}_{j,1}$ a little larger so that (C.1)′ holds under a weaker condition than $I \not\subset G$. But, if we make $\mathscr{E}_{j,1}$ too large, then (C.2)′ will not hold. This is the difficult point. Set

$$G = \sum_m I(2, m) , \tag{5.6}$$

where $\{I(2, m)\}_{m=1}^{\infty}$ are disjoint open intervals. In the second step we repeat the above argument for each $I(2, m)$. In the first step, we had only to consider the intervals included in I_1. But, this time, we cannot restrict our attention to the intervals included in $I(2, m)$ since the condition (C.2)′ is very delicate. We have to pay attention to the relations among $\{I(2, m)\}_m$. This is why we will introduce the intervals $\{J(2, m)\}_m$ in the following. See Fig. 3.

LEMMA 5.2. *We can inductively construct open intervals* $\{I(h, m)\}$, $\{J(h, m)\}$, *measurable sets* $\{\mathscr{E}(h, m)\}$ *and integers* $\{p(h, m)\}$, *where* $1 \leqq h$ *and* $1 \leqq m$, *having following properties*:

(i) $I(1, 1) = I_1$, $\mathscr{E}(1, 1) = \mathscr{E}(1)$, $p(1, 1) = p(1)$, $J(1, 1) = (-\varepsilon^{-1/100}, \varepsilon^{-1/100})$, $I(1, m) = \varnothing$, $\mathscr{E}(1, m) = \varnothing$, $p(1, m) = 0$, $J(1, m) = \varnothing$ *for* $m \geqq 2$, $\{I(2, m)\}_m$ *are defined by* (5.6),

(ii) $\sum_m I(h + 1, m) \subset \sum_m I(h, m)$, *where* $\{I(h, m)\}_m$ *are disjoint*,

(iii) $\sum_m |I(h + 1, m)| \leqq \varepsilon^{1/25} \sum_m |I(h, m)|$,

(iv) $\sum_m J(h, m) = \{x : M_{\sum_n I(h,n)}(x) > \varepsilon^{1/100}\}$, *where* $\{J(h, m)\}_m$ *are disjoint*,

(v) $\mathscr{E}(h, m) \subset I(h, m)$,

(vi) *if* $I(h, m) \neq \varnothing$, *then* $p(h, m) \in \{1, \cdots, N\}$,

(vii) *if* $I \subset I_1$ *and if* $I \not\subset \sum_m I(h + 1, m)$, *then there exist* $h' \leqq h$ *and* $n \geqq 1$ *such that*

$$|I \cap \mathscr{E}(h', n)|/|I| \geqq 1 - \varepsilon^{1/25} , \tag{5.7}$$

(viii) *if* I, h *and* n *satisfy* $I \subset \sum_m J(h, m)$, $p(h, n) \in \{1, \cdots, N\}$ *and* $F_{p(h,n)}(I) < 1 - \varepsilon^{1/3}$, *then* $\mathscr{E}(h, n) \cap I = \varnothing$.

Let us accept Lemma 5.2 for the moment. Set

$$\mathscr{E}_{j,h} = \bigcup_{k,m : k \leq h, p(k,m) = j} \mathscr{E}(k, m) . \tag{5.8}$$

Note that when $h = 1$, this definition concides with (5.2). Note that

$$\mathscr{E}_{j,1} \subset \mathscr{E}_{j,2} \subset \cdots \subset \mathscr{E}_{j,h} \subset \cdots . \tag{5.9}$$

LEMMA 5.3.

(C.1)‴ $$\max_{1 \le j \le N} |I \cap \mathscr{E}_{j,h}|/|I| \geqq 1 - \varepsilon^{1/25}$$
$$\text{if} \quad I \subset I_1 \quad \text{and if} \quad I \not\subset \sum_m I(h+1, m) ,$$

(C.2)‴ $$|I \cap \mathscr{E}_{j,h}|/|I| \leqq \varepsilon^{1/100} \quad \text{if} \quad I \subset I_1 \quad \text{and if (4.8)} .$$

Proof. If $I \subset I_1$ and if $I \not\subset \sum_m I(h+1, m)$, then by (vii) there exist $h' \leqq h$ and $n \geqq 1$ such that (5.7). Since $\mathscr{E}_{p(h',n),h} \supset \mathscr{E}(h', n)$,

$$|I \cap \mathscr{E}_{p(h',n),h}|/|I| \geqq 1 - \varepsilon^{1/25} .$$

This shows (C.1)‴.

Note that by (ii) and (iv)

$$\sum_m J(k+1, m) \subset \sum_m J(k, m) . \tag{5.10}$$

Let $I \subset I_1$ and $F_j(I) < 1 - \varepsilon^{1/3}$. If $I \subset \sum_m J(h, m)$, then by (5.10) $I \subset \sum_m J(h', m)$ for any $h' \in \{1, \cdots, h\}$. By (viii),

$$\mathscr{E}(h', n) \cap I = \varnothing$$

for any $h' \leqq h$ and $n \geqq 1$ such that $p(h', n) = j$. So, by (5.8),

$$\mathscr{E}_{j,h} \cap I = \varnothing . \tag{5.11}$$

If $k_I < h$, $I \subset \sum_m J(k_I, m)$ and $I \not\subset \sum_m J(k_I + 1, m)$, then by the same argument as above

$$\mathscr{E}_{j,k_I} \cap I = \varnothing .$$

By (iv)

$$|I \cap \sum_m I(k_I + 1, m)|/|I| \leqq \varepsilon^{1/100} .$$

Since

$$\mathscr{E}_{j,h} \subset \mathscr{E}_{j,k_I} \cup (\sum_m I(k_I + 1, m))$$

by (5.8) and (v).

$$\begin{aligned} |I \cap \mathscr{E}_{j,h}|/|I| &\leqq |I \cap \mathscr{E}_{j,k_I}|/|I| + |I \cap \sum_m I(k_I + 1, m)|/|I| \\ &\leqq \varepsilon^{1/100} . \end{aligned} \tag{5.12}$$

So, (C.2)‴ follows from (5.11) and (5.12). This conclcudes the proof of Lemma 5.3. □

Set

$$\mathscr{E}_j = \bigcup_{k=1}^{\infty} \mathscr{E}_{j,k}\,, \quad 1 \leqq j \leqq N\,.$$

Let $I \subset I_1$. Since

$$|\sum_m I(h+1, m)| \longrightarrow 0 \quad \text{as} \quad h \longrightarrow \infty$$

by (iii), there exists h_I such that

$$I \not\subset \bigcup_m I(h+1, m) \quad \text{for any} \quad h \geqq h_I\,.$$

Thus,

$$\begin{aligned}\max_{1\leq j\leq N} |I \cap \mathscr{E}_j|/|I| &= \max \lim_{h\to\infty} |I \cap \mathscr{E}_{j,h}|/|I| \quad \text{by (5.9)} \\ &= \lim_{h\to\infty} \max |I \cap \mathscr{E}_{j,h}|/|I| \\ &\geqq 1 - \varepsilon^{1/25} \quad \text{by (C.1)}'''\,.\end{aligned}$$

If $I \subset I_1$ and if (4.8), then

$$\begin{aligned}|I \cap \mathscr{E}_j|/|I| &= \lim_{h\to\infty} |I \cap \mathscr{E}_{j,h}|/|I| \quad \text{by (5.9)} \\ &\leqq \varepsilon^{1/100} \quad \text{by (C.2)}'''\,.\end{aligned}$$

Thus, these $\mathscr{E}_j$ $(1 \leqq j \leqq N)$ satisfy (C.1)$'$ and (C.2)$'$. So, E_j^1 $(1 \leqq j \leqq N)$ defined by (5.1) satisfy (C.1)$''$ and (C.2)$''$.

Lastly, we remove the restriction $I \subset I_1$ in (C.1)$''$ and (C.2)$''$. By the same argument as above, for each positive integer L we get measurable sets $E_1^L, \cdots, E_N^L$ such that

(C.1)$''''$ $\qquad \min\limits_{1\leq j\leq N} |I \cap E_j^L|/|I| < \varepsilon^{1/25}$ if $I \subset (-L, L)$,

(C.2)$''''$ $\qquad |I \cap E_j^L|/|I| > 1 - \varepsilon^{1/100}$ if $I \subset (-L, L)$ and if (4.8).

There exists a sequence

$$1 \leqq L(1) < L(2) < \cdots$$

such that

$$\{\chi_{E_j^{L(k)}}\}_{k=1}^{\infty}\,, \quad 1 \leqq j \leqq N\,,$$

converge weakly * in L^∞. Let

$$E_j = \{x \in R\colon w^*\text{-}\lim_{k\to\infty} \chi_{E_j^{L(k)}}(x) > 1/2\}\,.$$

Then,

$$\begin{aligned}\min_{1\leq j\leq N} |I \cap E_j|/|I| &\leqq \min_{1\leq j\leq N} 2 \int_I w^*\text{-}\lim \chi_{E_j^{L(k)}} dy/|I| \\ &= 2 \lim_{k\to\infty} \min_{1\leq j\leq N} |I \cap E_j^{L(k)}|/|I| \leqq 2\varepsilon^{1/25} < \varepsilon^{1/26}\,.\end{aligned}$$

Thus, (C.1) follows. If $F_j(I) < 1 - \varepsilon^{1/3}$, then

$$\begin{aligned}|I \cap E_j|/|I| &= 1 - |I \cap E_j^c|/|I| \\ &\geqq 1 - 2\left\{|I| - \int_I w^*\text{-}\lim_{k\to\infty} \chi_{E_j^{L(k)}} dy\right\}\Big/|I| \\ &= 1 - 2\{|I| - \lim_k |I \cap E_j^{L(k)}|\}/|I| \\ &\geqq 1 - 2\{1 - (1 - \varepsilon^{1/100})\} \geqq 1 - \varepsilon^{1/101} .\end{aligned}$$

Thus, (C.2) follows. This concludes the proof of Theorem 4.

Proof of Lemma 5.2. Assume that $\{I(h, m)\}$, $(h = 2, \cdots, k;\ m = 1, 2, \cdots)$, $\{J(h, m)\}$, $\{\mathscr{E}(h, m)\}$, $\{p(h, m)\}$, $(h = 2, \cdots, k-1;\ m = 1, 2, \cdots)$, have been defined so that they satisfy (i)-(viii). Define $\{J(k, m)\}_m$ by (iv). We show how to define $\{\mathscr{E}(k, m)\}_m$, $\{p(k, m)\}_m$ and $\{I(k+1, m)\}_m$.

Let

$$t(I) = \min\{1 \leqq j \leqq N\colon \sup_{z \in T(I)} |F_j(z)| > 1 - \varepsilon\} .$$

By (4.1), $t(I)$ is well defined.

If $I(k, n) = \varnothing$, then set

$$\mathscr{E}(k, n) = \varnothing , \qquad p(k, n) = 0 .$$

If $I(k, n) \neq \varnothing$, then there exists unique $J(k, m_n)$ satisfying

$$I(k, n) \subset J(k, m_n)$$

by the definition of $\{J(k, m)\}_m$. Set

$$\begin{aligned}R(k, n) &= I(k, n) \cap R(J(k, m_n), F_{t(J(k,m_n))}, \varepsilon^{1/3}) , \\ \mathscr{E}(k, n) &= I(k, n) \backslash R(k, n) , \\ p(k, n) &= t(J(k, m_n)) .\end{aligned}$$

Note that

$$\sum_{n: I(k,n) \subset J(k,m)} \mathscr{E}(k, n) \subset J(k, m) \backslash R(J(k, m), F_{t(J(k,m))}, \varepsilon^{1/3}) . \tag{5.13}$$

Set

$$\sum_i I(k+1, i) = \sum_n \{x \in I(k, n)\colon M_{R(k,n)}(x) > \varepsilon^{1/25}\} \tag{5.14}$$

where $\{I(k+1, i)\}_i$ are disjoint open intervals. Then,

$$\begin{aligned}\sum_i |I(k+1, i)| &\leqq C\varepsilon^{-1/25} \sum_n |R(k, n)| \\ &\qquad \text{by the Hardy-Littlewood maximal theorem ,} \\ &\leqq C\varepsilon^{-1/25} \sum_m |R(J(k, m), F_{t(J(k,m))}, \varepsilon^{1/3})|\end{aligned}$$

$$
\begin{aligned}
& \text{by the definition of } \{R(k, n)\}_n , \\
(5.15) \qquad & \leqq C\varepsilon^{-1/25+1/4} \sum_m |J(k, m)| \quad \text{by Lemma D} , \\
& \leqq C\varepsilon^{-1/25+1/4-1/100} \sum_n |I(k, n)| \\
& \text{by the definition of } \{J(k, m)\}_m \text{ and} \\
& \text{the Hardy-Littlewood maximal theorem} , \\
& \leqq \varepsilon^{1/25} \sum_n |I(k, n)| .
\end{aligned}
$$

Lastly, we show that the above defined $\{J(k, m)\}_m$, $\{\mathscr{E}(k, m)\}_m$, $\{p(k, m)\}_m$ and $\{I(k+1, m)\}_m$ satisfy (ii)-(viii). (ii) and (iv)-(vi) are clear. (iii) follows from (5.15).

Let

$$I \subset I_1 \quad \text{and} \quad I \not\subset \sum_m I(k+1, m) .$$

If $I \not\subset \sum_m I(k, m)$, then (vii) follows from the hypothesis of induction. Let

$$I \subset I(k, n) .$$

Then, by (5.14)

$$|I \cap R(k, n)|/|I| \leqq \varepsilon^{1/25} .$$

So

$$|I \cap \mathscr{E}(k, n)|/|I| > 1 - \varepsilon^{1/25} .$$

Thus, (vii) follows.

Let

$$
\begin{aligned}
& I \subset J(k, m) , \qquad p(k, n) \in \{1, \cdots, N\} \quad \text{and} \\
(5.16) \qquad & F_{p(k,n)}(I) < 1 - \varepsilon^{1/3} .
\end{aligned}
$$

If $I(k, n) \cap I \neq \varnothing$, then

$$I(k, n) \subset J(k, m)$$

by the definition of $\{J(k, m)\}_m$ and

$$p(k, n) = t(J(k, m)) \tag{5.17}$$

by the definition of $p(k, n)$. So, by (5.16)-(5.17) and Lemma 5.1,

$$I \subset R(J(k, m), F_{t(J(k,m))}, \varepsilon^{1/3}) .$$

Thus, by (5.13)

$$I \cap \mathscr{E}(k, n) = \varnothing .$$

Hence, (viii) holds. This concludes the proof of Lemma 5.2. □

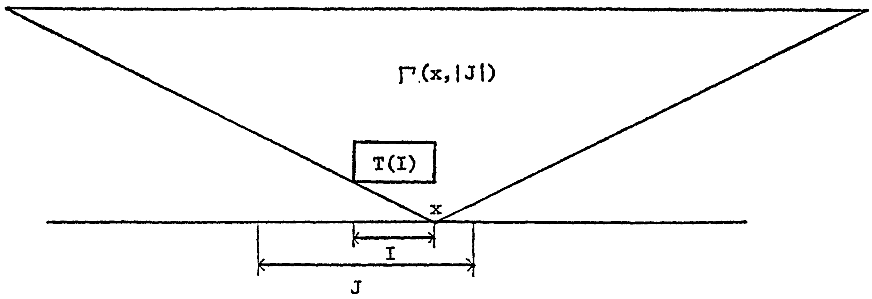

FIGURE 1

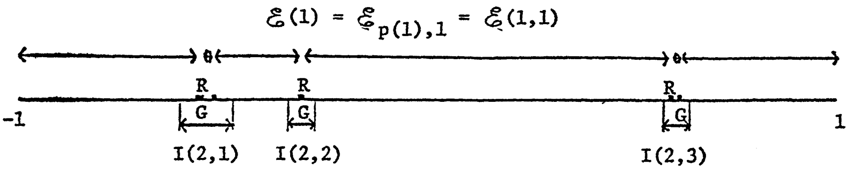

FIGURE 2

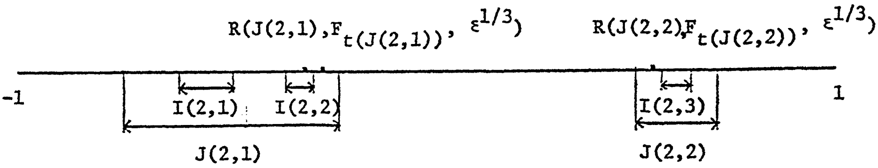

FIGURE 3

6. Further discussion. Jones [14] showed that for the case $d = 1$ Corollary 1 follows from Theorem A. By the same argument, we can show that for the case $d = 1$ Theorem 1 follows from Theorem 2.

The following is completely due to [14].

Let $E_1, \cdots, E_N \subset R^1$ be such that (1.1). Let $h_j(z)$ be the harmonic extension to R^2_+ of $\chi_{E_j}(x)$ and $Hh_j(z)$ be the harmonic extension to R^2_+ of the Hilbert transform of $\chi_{E_j}(x)$. If

$$|(x - 2^\lambda y, x + 2^\lambda y) \cap E_j|/|(x - 2^\lambda y, x + 2^\lambda y)| \leqq 2^{-2\lambda}$$

and if λ is large enough, then

$$\begin{aligned} h_j(x, y) &= \int_{E_j} (y/((x-t)^2 + y^2))dt/\pi \\ &\leqq \int_{|x-t|>2^\lambda y} (y/((x-t)^2 + y^2))dt/\pi + \int_{(x-2^\lambda y, x+2^\lambda y)\cap E_j} dt/(\pi y) \qquad (6.1) \\ &\leqq 2^{-\lambda/2}\,. \end{aligned}$$

Set

$$F_j(z) = 2^{-2N(h_j(z)+iHh_j(z))}\,, \quad \text{where} \quad i = \sqrt{-1}\,.$$

Then,

$$F_j \in H^\infty ,$$
$$\|F_j\|_\infty \leqq 1 ,$$
$$\max_{1\le j\le N} |F_j(z)| > 1 - 2N2^{-\lambda/2} \quad \text{for any} \quad z \in R^2_+ \quad \text{by (6.1)} .$$

Let $G_1, \cdots, G_N$ be corona solutions guaranteed by Theorem 2. Since

$$\|G_j\|_\infty \leqq 2$$
$$|F_j(x, 0)| \leqq 2^{-2N} \quad \text{a.e. on} \quad E_j ,$$

we get

$$|G_j(x, 0)F_j(x, 0)| \leqq 2\cdot 2^{-2N} \leqq 1/2N \quad \text{a.e. on} \quad E_j . \tag{6.2}$$

Since

$$\|\mathrm{Im}\,(F_j(\cdot, 0)G_j(\cdot, 0))\|_\infty \leqq A(N, 2N2^{-\lambda/2}) \leqq C_N/\lambda$$

by Theorem 2 and since the Hilbert transform is a bounded operator from L^∞ to BMO, we get

$$\|\mathrm{Re}\,(F_j(\cdot, 0)G_j(\cdot, 0))\|_{\mathrm{BMO}} \leqq C_N/\lambda . \tag{6.3}$$

Set

$$\tilde{f}_j(x) = \max\,(\mathrm{Re}\,(F_j(x, 0)G_j(x, 0) - 1/2N), 0) .$$

Then,

$$\tilde{f}_j(x) = 0 \quad \text{on} \quad E_j \quad \text{by (6.2)}$$

and

$$\|\tilde{f}_j\|_{\mathrm{BMO}} \leqq C_N/\lambda \quad \text{by (6.3)} .$$

Since

$$\sum_{j=1}^N \mathrm{Re}\,(F_jG_j) \equiv 1 ,$$
$$\sum_{j=1}^N \tilde{f}_j(x) \geqq 1/2 \quad \text{for any} \quad x \in R^1 .$$

Set

$$f_j(x) = \tilde{f}_j(x) \Big/ \sum_{k=1}^N \tilde{f}_k(x) .$$

Then, these satisfy (1.2)–(1.5).

REMARK. Recently, J. B. Garnet and P. W. Jones found a simple proof of [15]. And their method simplifies the proof of Theorem 1 in this paper. I would like to thank Professor P. W. Jones for valuable information and for his encouragement.

REFERENCES

1. L. Carleson, *Interpolation by bounded analytic functions and the corona theorem*, Ann. of Math., **76** (1962), 547-559.
2. L. Carleson, *The corona theorem*, Lecture Notes in Math., **118** (1968), 121-132.
3. ———, *Two remarks on H^1 and* BMO, Advances in Math., **22** (1976), 269-275.
4. S. Y. Chang, *A characterization of Douglas subalgebras*, Acta Math., **137** (1976), 81-89.
5. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc., **79** (1980), 249-254.
6. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), 569-645.
7. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math., **129** (1972), 137-193.
8. T. W. Gamelin, *Wolff's proof of the corona theorem*, preprint.
9. J. B. Garnett, *Two constructions in* BMO, Proceedings of Symposia in Pure Mathematics, **35** (1978), 295-302.
10. J. B. Garnett and P, W. Jones, *The distance in* BMO *to L^∞*, Ann. of Math., **108** (1978), 373-393.
11. L. Hörmander, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc., **73** (1967), 943-949.
12. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415-426.
13. P. W. Jones, *Constructions with functions of bounded mean oscillation*, Ph.D. thesis, University of California, 1978.
14. ———, *Estimates for the corona problem*, to appear in J. Functional Anal.
15. ———, *Factorization of A_p weights*, Ann. of Math., **111** (1980), 511-530.
16. ———, *Carleson measures and the Fefferman-Stein decomposition of* BMO(R), Ann. of Math., **111** (1980), 197-208.
17. D. E. Marshall, *Subalgebras of L^∞ containg H^∞*, Acta Math., **137** (1976), 91-98.
18. M. Rosenblum, *A corona theorem for countably many functions*, Integral Equations and Operator Theory, **3** (1980), 125-137.
19. N. Th. Varopoulos, BMO *functions and the $\bar{\partial}$-equations*, Pacific J. Math., **71** (1977), 221-273.
20. ———, *A remark on functions of bounded mean oscillation and bounded harmonic functions*, Pacific J. Math., **74** (1978), 257-259.
21. ———, *A probabilistic proof of the Garnett-Jones theorem on* BMO, preprint.

Received April 20, 1980. Supported in part by Science Research Foundation of Japan (General Reserch (C) 1980).

UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90024
AND
TOHOKU UNIVERSITY
KAWAUCHI, SENDAI, JAPAN

Current address: University of Chicago
Chicago, IL 60637