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AN INDEX THEOREM AND HYPOELLIPTICITY ON NILPOTENT LIE GROUPS

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Extending results of Grushin we determine the index of $p(x, D)$ where $p(x, \xi)$ is a polynomial homogeneous with respect to some family of dilations on R^{2d} and $p(x, \xi) \neq 0$ if $(x, \xi) \neq (0, 0)$. In general these operators are not elliptic. If G is a step two nilpotent Lie group and P is a left invariant differential operator on G which is homogeneous with respect to some family of dilations, we apply this index theorem to prove that P is hypoelliptic if and only if P^* is hypoelliptic. This extends a result of Helffer and Nourrigat.

1. **An index theorem.** A family of dilations on a Lie algebra $\mathscr{G}$ is a one parameter family of automorphisms $\{\delta_r : r > 0\}$ of $\mathscr{G}$ of the form $\delta_r = \exp((\log r)A)$, where A is a diagonalizable automorphism of $\mathscr{G}$ with positive real eigenvalues. There is no loss of generality in assuming that the smallest eigenvalue is 1. A finite dimensional normed vector space V with norm $|\ |$ determines an abelian Lie algebra. Let $\{\delta_r\}$ be a family of dilations on V. For $w \in V$ define $\|w\|$ by $\|w\| = r$ if $|\delta_r^{-1}(w)| = 1$. Then $w \to \|w\|$ is continuous on V and C^∞ on $V - \{0\}$ by the implicit function theorem. Let $\mathscr{B} = \{w_1, w_2, \cdots, w_n\}$ be a basis for V consisting of eigenvectors of A with corresponding eigenvalues $\mu_1, \cdots, \mu_n$. If $w = a_1 w_1 + \cdots + a_n w_n$, then

$$\delta_r w = \sum r^{\mu_j} a_j w_j \quad \text{and} \tag{1.1}$$

$$\|w\| \approx \sum |a_j|^{1/\mu_j} . \tag{1.2}$$

Throughout this section we will be considering a family of dilations on the abelian Lie algebra $R^{2d} = R^d_x \oplus R^d_\xi$. We do not necessarily assume that either R^d_x or R^d_ξ is invariant under $\{\delta_r\}$. Let $f \in C^\infty(R^{2d})$, $f(w) = 0$ for $\|w\| \leqq 1/2$, and $f(w) = 1$ for $\|w\| \geqq 1$. Define $\Phi(w) = 1 + f(w)\|w\|$ and $\varphi(w) = 1$ for all $w = (x, \xi) \in R^{2d}$. Note that there is a C such that if $|w - w'| \leqq \Phi(w)$ then $\Phi(w') \leqq C\Phi(w)$. Thus (Φ, φ) is a pair of weight functions on R^d_x as defined in Beals [1]. We will usually not mention φ and will refer to Φ as the weight function for the family of dilations $\{\delta_r\}$. Note that Φ satisfies the coercive estimate

$$|w| \leqq C\Phi(w)^{\bar{\mu}} \tag{1.3}$$

where $\bar{\mu} = \max\{\mu_1, \cdots, \mu_{2d}\}$.

For $m \in \boldsymbol{R}$, let S_Φ^m denote the set of all smooth functions p on $\boldsymbol{R}^{2d}$ such that for each α and $\beta \in \boldsymbol{N}^d$

$$\sup \{\Phi(x, \xi)^{-m+|\alpha|} |D_\xi^\alpha D_x^\beta p(x, \xi)| : (x, \xi) \in \boldsymbol{R}^{2d}\} < \infty .$$

$\mathscr{L}_\Phi^m$ is the set of pseudodifferential operators with symbols in S_Φ^m, H_Φ^m is the associated (global) Sobolev space as defined in [1] and $\| \ \|_{m,\Phi}$ is a norm for the topology on H_Φ^m. We note that in the special case where $m \in \boldsymbol{N}$ and $m/\mu_j \in \boldsymbol{N}$ for all j (this is necessarily the case in the context of Theorem 2 below, by Proposition 1.3 of [7]), then $\| \ \|_{m,\Phi}$ can be given explicitly as follows: Let $\mathscr{B}$ be a basis for $\boldsymbol{R}^{2d}$ consisting of eigenvectors for $\{\delta_r\}$ and let $a_j(x, \xi)$ be the jth coordinate of (x, ξ) with respect to the basis $\mathscr{B}$. By (1.2) above and 6.17 of [1]

$$\|u\|_{m,\Phi} \approx \sum \|a_j(x, D)^{m/\mu_j} u\| + \|u\| \tag{1.4}$$

where $\| \ \|$ is the L^2 norm.

We shall denote by $\widetilde{S}_\Phi^m$ the subset of S_Φ^m consisting of functions p such that for all α and β in $\boldsymbol{N}^d$

$$\sup \{\Phi(x, \xi)^{-m+|\alpha|+|\beta|} |D_\xi^\alpha D_x^\beta p(x, \xi)| : (x, \xi) \in \boldsymbol{R}^{2d}\} < \infty .$$

We say that $p \in C^\infty(\boldsymbol{R}^{2d})$ is homogeneous of degree m with respect to $\{\delta_r\}$ for large w if there is a c, $0 < c < 1$, such that $p(\delta_r w) = r^m p(w)$ for all $r \geqq 1$ and all w for which $\|w\| \geqq c$. If p is homogeneous of degree m with respect to $\{\delta_r\}$ for large w and if v is an eigenvector for the generator A of $\{\delta_r\}$ with eigenvalue μ, then

$$r^\mu D_v p(\delta_r w) = r^m D_v p(w) .$$

If $\|w\| \geqq 1$, let $r = \|w\|$ and $w' = \delta_r^{-1}(w)$. Then $\|w'\| = 1$ and $D_v p(w) = \|w\|^{m-\mu} D_v p(w')$. Thus there is a C such that

$$|D_v p(w)| \leqq C\|w\|^{m-\mu} \leqq C\|w\|^{m-1} \tag{1.5}$$

for all w, $\|w\| \geqq 1$. Consequently if p is homogeneous of degree m with respect to $\{\delta_r\}$ for large w, then $p \in \widetilde{S}_\Phi^m$. It follows from this remark that $\Phi \in \widetilde{S}_\Phi^1$ and hence $\Phi^m \in \widetilde{S}_\Phi^m$ for all $m \in \boldsymbol{R}$.

We say that $p \in S_\Phi^m$ is Φ-elliptic if there is a C such that $\Phi(w)^m \leqq C|p(w)|$ for $|w| \geqq C$. Note that if p is a polynomial and p is homogeneous of degree m with respect to $\{\delta_r\}$, then p is Φ-elliptic if and only if $p(w) \neq 0$ for $|w| \neq 0$. Note that in general Φ-ellipticity does not imply ellipticity in the usual sense. For example on $\boldsymbol{R}^2 \times \boldsymbol{R}^2$, $p(x, \xi) = \xi_1^4 + x_1^2 + 2x_1\xi_1 + \xi_1^2 + \xi_2^2 + x_2^2$ is Φ-elliptic and homogeneous of degree two, where the dilations are given in terms of coordinates $a_1 = \xi_1$, $a_2 = x_1 + \xi_1$, $a_3 = \xi_2$ and $a_4 = x_2$, with $\mu_1 = 2$, $\mu_2 = \mu_3 = \mu_4 = 1$.

If Γ is an oriented curve and p maps the range of Γ into

$C - \{0\}$, let $\Delta_\Gamma \arg p$ denote the change in the argument of p along Γ. In the following theorem Γ is the curve in $\boldsymbol{R}_x \oplus \boldsymbol{R}_\xi$ given by $x(\theta) = \cos\theta$, $y(\theta) = \sin\theta$, $0 \leqq \theta \leqq 2\pi$. In the case where $\boldsymbol{R}_\xi^d$ and $\boldsymbol{R}_x^d$ are eigenspaces for A with eigenvalues 1 and $1 + \delta$ respectively, $\delta > 0$, this theorem was proved in [2].

THEOREM 1. *Let* $\delta_r = \exp((\log r)A)$, $r > 0$, *be a family of dilations on* $\boldsymbol{R}^{2d}$, Φ *the weight function for* $\{\delta_r\}$. *Let* $p = p_0 + p_1$ *where* p_0 *is* Φ*-elliptic and homogeneous of degree* m *with respect to* $\{\delta_r\}$ *for large* w *and* $p_1 \in S_\Phi^{m_1}$ *for some* $m_1 < m$. *Then* $p(x, D): H_\Phi^m \to L^2$ *is Fredholm. If* $d > 1$, *then* $\operatorname{ind} p(x, D) = 0$. *If* $d = 1$, *then* $2\pi \operatorname{ind} p(x, D) = \Delta_\Gamma \arg p_0$. *If* $d = 1$ *and* p_0 *is a polynomial, then* $\operatorname{ind} p(x, D)$ *is also given by* (1.6) *below.*

Proof. By Theorem 7.2 of [1] and (1.3) above, $p(x, D): H_\Phi^m \to L^2$ is Fredholm. By Corollary 6.13 of [1], $p_1(x, D): H_\Phi^m \to L^2$ is compact. Hence $\operatorname{ind} p_0(x, D) = \operatorname{ind} p(x, D)$. Let $f \in C^\infty(\boldsymbol{R}^{2d})$ be real valued, $f(w) = 0$ for $\|w\| \leqq 1/2$, $f(w) = 1$ for $\|w\| \geqq 1$. Let $a(w) = f(w)/\|w\|^{m/2}$, $q = p_0 a^2$. Then $A = a(x, D) \in \mathscr{L}_\Phi^{-m/2}$, and by the pseudo-differential operator calculus $p_0(x, D)A^*A = q(x, D) + R$ where $R \in \mathscr{L}_\Phi^{-1}$. Thus $\operatorname{ind} q(x, D) = \operatorname{ind} p_0(x, D)$. Also $q(\delta_r w) = p_0(w) \neq 0$ for all $r \geqq 1$ and all w, $\|w\| = 1$. If $d > 1$, $\{w \in \boldsymbol{R}^{2d}: \|w\| = 1\}$ is simply connected, so q can be continuously deformed to a nonzero constant through Φ-elliptic symbols which are homogeneous of degree 0 for large w. Hence $\operatorname{ind} q(x, D) = 0$.

Now consider the case $d = 1$. Although q is not elliptic in the classic sense, q is included in the class of symbols for which Hormander proves the index theorem in §7 of [5]. In [5] it is shown that $2\pi \operatorname{ind} q^w(x, D) = \Delta_\Gamma \arg q$, where $q^w(x, D)$ is the Weyl pseudo-differential operator with symbol q. By (4.10) of [5] $q^w(x, D) = a(x, D)$ where $a = q + r$, $r \in S_\Phi^{-1}$. Thus $\operatorname{ind} q(x, D) = \operatorname{ind} q^w(x, D)$. Clearly $\Delta_\Gamma \arg q = \Delta_\Gamma \arg p_0$.

If $d = 1$ and p_0 is a polynomial, then $\operatorname{ind} p(x, D)$ can also be computed as follows: Let v_1 and v_2 be eigenvectors for the generator A of $\{\delta_r\}$, chosen so that if (x_1, ξ_1) and (x_2, ξ_2) are the respective x, ξ coordinates of v_1 and v_2, then $x_1\xi_2 - x_2\xi_1 > 0$. Let Γ_+ be the line $t \to v_1 + tv_2$ and Γ_- the line $t \to -v_1 + tv_2$, $t \in \boldsymbol{R}$. Let $m_2 = m/\mu_2$. Let ν_+ be the number of complex roots z of $p_0(v_1 + zv_2)$ with positive imaginary part and ν_- the number of complex roots of $p_0(-v_1 + zv_2)$ with negative imaginary part. By the homogeneity of p_0,

$$\Delta_\Gamma \arg p_0 = \Delta_{\Gamma_+} \arg p_0 - \Delta_{\Gamma_-} \arg p_0 \quad \text{and}$$

$$\Delta_{\Gamma_+} \arg p_0 = -i \int_{-\infty}^{\infty} \frac{d}{dt} |p_0(v_1 + tv_2)| dt = 2\pi(\nu_+ - m_2/2) .$$

$$\Delta_{\Gamma_-} \arg p_0 = -i\int_{-\infty}^{\infty} \frac{d}{dt}|p_0(tv_2 - v_1)|\,dt = 2\pi(m_2/2 - \nu_-)\ .$$

Thus

$$\operatorname{ind} p(x, D) = \nu_+ + \nu_- - m_2\ . \tag{1.6}$$

2. Hypoellipticity of P^*. Let $\mathscr{G}$ be a nilpotent Lie algebra of step 2; i.e., $[\mathscr{G}, \mathscr{G}_2] = 0$ where $\mathscr{G}_2 = [\mathscr{G}, \mathscr{G}]$. Let G be the corresponding connected, simply connected Lie group. A family of dilations $\{\delta_r\}$ on $\mathscr{G}$ induces a family of algebra automorphisms, also denoted $\{\delta_r\}$, of $\mathscr{U}(\mathscr{G})$, the complexified universal enveloping algebra of $\mathscr{G}$. An element P of $\mathscr{U}(\mathscr{G})$ is said to be homogeneous of degree m with respect to $\{\delta_r\}$ if $\delta_r(P) = r^m P$ for all $r > 0$. The set of all $P \in \mathscr{U}(\mathscr{G})$ such that P is homogeneous of degree m with respect to a given family of dilations $\{\delta_r\}$ will be denoted $\mathscr{U}_m(\mathscr{G}, \{\delta_r\})$ or simply $\mathscr{U}_m(\mathscr{G})$ when there is no chance of confusion. We consider the elements of $\mathscr{U}(\mathscr{G})$ as left invariant differential operators on G.

THEOREM 2. *Let $\mathscr{G}$ be a nilpotent Lie algebra of step two and $\{\delta_r\}$ a family of dilations on $\mathscr{G}$. If $P \in \mathscr{U}_m(\mathscr{G}, \{\delta_r\})$ is hypoelliptic, then P^* is hypoelliptic.*

When $\{\delta_r\}$ is the natural family of dilations for a grading $\mathscr{G} = \mathscr{G}_1 \oplus \mathscr{G}_2$ of $\mathscr{G}$, then this result was proved in Helffer and Nourrigat [4]. For the Heisenberg group such a result was proved in Miller [6]. It follows from this theorem that any hypoelliptic $P \in \mathscr{U}_m(\mathscr{G})$ is locally solvable.

The proof is based on the Helffer-Nourrigat-Rockland characterization of the hypoelliptic operators in $\mathscr{U}_m(\mathscr{G})$: $P \in \mathscr{U}_m(\mathscr{G})$ is hypoelliptic if and only if $\pi(P)$ is injective in $\mathscr{S}_\pi$ for every nontrivial irreducible unitary representation π of G. (See [3] and [8]. That this result holds for arbitrary dilations is shown in [7].) We shall also need some other preliminary information before beginning the proof of Theorem 2.

By Lemma 1.2 of [7] there is a basis $\{X_1, \cdots, X_N; \cdots, X_n\}$ of $\mathscr{G}$ such that each X_j is an eigenvector for the generator A of $\{\delta_r\}$, $\{X_{N+1}, \cdots, X_n\}$ spans $\mathscr{G}_2$, and for each $k > N$ there are i and $j \leq N$ such that $[X_i, X_j] = X_k$. Let μ_j be the eigenvalue of A corresponding to X_j. If $\alpha \in N^n$, let $\alpha\mu = \sum \alpha_j \mu_j$ and $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Then $P \in \mathscr{U}_m(\mathscr{G})$ if and only if

$$P = \sum_{\alpha\mu = m} a_\alpha X^\alpha \tag{2.1}$$

for some $a_\alpha \in \boldsymbol{C}$.

Let $\mathscr{G}_1$ be the subspace of $\mathscr{G}$ spanned by $\{X_1, \cdots, X_N\}$. Letting

$\mathscr{G}^*$ denote the vector space dual of $\mathscr{G}$, we define δ_r on $\mathscr{G}^*$ to be the transpose of δ_r on $\mathscr{G}$ for each $r > 0$. Since $\mathscr{G}_1$ is invariant under $\{\delta_r\}$, $\{\delta_r\}$ (on $\mathscr{G}^*$) restricts to a family of dilations on the vector space $\mathscr{G}_1^*$. For $\eta \in \mathscr{G}_1^*$ define $\|\eta\|$ as in §1. If $X \in \mathscr{G}$, let $X = X' + X''$ where $X' \in \mathscr{G}_1$, $X'' \in \mathscr{G}_2$. For $\eta \in \mathscr{G}_1^*$,

$$\pi_\eta(\exp X) = \exp i\langle \eta, X'\rangle \tag{2.2}$$

defines a unitary representation of G on $\boldsymbol{C}$. It follows from (2.1) that if $P \in \mathscr{U}_m(\mathscr{G})$, then

$$\pi_{\delta_r\eta}(P) = r^m \pi_\eta(P) = \pi_\eta(\delta_r P)\,; \qquad \eta \in \mathscr{G}_1^*\,. \tag{2.3}$$

We next recall some facts about the representation theory for G. More details are given in [7]. Let $\zeta \in \mathscr{G}_2^*$. Then there is a $d = d(\zeta) \leqq N/2$ and a basis $\mathscr{B}(\zeta) = \{Y_1(\zeta), \cdots, Y_N(\zeta)\}$ for $\mathscr{G}_1$ such that $\mathscr{B}(\zeta)$ is orthogonal with respect to the inner product determined by the basis $\{X_1, \cdots, X_N\}$ and such that

$$\begin{aligned} &\langle \zeta, [Y_j(\zeta), Y_{j+d}(\zeta)]\rangle = 1 \qquad \text{for} \quad j \leqq d \\ &\langle \zeta, [Y_j(\zeta), Y_k(\zeta)]\rangle = 0 \end{aligned} \tag{2.4}$$

for all other choices $j < k \leqq N$. (In [7] we had $[Y_j(\zeta), Y_{j+d}(\zeta)] = \lambda_j > 0$. This was necessary because we wanted the basis to be orthonormal, but that is not needed here.) For any $\rho \in \boldsymbol{R}^{N-2d}$ there is an irreducible unitary representation $\pi_{\rho,\zeta}$ of G on $L^2(\boldsymbol{R}^d)$ such that

$$\begin{aligned} &\pi_{\rho,\zeta}(Y_j(\zeta))u(t) = \partial u/\partial t_j\,, && j \leqq d\,; \\ &\pi_{\rho,\zeta}(Y_{j+d}(\zeta))u(t) = it_j u(t)\,, && j \leqq d\,; \\ &\pi_{\rho,\zeta}(Y_{j+2d}(\zeta))u(t) = i\rho_j u(t)\,, && j \leqq N-2d\,; \\ &\pi_{\rho,\zeta}(Z)u(t) = i\langle \zeta, Z\rangle u(t)\,, && Z \in \mathscr{G}_2\,. \end{aligned} \tag{2.5}$$

Furthermore every irreducible unitary representation of G is unitarily equivalent to $\pi_{\rho,\zeta}$ for some $\zeta \in \mathscr{G}_2^*$ and some $\rho \in \boldsymbol{R}^{N-2d(\zeta)}$. Note that if $\zeta = 0$ we obtain the representation defined by (2.2).

For $\zeta \in \mathscr{G}_2^*$, $t \in \boldsymbol{R}^d$, $\tau \in \boldsymbol{R}^d$ and $\rho \in \boldsymbol{R}^{N-2d}$, $d = d(\zeta)$, let $\eta(t, \tau; \rho, \zeta)$ be that element η of $\mathscr{G}_1^*$ such that

$$\begin{aligned} &\langle \eta, Y_j(\zeta)\rangle = \tau_j\,, \qquad \langle \eta, Y_{j+d}(\zeta)\rangle = t_j\,, \qquad j \leqq d\,; \\ &\langle \eta, Y_{j+2d}(\zeta)\rangle = \rho_j\,, \qquad j \leqq N-2d\,. \end{aligned} \tag{2.6}$$

Let $f \in C^\infty(\boldsymbol{R}^N)$ satisfy $f \equiv 0$ in a neighborhood of 0 and $f \equiv 1$ outside some bounded set. Define

$$\Phi_{\rho,\zeta}(t, \tau) = 1 + f(t, \tau, \rho)\|\eta(t, \tau; \rho, \zeta)\|\,.$$

Let $\zeta \in \mathscr{G}_2^*$, $\zeta \neq 0$, be fixed. If for all $\rho \in \boldsymbol{R}^{N-2d}$, $q_\rho \in C^\infty(\boldsymbol{R}^{2d})$ and for all multi-indices α and β there is a $C_{\alpha\beta}$ such that

$$|D_\tau^\alpha D_t^\beta q_\rho(t, \tau)| \leqq C_{\alpha\beta}\Phi_{\rho,\zeta}(t, \tau)^{k-|\alpha|}$$

for all $(t, \tau, \rho) \in \boldsymbol{R}^N$ we will write "$q_\rho \in S_{\rho,\zeta}^k$ uniformly in ρ". $\mathscr{L}_{\rho,\zeta}^k$ is the space of pseudodifferential operators with symbols in $S_{\rho,\zeta}^k$; $H_{\rho,\zeta}^k$ the corresponding global Sobolev space as defined in [1].

It follows from (2.5), (2.6) and (2.2) that, for $X \in \mathscr{G}_1$,

$$\operatorname{sym} \pi_{\rho,\zeta}(X)(t, \tau) = \pi_{\eta(t,\tau,\rho,\zeta)}(X)\ , \tag{2.7}$$

where $\operatorname{sym} Q$ denotes the symbol of the operator Q. Let $\zeta \in \mathscr{G}_2^*$ be fixed and let $\{X_1, \cdots, X_n\}$ be the basis for $\mathscr{G}$ described at the beginning of this section. By (2.7) and (1.2),

$$\pi_{\rho,\zeta}(X_j) \in \mathscr{L}_{\rho,\zeta}^{\mu_j} \text{ uniformly in } \rho \text{ if } \quad j \leqq N\,, \tag{2.8}$$

$$\pi_{\rho,\zeta}(X_j) \in \mathscr{L}_{\rho,\zeta}^{0} \text{ uniformly in } \rho \text{ if } \quad j > N\,. \tag{2.9}$$

Thus if $P \in \mathscr{U}_m(\mathscr{G})$, then $\pi_{\rho,\zeta}(P) \in \mathscr{L}_{\rho,\zeta}^m$ uniformly in ρ.

LEMMA. *Let $P \in \mathscr{U}_m(\mathscr{G})$ satisfy $\pi_\eta(P) \neq 0$ for each of the one dimensional unitary representations π_η, $\eta \in \mathscr{G}_1^*$, $\eta \neq 0$. Then for fixed $\zeta \in \mathscr{G}_2^*$, $\zeta \neq 0$, there is a $c > 0$ and a $C > 0$ such that*

$$|\operatorname{sym} \pi_{\rho,\zeta}(P)(t, \tau)| \geqq c\Phi_{\rho,\zeta}(t, \tau)^m$$

for all $\rho \in \boldsymbol{R}^{N-2d}$ and all $(t, \tau) \in \boldsymbol{R}^{2d}$ such that $|t| + |\tau| \geqq C$.

Proof. Let $S = \{\eta \in \mathscr{G}_1^*\colon \|\eta\| = 1\}$ and let $c_1 = \min\{\pi_\eta(P)\colon \eta \in S\}$. For arbitrary $\eta \in \mathscr{G}_1^*$, $\eta \neq 0$, let $r = \|\eta\|^{-1}$. Then $\|\delta_r\eta\| = 1$. (2.3) implies that $|\pi_\eta(P)| \geqq c_1\|\eta\|^m$. Thus letting $p'_{\rho,\zeta}(t, \tau) = \pi_{\eta(t,\tau,\rho,\zeta)}(P)$, we have

$$|p'_{\rho,\zeta}(t, \tau)| \geqq c_1\|\eta(t, \tau; \rho, \zeta)\|^m\ . \tag{2.10}$$

Let $p_{\rho,\zeta} = \operatorname{sym} \pi_{\rho,\zeta}(P)$. By (2.7), the pseudodifferential operator calculus, (2.9) and the remark following (2.9),

$$p_{\rho,\zeta} - p'_{\rho,\zeta} \in S_{\rho,\zeta}^{m-1} \text{ uniformly in } \rho\ . \tag{2.11}$$

Now there exist $c_2 > 0$ and C_2 such that if $|t| + |\tau| \geqq C_2$ then $\|\eta(t, \tau; \rho, \zeta)\|^m \geqq c_2(|t| + |\tau|)$ for all ρ. Thus, by (2.10), there exist $c_3 > 0$ and C_3 such that if $|t| + |\tau| \geqq C_3$, then $|p'_{\rho,\zeta}(t, \tau)| \geqq c_3\Phi_{\rho,\zeta}(t, \tau)^m$ for all ρ. Also, by (2.11), it follows that given $\varepsilon > 0$ there is a $C_4(\varepsilon)$ such that if $|t| + |\tau| \geqq C_4(\varepsilon)$, then for all ρ

$$|p_{\rho,\zeta}(t, \tau) - p'_{\rho,\zeta}(t, \tau)| < 1/2\varepsilon\Phi_{\rho,\zeta}(t, \tau)^m\ .$$

The lemma follows by taking $C = \max\{C_3, C_4(c_3)\}$.

Proof of Theorem 2. By the theorem of Helffer-Nourrigat-Rockland, to prove P^* hypoelliptic it suffices to show that $\ker \pi_{\rho,\zeta}(P^*) = 0$ for all $\zeta \in \mathscr{G}_2^*$ and all $\rho \in \boldsymbol{R}^{N-2d(\zeta)}$, except $\zeta = 0$, $\rho = 0$. (We consider $\pi_{\rho,\zeta}(P)$ and $\pi_{\rho,\zeta}(P^*)$ as bounded operators from $H^m_{\rho,,\zeta}$ to $H^0_{\rho,\zeta}$). If $\zeta = 0$, then

$$\pi_{\rho,\zeta}(P^*) = \overline{\pi_{\rho,\zeta}(P)} \neq 0 \tag{2.12}$$

for all $\rho \neq 0$. If $\zeta \neq 0$, then by Theorem 7.2 of [1] and the above lemma, $\pi_{\rho,\zeta}(P)$ is Fredholm for all ρ. Also by Remark 1.4 of [4] and the Helffer-Nourrigat-Rockland Theorem, $\ker \pi_{\rho,\zeta}(P) = \ker \pi_{\rho,\zeta}(P) \cap \mathscr{S}_\pi = 0$. Hence it suffices to prove that $\operatorname{ind} \pi_{\rho,\zeta}(P) = 0$.

We consider first the case when $d = d(\zeta) < N/2$. Let $q_{\rho,\zeta} = \operatorname{sym} \pi_{\rho,\zeta}(P^*)$. By (2.12) and the above lemma there is a $c > 0$ and a C such that $|q_{\rho,\zeta}(t, \tau)| \geqq c\Phi_{\rho,\zeta}(t, \tau)^m$ for all $(t, \tau, \rho) \in \boldsymbol{R}^N$ with $|t| + |\tau| \geqq C$. Choose $f \in C^\infty(\boldsymbol{R}^{2d})$ such that $f(t, \tau) \equiv 0$ if $|t| + |\tau| \leqq C$, $f(t, \tau) \equiv 1$ if $|t| + |\tau| \geqq 2C$. Let $a_{\rho,\zeta} = fq^{-1}_{\rho,\zeta}$. Then $a_{\rho,\zeta} \in S^{-m}_{\rho,\zeta}$ uniformly in ρ and $b_{\rho,\zeta} = 1 - a_{\rho,\zeta} \circ q_{\rho,\zeta} \in S^{-1}_{\rho,\zeta}$ uniformly in ρ, where $p \circ q$ denotes the symbol of $p(t, D)q(t, D)$. Let $\psi(\tau) = (1 + |\tau|^2)^{1/2m}$. There is a $C > 0$ (depending on ζ), such that $\psi(\tau) \leqq C\Phi_{\rho,\zeta}(t, \tau)$ and, by (2.8), such that $|\rho|^\varepsilon \leqq C\Phi_{\rho,\zeta}(t, \tau)$ for all $(t, \tau, \rho) \in \boldsymbol{R}^N$, where $\varepsilon = \min\{1/\mu_j: i \leqq j \leqq N\}$. Thus $a_{\rho,\zeta} \in S^0_\psi$ uniformly in ρ and $|\rho|^\varepsilon b_{\rho,\zeta} \in S^0_\psi$ uniformly in ρ. By the L^2 boundedness theorem for pseudodifferential operators there is a C_1 such that $\|a_{\rho,\zeta}(t, D)u\| \leqq C_1\|u\|$ and $|\rho|^\varepsilon \|b_{\rho,\zeta}(t, D)u\| \leqq C_1\|u\|$, for all $u \in L^2(\boldsymbol{R}^d)$ and all ρ. Thus if $|\rho|^\varepsilon \geqq 2C_1$,

$$\begin{aligned}\|u\| &\leqq \|a_{\rho,\zeta}(t, D)\pi_{\rho,\zeta}(P^*)u\| + \|b_{\rho,\zeta}(t, D)u\| \\ &\leqq C_1\|\pi_{\rho,\zeta}(P^*)u\| + 1/2\|u\| .\end{aligned}$$

Hence $\pi_{\rho,\zeta}(P^*)$ is injective and thus $\operatorname{ind}\pi_{\rho,\zeta}(P) = 0$ if $|\rho|^\varepsilon \geqq 2C_1$. Since $\operatorname{ind}\pi_{\rho,\zeta}(P)$ is independent of ρ, $\operatorname{ind}\pi_{\rho,\zeta}(P) = 0$ for all $\rho \in \boldsymbol{R}^{N-2d}$.

If $d = d(\zeta) = N/2$, we write π_ζ for $\pi_{0,\zeta}$. Define $\varphi: \boldsymbol{R}^d_t \oplus \boldsymbol{R}^d_\tau \to \mathscr{G}_1^*$ by $\varphi(t, \tau) = \eta(t, \tau; 0, \zeta)$, as defined before (2.6). Let $\delta'_r = \varphi^{-1} \circ \delta_r \circ \varphi$. Then $\{\delta'_r\}$ is a family of dilations on $\boldsymbol{R}^{2d}$. Let $p'_\zeta(t, \tau) = \pi_{\eta(t,\tau;0,\zeta)}(P)$. It follows from (2.3) that p'_ζ is homogeneous of degree m with respect to $\{\delta'_r\}$ and by (2.12) p'_ζ is Φ_ζ-elliptic. Since $p'_\zeta - \operatorname{sym}\pi_\zeta(P) \in S^{m-1}_\zeta$ we can apply Theorem 1 to find $\operatorname{ind}\pi_\zeta(P)$. If $d > 1$, then $\operatorname{ind}\pi_\zeta(P) = 0$.

If $d = 1$ and $\mathscr{B}(\zeta) = \{Y_1(\zeta), Y_2(\zeta)\}$, set $Y_1(-\zeta) = Y_2(\zeta)$, $Y_2(-\zeta) = Y_1(\zeta)$. Then $\mathscr{B}(-\zeta) = \{Y_1(-\zeta), Y_2(-\zeta)\}$ satisfies (2.4) for $-\zeta$. Also $\eta(t, \tau; -\zeta) = \eta(\tau, t; \zeta)$ and $p'_{-\zeta}(t, \tau) = p'_\zeta(\tau, t)$. By Theorem 1

$$2\pi \operatorname{ind}\pi_{-\zeta}(P) = \Delta_\Gamma \arg p'_{-\zeta} = -\Delta_\Gamma \arg p'_\zeta = -2\pi \operatorname{ind}\pi_\zeta(P) .$$

But $\ker\pi_\zeta(P) = \ker\pi_{-\zeta}(P) = 0$ implies $\operatorname{ind}\pi_\zeta(P) \geqq 0$ and $\operatorname{ind}\pi_{-\zeta}(P) \geqq 0$. Thus $\operatorname{ind}\pi_\zeta(P) = 0$.

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