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COUNTER-EXAMPLES TO SOME CONJECTURES ABOUT DOUBLY STOCHASTIC MEASURES

V. LOSERT

Some new types of doubly stochastic measures are constructed. Using measure preserving transformations, one can construct examples of nontrivial extreme doubly stochastic measures which are absolutely continuous with respect to another extreme doubly stochastic measure (disproving a conjecture by Feldman). By combinatorial arguments, one gets an extreme doubly stochastic measure that is not concentrated on a countable union of function graphs and whose support is the whole unit square.

0. Let I be the unit interval, m the ordinary Lebesgue measure on I. A probability measure μ on $I \times I$ is called *doubly stochastic*, if its marginal distributions coincide with m (i.e., $\mu(A \times I) = \mu(I \times A) = m(A)$ for each Borel set $A \subseteq I$). This is thought of as a continuous analogue of the notion of a doubly stochastic matrix (see [8] for a survey of results about doubly stochastic matrices). By a theorem of G. Birkhoff and von Neumann, the extreme points of the set of doubly stochastic matrices are the permutation matrices. The continuous analogue of a permutation matrix would be the graph of a bijective, measure preserving function, but it is easy to construct extreme points that are not of this type (a deep study, showing that for some purposes these special measures may well suffice has been given in [13]). On the other hand, all concrete examples of extreme doubly stochastic measures (e.d.s.m.) that can be found in the literature are concentrated on the graphs (or inverse graphs) of functions, mostly even linear functions and there was some common belief that any double stochastic measure must in some sense be made up from graphs (cf. the beginning of §2 of [2]). A functional analytic characterization of the e.d.s.m. has been given by Douglas [4] and Lindenstrauss [7]. Several authors have tried to generalize properties of permutation matrices to these measures, see e.g., [1], [2], [10]. One of the aims of this paper is to present some new constructions of e.d.s.m. which will also disprove some natural conjectures. The second construction yields a measure that is not concentrated on graphs. While the measures in the first construction are still concentrated on two graphs, it turns out that even in this case the geometric interrelations are much more complicated. In [2] the following conjecture (attributed to J. Feldman) was mentioned: if μ is an e.d.s.m. and if ν is a doubly stochastic measure which is

absolutely continuous with respect to μ, then $\nu = \mu$. In [2] this conjecture was confirmed by Brown and Shiflett for a class of extreme doubly stochastic measures which is geometrically related to permutation matrices. In Theorem 1 of our paper we will give a functional analytic characterization of measures μ that satisfy a slightly stronger property as above (ν need not be positive — the result of [2] holds also for this stronger property). It is along the lines of the results by Douglas in [5]. Then we study a special class of doubly stochastic measures, defined in a certain way by a measure preserving transformation T. A similar type of measures (but with a different behavior) has been studied in [12]. In Theorem 2 we give conditions on T that ensure that the corresponding measure be extremal (resp. satisfies the properties of Theorem 1). Then it is easy to give examples of transformations for which the properties of Theorem 1 do not hold and which disprove also Feldman's conjecture mentioned above. The idea to use ergodicity properties of transformations for the construction of doubly stochastic measures was first used in [13] p. 87.

The second part of the paper concerns the support of an e.d.s.m. It has been proved in [7] that any such measure is singular with respect to the ordinary Lebesgue measure $m \otimes m$ on the unit square, i.e., it is concentrated on a set of $m \otimes m$ – measure zero. Nevertheless, we will give an example showing that the support of the measure may be the whole unit square, i.e., the measure is not concentrated on a closed set whose $m \otimes m$-measure is less than one. This measure has also the property that any graph (or inverse graph) of some measurable function has measure zero (Theorem 3). I would like to thank S. Graf for bringing these problems to my attention and also for several references to the literature.

1. If m' is an arbitrary probability measure on $I = [0, 1]$, we write $E_{m'}$ for the set of all probability measures on $I \times I$ whose marginal distributions are equal to m'. We write $\mu' \prec \mu$, if μ' is absolutely continuous with respect to μ.

THEOREM 1. *If $\mu \in E_m$, then the following statements are equivalent:*

(i) *If ν is an arbitrary (complex) measure which is absolutely continuous with respect to μ and whose marginal distributions are equal to m, then $\nu = \mu$.*

(ii) *The space of functions $F = \{(x, y) \to f(x) + g(y): f, g \in L^\infty(m)\}$ is weak $*$-dense in $L^\infty(\mu)$.*

(iii) *If m' is a probability measure which is absolutely continuous with respect to m and $\mu' \in E_{m'}$ is absolutely continuous with* *respect to* μ, *then* μ' *is an extreme point in* $E_{m'}$ (i.e., μ *is a sort of "hereditary extreme point"*).

The main content of the theorem may be rephrased as follows: if F *is* w^**-dense in* $L^\infty(\mu)$, $m' \prec m$, $\mu' \prec \mu$, *and* $\mu' \in E_{m'}$, *then* F *is also* w^**-dense in* $L^\infty(\mu')$.

Proof. (ii) $\Rightarrow$ (i) is Theorem 4 of [5].

(i) $\Rightarrow$ (ii) follows also from the methods of [5]: if $u \in L^1(\mu)$ annihilates the space F, then $\nu = (1 + u)\mu$ has the same marginal distributions as μ.

(iii) $\Rightarrow$ (ii). Given finitely many elements $u_1, \cdots, u_k \in L^1(\mu)$, $\varepsilon > 0$ and $h \in L^\infty(\mu)$, there exists a nonnegative function $u \in L^1(\mu)$ and $c > 0$ such that $u(x, y) = u(y, x)$ for all $x, y \in I$, $\int u(t)d\mu(t) = 1$ and $|u_i| \leqq cu$ for $i = 1, \cdots, k$. Put $\mu' \doteq u\mu$ and $m' = \left(\int u(\cdot, y)dm(y)\right)m$; then $m' \prec m$ and $\mu' \in E_{m'}$. By (iii) and Theorem 1 of [4], there exists a function $h_0 \in F$ such that $\|h - h_0\|_{L^1(\mu')} < \varepsilon/c$. This implies clearly $\left|\int u_i(t)(h(t) - h_0(t))d\mu(t)\right| < \varepsilon$ for $i = 1, \cdots, k$.

(ii) $\Rightarrow$ (iii). By Theorem 1 of [4] it is sufficient to show that F is dense in $L^\infty(\mu)$ with respect to the norm topology induced by $L^1(\mu')$.

The result follows from a general lemma on convex subsets of $L^\infty(\mu)$.

LEMMA. *If* F *is a convex subset of* $L^\infty(\mu)$ *and* $\mu' \prec \mu$, *then the* w^**-closure of* F *in* $L^\infty(\mu)$ *is contained in the closure taken with respect to the norm topology induced by* $L^1(\mu')$.

Proof. Since $\mu' \prec \mu$, we get $\mu' = u\mu$ for some $u \in L^1(\mu)$. The continuous functionals for the w^*-topology are given by elements of $L^1(\mu)$. The continuous functionals for the norm topology induced by $L^1(\mu')$ are represented by elements of $\{hu: h \in L^\infty(\mu)\} \subseteqq L^1(\mu)$ (if one uses the same duality as above). Now the lemma follows easily from the Hahn Banach theorem ([11], Ch. II, 9.2).

Now let $T: [0, 1/2] \to [0, 1/2]$ be an arbitrary measure preserving transformation. We consider measures μ supported by the following four sets: $F_1 = \{(x, Tx); 0 \leqq x \leqq 1/2\}$, $F_2 = \{(x, x - 1/2); 1/2 \leqq x \leqq 1\}$, $F_3 = \{(x, x + 1/2); 0 \leqq x \leqq 1/2\}$, $F_4 = \{(x, x): 1/2 \leqq x \leqq 1\}$ (i.e., the unit square is partitioned into four congruent sub squares. In the first of them we consider the graph of T, in the other squares, the diagonal). Let $j: [0, 1/2] \to F_1$ be defined by $j(x) = (x, Tx)$. j induces a bijective correspondence between measures on $[0, 1/2]$ and F_1.

The corresponding measures will be denoted by the same letter. If μ is supported by $F = F_1 \cup F_2 \cup F_3 \cup F_4$, then μ_1 shall denote its restriction to F_1.

THEOREM 2. *If μ is a doubly stochastic measure, supported by F, then the following statements hold:*

(i) *μ is uniquely determined by $\mu_1 = \mu \mid F_1$. Conversely, a measure μ_1 on $[0, 1/2]$ induces a doubly stochastic measure on F, iff μ_1 is T-invariant and $0 \leqq \mu_1 \leqq m$.*

(ii) *Write $\mu_1 = h_1 m (h_1 \in L^\infty(m),\ 0 \leqq h_1 \leqq 1)$. μ is an extreme doubly stochastic measure iff μ_1* (resp. h_1) *has the following property: if $A \subseteqq [0, 1/2]$ is a T-invariant, measurable set with $\mu_1(A) > 0$, then* $\operatorname{ess\,sup}_{x \in A} h_1(x) = 1$.

(iii) *μ satisfies the conditions of Theorem 1, iff μ_1* (resp. h_1) *has the following property: if $A \subseteqq [0, 1/2]$ is a T-invariant, measurable set with $\mu_1(A) > 0$, then $m\{x \in A : h_1(x) = 1\} > 0$. In particular, if $\{x : h_1(x) = 1\}$ has m-measure zero, then μ does not satisfy the conditions of Theorem 1.*

Proof. Assume that μ is given. Let A be a measurable subset of $[0, 1/2]$.

Since μ is doubly stochastic, we get:

$$\begin{aligned} m(A) = \mu(I \times A) &= \mu((I \times A) \cap F_1) + \mu((I \times A) \cap F_2) \\ &= \mu(T^{-1}A \times A) + \mu([1/2, 1] \times A) \\ &= \mu_1(T^{-1}A) + \mu((A + 1/2) \times [0, 1/2]) \,. \end{aligned} \tag{1}$$

Similarly

$$\begin{aligned} m(A) &= \mu((A + 1/2) \times I) = \mu((A + 1/2) \times [0, 1/2]) + \mu((A + 1/2) \times [1/2, 1]) \\ &\underset{(1)}{=} m(A) - \mu_1(T^{-1}A) + \mu([1/2, 1] \times (A + 1/2)) \,. \end{aligned}$$

This gives:

$$\mu([1/2, 1] \times (A + 1/2)) = \mu_1(T^{-1}A) \,. \tag{2}$$

Furthermore

$$m(A) = \mu(I \times (A + 1/2)) = \mu([0, 1/2] \times (A + 1/2)) + \mu([1/2, 1] \\ \times (A + 1/2)$$

gives

$$\mu([0, 1/2] \times (A + 1/2)) = m(A) - \mu_1(T^{-1}A) \,. \tag{3}$$

And finally

$$m(A) = \mu(A \times I) = \mu(A \times [0, 1/2]) + \mu(A \times [1/2, 1])$$
$$\underset{(3)}{=} \mu_1(A) + m(A) - \mu_1(T^{-1}A) .$$

The last equation shows that $\mu_1(A) = \mu_1(T^{-1}A)$, i.e., μ_1 is T-invariant. (1), (2), (3) show that the values of μ on F_2, F_3, F_4 are uniquely determined by μ_1. $\mu_1 \leqq m$ holds by (3).

Conversely, if one defines the measure μ on F_2, F_3, F_4 by the formulas (1), (2), (3), it is easily seen that μ is a doubly stochastic measure. This proves (i).

(ii). Assume that $A \subseteqq [0, 1/2]$ is T-invariant $\mu_1(A) > 0$, $0 < \alpha \leqq 1/2$ and $h_1 \leqq 1 - \alpha$ a.e. on A. Let c_A be the characteristic function of A. If we put $\mu_1' = (1 + \alpha(1-\alpha)^{-1}c_A)\mu_1$ and $\mu_1'' = (1 - \alpha(1-\alpha)^{-1}c_A)\mu_1$, then it follows easily from (i) that μ_1' and μ_1'' define doubly stochastic measures μ' and μ'' such that $\mu = (\mu' + \mu'')/2$.

For the converse assume that $\mu = (\mu' + \mu'')/2$ and $\mu' \neq \mu''$. Then we get measures μ_1' and μ_1'' such that $\mu_1 = (\mu_1' + \mu_1'')/2$. Put $\mu_1' = f'\mu_1$ and $\mu_1'' = f''\mu_1$. Then $f' + f'' = 2$ and $f' \neq f''$. Therefore we may assume that there exists $\beta > 1$ such that $A = \{x: f'(x) > \beta\}$ satisfies $\mu_1(A) > 0$. Since μ_1 and μ_1' are T-invariant, the same holds for A. Since $\mu_1' = f'hm \leqq m$ by (i), we have $f'h \leqq 1$ and therefore $h \leqq \beta^{-1}$ on A.

(iii). By an argument similar to (i), it can be shown that an arbitrary (complex) measure μ' on $F_1 \cup F_2 \cup F_3 \cup F_4$ with marginal distributions m is uniquely determined by its restriction μ_1' to F_1. μ_1' has to be T-invariant furthermore $\mu' \prec \mu$ if $\mu_1' \prec \mu_1$ and $m - \mu_1' \prec m - \mu_1$ Now the same construction as in (ii) gives the result.

EXAMPLE. It is now easy to give examples of extreme doubly stochastic measures which do not satisfy the properties of Theorem 1. Let $T_0: I \to I$ be an arbitrary ergodic transformation (with respect to m), e.g., $T_0(x) = x + \alpha (\mathrm{mod}\ 1)$, α irrational. We consider the mapping $g: I \to [0, 1/2]$ defined by $g(x) = x^{1/2}/2$. Then $T = g \circ T_0 \circ g^{-1}$ is an ergodic transformation of $[0, 1/2]$ with invariant measure $g(m) = 8ym$. Now define $h_1(y) = 2y$ for $y \in [0, 1/2]$. By Theorem 2, $\mu_1 = h_1 m$ defines a doubly stochastic measure μ, which is extremal (by (ii) — observe that $A = [0, 1/2]$ is the only invariant set) but does not satisfy the properties of Theorem 1 (by (iii)). It is easily seen that there exists even a probability measure $\mu' \neq \mu$ which is doubly stochastic and absolutely continuous with respect to μ (e.g., the diagonal measure on $F_2 \cup F_3$).

2. In the next section we present another method to construct extreme doubly stochastic measures. It uses approximations by

measures on finite subalgebras. Let Σ be the σ-algebra of Borel sets on I. We will consider increasing sequences (Σ_n) resp. (Σ'_n) of finite subalgebras of I, where $\Sigma_n(\Sigma'_n)$ is generated by a partition $P_n(Q_n)$ of I into finitely many subintervals. We will define a measure μ successively on the subalgebras $\Sigma_n \otimes \Sigma'_n$. To assure that the limit measure is extremal, we will use an idea similar to [2] Thm. 1 and [3] Thm. 2. There is a connection between extremality and the existence of "loops". If μ is not extremal, one will also get such loops in one of the approximating algebras $\Sigma_n \otimes \Sigma'_n$. The aim of the following construction will be, to cut off each such loop at a later step.

THEOREM 3. *There exists an extreme doubly stochastic measure μ whose support is the whole space. In addition, μ has the following property: if $\mu = \int \mu_{1,x}\, dm(x) = \int \mu_{2,x} dm(x)$ are the disintegrations of μ with respect to the two coordinate projections ($\mu_{1,x}$ is concentrated on $\{x\} \times I$, $\mu_{2,x}$ on $I \times \{x\}$), then the measures $\mu_{1,x}$ and $\mu_{2,x}$ are continuous* a.e. (i.e., *each point has measure zero*). *In particular any graph or inverse graph of some measurable function $f: I \to I$ has μ-measure zero.*

Proof. By induction we will define partitions $P_n = \{I_\alpha^{(n)}\}_{\alpha \in A^{(n)}}$ and $Q_n = \{J_\alpha^{(n)}\}_{\alpha \in A^{(n)}}$ of I. For $n = 0$ we put $A^{(0)} = \{1\}$, $I_1^{(0)} = J_1^{(0)} = I$, $\mu(I \times I) = 1$. Now assume that P_n, Q_n have already been defined, and $\mu(I_\alpha^{(n)} \times J_\beta^{(n)}) > 0$ for all $\alpha, \beta \in A^{(n)}$. We consider two different cases:

(a) if n is even, we choose $k > \max\{(n+1)\mu(I_\alpha^{(n)} \times J_\beta^{(n)})(m(I_\alpha^{(n)})^{-1} + m(J_\beta^{(n)})^{-1}) : \alpha, \beta \in A^{(n)}\}$. We put $A^{(n+1)} = A^{(n)} \times \{1, \cdots, k\}$. Each interval $I_\alpha^{(n)}$ is partitioned into k-subintervals $I_{\alpha,1}^{(n+1)}, \cdots, I_{\alpha,k}^{(n+1)}$ of equal length. We put $P_{n+1} = \{I_{\alpha,j}^{(n+1)} : \alpha \in A^{(n)}, j = 1, \cdots, k\}$. Q_{n+1} is constructed in the same way from Q_n. Finally $\mu(I_{\alpha,i}^{(n+1)} \times J_{\beta,j}^{(n+1)}) = k^{-2}\mu(I_\alpha^{(n)} \times J_\beta^{(n)})$. Then it is easily seen that

$$\mu(J_\alpha^{(n+1)} \times J_\beta^{(n+1)}) < (n+1)^{-1} \min(m(I_\alpha^{(n+1)}), m(J_\beta^{(n+1)})) \quad \text{for} \quad \alpha, \beta \in A^{(n+1)}, n+1 \text{ odd} . \tag{4}$$

(b) if n is odd, let $r = r(n)$ be the cardinality of $A^{(n)} \times A^{(n)}$ (i.e., the number of atoms in $\Sigma_n \otimes \Sigma'_n$). We choose $k > 2^n r$ and put $A^{(n+1)} = A^{(n)} \times \{1, \cdots, k\}^r$. Let $\varphi_n: A^{(n)} \times A^{(n)} \to \{1, \cdots, r\}$ be a fixed bijection. Finally put $c_n = \min\{\mu(I_\alpha^{(n)} \times J_\beta^{(n)}) : \alpha, \beta \in A^{(n)}\}$. To get P_{n+1} (resp. Q_{n+1}) we partition each interval $I_\alpha^{(n)}$ (resp. $J_\alpha^{(n)}$) into k^r subintervals $I_{\alpha i_1 \cdots i_r}^{(n+1)}$ (resp. $J_{\alpha i_1 \cdots i_r}^{(n+1)}$) where $1 \leqq i_1 \leqq k$ for $1 = 1, \cdots, r$, whose length will be decided later on. We put

$$\mu(I^{(n+1)}_{\alpha i_1\cdots i_r} \times J^{(n+1)}_{\beta i_1\cdots i_s+1\cdots i_r}) = (\mu(I^{(n)}_\alpha \times J^{(n)}_\beta) - c_n k^{-2r} r^{-2}) k^{1-r}(k-1)^{-1} + c_n k^{-4r} r^{-2} \quad \text{if} \quad s = \varphi_n(\alpha, \beta) \tag{5}$$

and

$$\mu(I^{(n+1)}_{\alpha i_1\cdots i_r} \times J^{(n+1)}_{\beta j_1\cdots j_r}) = c_n k^{-4r} r^{-2} \tag{6}$$

for all other choices of the indices ($1 \leqq i_l, j_l \leqq k, l = 1, \cdots, r, \alpha, \beta \in A^{(n)}$). It is easily seen that the sum of the measures of all subrectangles of $I^{(n)}_\alpha \times J^{(n)}_\beta$ equals $\mu(I^{(n)}_\alpha \times J^{(n)}_\beta)$. The length of the interval $I^{(n+1)}_\alpha (\alpha \in A^{(n+1)})$ is now determined by $m(I^{(n+1)}_\alpha) = \sum_{\beta \in A(n+1)} \mu(I^{(n+1)}_\alpha \times J^{(n+1)}_\beta)$, similarly for $J^{(n+1)}_\beta$.
If $1 \leqq i_1 < k$ for $l = 1, \cdots, r$ and $\alpha, \beta \in A^{(n)}$, then we have by (5), (6):

$$\mu(I^{(n+1)}_{\alpha, i_1\cdots i_r} \times J^{(n)}_\beta) - k^{1-r}(k-1)^{-1}\mu(I^{(n)}_\alpha \times J^{(n)}_\beta) = -c_n k^{-3r} r^{-2}(k-1)^{-1} \leqq 0 . \tag{7}$$

Summation over β gives

$$m(I^{(n+1)}_{\alpha, i_1\cdots i_r}) - k^{1-r}(k-1)^{-1} m(I^{(n)}_\alpha) = -c_n k^{-3r} r^{-3/2}(k-1)^{-1} . \tag{8}$$

Since by (4) $m(I^{(n)}_\alpha) \geqq n\mu(I^{(n)}_\alpha \times J^{(n)}_\beta)$ and $c_n \leqq \mu(I^{(n)}_\alpha \times J^{(n)}_\beta)$, we get:

$$\begin{aligned} m(I^{(n+1)}_{\alpha i_1\cdots i_r}) &\geqq \mu(I^{(n)}_\alpha \times J^{(n)}_\beta)(nk^{1-r}(k-1)^{-1} - k^{-3r} r^{-3/2}(k-1)^{-1}) \\ &\geqq \mu(I^{(n)}_\alpha \times J^{(n)}_\beta)(n-1)k^{1-r}(k-1)^{-1} . \end{aligned}$$

Combined with (7), we find that: $m(I^{(n+1)}_{\alpha i_1\cdots i_r}) \geqq (n-1)\mu(I^{(n+1)}_{\alpha i_1\cdots i_r} \times J^{(n)}_\beta) \geqq (n-1)\mu(I^{(n+1)}_{\alpha i_1\cdots i_r} \times J^{(n+1)}_{\beta j_1\cdots j_r})$ for all $1 \leqq j_l \leqq k, l = 1, \cdots, r$. Now put $I^{(n)} = \bigcup \{I^{(n+1)}_{\alpha i_1\cdots i_r}: \alpha \in A^{(n)}, i_1 = k$ for at least one l). Then

$$m(I^{(n+1)}_\alpha \times J^{(n+1)}_\beta) \leqq (n-1)^{-1} m(I^{(n+1)}_\alpha) \quad \text{for all} \quad \beta \in A^{(n+1)} , \tag{9}$$

if $I^{(n+1)}_\alpha$ is not contained in $I^{(n)}$. If $\alpha \in A^{(n)}$ is fixed, then $I^{(n)}_\alpha \backslash I^{(n)}$ is a union of $(k-1)^r$ intervals for which (8) holds. Therefore

$$\begin{aligned} m(I^{(n)}_\alpha \backslash I^{(n)}) \geqq (1-k^{-1})^{r-1} m(I^{(n)}_\alpha)(1 - k^{-2r-1} r^{-3/2}) &\geqq (1-2^{-n})^2 m(I^{(n)}_\alpha) \\ &\geqq (1-2^{-n+1}) m(I^{(n)}_\alpha) \end{aligned}$$

(by our choice of k and since clearly $r \geqq 2^n$). Summing over $\alpha \in A^{(n)}$, we get:

$$m(I^{(n)}) \leqq 2^{1-n} . \tag{10}$$

In the same way, we get an exceptional set $J^{(n)}$ with $m(J^{(n)}) < 2^{1-n}$, such that $\mu(I^{(n+1)}_\alpha \times J^{(n+1)}_\beta) < (n-1)^{-1} m(J^{(n+1)}_\beta)$ for all rectangles not contained in $I \times J^{(n)}$.

Having defined all partitions P_n, Q_n, we want to extend μ to a measure on $\Sigma \otimes \Sigma$. This is done as follows: Let $\mu^{(n)}$ be the probability measure on $I \times I$, whose restriction to each rectangle $I_\alpha^{(n)} \times J_\beta^{(n)} (\alpha, \beta \in A^{(n)})$ is a multiple of the ordinary Lebesgue measure and such that $\mu^{(n)}(I_\alpha^{(n)} \times J_\beta^{(n)}) = \mu(I_\alpha^{(n)} \times J_\beta^{(n)})$. Let μ' be a cluster point of the sequence $(\mu^{(n)})$ in the weak topology of the set of all Radon measures on $I \times I$ with respect to continuous functions). Since each $\mu^{(n)}$ is doubly stochastic, the same is true for μ'. In particular, the boundary of each rectangle $I_\alpha^{(n)} \times J_\beta^{(n)}$ has μ'-measure zero. If $(\mu^{(n_i)})$ is a subsequence of $(\mu^{(n)})$ converging towards μ', it follows from [9] Ch. II, Thm. 6.1 that $\mu^{(n_i)}(I_\alpha^{(n)} \times J_\beta^{(n)})$ converges to $\mu'(I_\alpha^{(n)} \times J_\beta^{(n)})$. This shows that $\mu'(I_\alpha^{(n)} \times J_\beta^{(n)}) = \mu(I_\alpha^{(n)} \times J_\beta^{(n)})$, i.e., μ' is an extension of μ. (From now on we will again write μ for μ'.) We claim that μ has the properties stated in Theorem 3.

We have already remarked that μ is doubly stochastic. Let $\mu = \int \mu_{1,x} dm(x)$ be the disintegration of μ with respect to the first coodinate projection ($\mu_{1,x}$ is concentrated on $\{x\} \times I$). Similarly $\mu^{(n)} = \int \mu_{1,x}^{(n)} dm(x)$. If $x \in I_\alpha^{(n)} (\alpha \in A^{(n)})$ and M is a Borel subset of I, then $\mu_{1,x}^{(n)}(M) = \mu^{(n)}(I_\alpha^{(n)} \times M)/m(I_\alpha^{(n)})$. Put $I^\infty = \bigcap_m \bigcup_{n \geqq m} I^{(n)}$. Then $m(I^\infty) = 0$ by (10). If $x \notin I^\infty$, then by (4) and (9) $\mu_{1,x}^{(n)}(J_\alpha^{(m)}) = \mu_{1,x}^{(m)}(J_\alpha^{(m)}) < (m-2)^{-1}$ for all $\alpha \in A^{(m)}$, $m \geqq n_0(x)$, $n \geqq m$. It follows that any cluster point $\mu'_{1,x}$ of the sequence $(\mu_{1,x}^{(n)})$ is a continuous measure and by the same argument as used for $(\mu^{(n)})$, the sequence $(\mu_{1,x}^{(n)})$ converges in the weak topology towards $\mu'_{1,x}$. By dominated convergence, we get $\mu = \int \mu'_{1,x} dm(x)$ and therefore $\mu_{1,x} = \mu'_{1,x}$ a.e. The same argument works for the second coordinate.

The interiors of the rectangles $I_\alpha^{(n)} \times J_\beta^{(n)} (\alpha, \beta \in A^{(n)}, n \geqq 0)$ form clearly a basis for the topology of $I \times I$. By our construction $\mu(I_\alpha^{(n)} \times J_\beta^{(n)}) = \mu^{(n)}(I_\beta^{(n)} \times J_\alpha^{(n)}) > 0$ and therefore supp $\mu = I \times I$.

The last thing that we have to show is the extremality of μ. Equivalently: if $f \in L^\infty(\mu)$ and the conditional expectations $E_1 f$ and $E_2 f$ with respect to the two coordinate projections are zero, then f equals zero. Assume the contrary, i.e., $\|f\|_\infty = 1$ and $E_1 f = E_2 f = 0$. By the martingale convergence theorem, there exists an odd integer n and $\alpha', \beta' \in A^{(n)}$ such that

$$\left| \int_{I_{\alpha'}^{(n)} \times J_{\beta'}^{(n)}} f d\mu \right| \geqq 2^{-1} \mu(I_{\alpha'}^{(n)} \times J_{\beta'}^{(n)}) . \tag{11}$$

Put

$$B^{(n+1)} = \{(\alpha i_1 \cdots i_r, \beta i_1 \cdots i_s + 1 \cdots i_r) \colon \alpha, \beta \in A^{(n)}, \varphi_n(\alpha, \beta) = s ,$$
$$1 \leqq i_s < k, 1 \leqq i_l \leqq k \quad \text{for} \quad l = 1, \cdots, r, l \neq s\} \subseteqq A^{(n+1)} \times A^{(n+1)}$$

and put

$$a_{\alpha\beta} = \int_{I_\alpha^{(n+1)} \times J_\beta^{(n+1)}} f d\mu \quad \text{for} \quad \alpha, \beta \in A^{(n+1)} .$$

By (6) we have $\mu(\cup \{I_\alpha^{(n+1)} \times J_\beta^{(n+1)}, (\alpha, \beta) \notin B^{(n+1)}\}) < c_n k^{-2r} r^{-1}$. Since $\|f\|_\infty = 1$ and $E_1 f = E_2 f = 0$ we conclude that:

$$\sum_{\beta \in A^{(n+1)}} |\sum \{a_{\alpha\beta}: \alpha \in A^{(n+1)}, (\alpha, \beta) \in B^{(n+1)}\}| < c_n k^{-2r} 8^{-1} \tag{12}$$

$$\sum_{\alpha \in A^{(n+1)}} |\sum \{a_{\alpha\beta}: \beta \in A^{(n+1)}, (\alpha, \beta) \in B^{(n+1)}\}| < c_n k^{-2r} 8^{-1} . \tag{13}$$

Since $2^{-1}\mu(I_{\alpha'}^{(n)} \times J_{\beta'}^{(n)}) - c_n k^{-2r} r^{-1} \geqq 4^{-1} c_n$, (11) can be reformulated as follows:

$$\sum \{a_{\alpha\beta}: I_\alpha^{(n\times 1)} \subseteqq I_{\alpha'}^{(n)}, J_\beta^{(n+1)} \subseteqq J_{\beta'}^{(n)}, (\alpha, \beta) \in B^{(n+1)}\} \geqq 4^{-1} c_n \quad \text{for some } \alpha', \beta' \in A^{(n)} . \tag{14}$$

Now choose numbers $(a_{\alpha\beta})_{(\alpha,\beta) \in B(n+1)}$ such that (12), (13), (14) are fulfilled and the cardinality of $C = \{(\alpha, \beta): a_{\alpha\beta} \neq 0\}$ becomes minimal. By (14), there exists $(\alpha, \beta) \in C$ such that $|a_{\alpha\beta}| \geqq 4^{-1} c_n k^{-2r}$. Put $D_0 = \{(\alpha, \beta)\}$ and then inductively:

$$D_{2i+1} = \{(\delta_1, \delta_2) \in C \backslash D_{2i}: \exists \delta_3 \in A^{(n+1)}: (\delta_1, \delta_3) \in D_{2i}\}$$
$$D_{2i} = \{(\delta_1, \delta_2) \in C \backslash D_{2i-1}: \exists \delta_3 \in A^{(n+1)}: (\delta_3, \delta_2) \in D_{2i-1}\}$$

(e.g., D_{2i} stands for those rectangles that belong to a row determined by some rectangle from D_{2i-1}, but do not belong to D_{2i-1} itself).

Now assume that $D_i \cap D_j \neq \emptyset$ for some $i < j$ and choose i minimal. Assume that $i \neq 0$. Take $(\delta_1, \delta_2) \in D_i \cap D_j$. If i is odd, there exists $\delta_3 \in A^{(n+1)}$ such that $(\delta_1, \delta_3) \in D_{i-1}$. If j is also odd, then $(\delta_1, \delta_3) \in D_j \cup D_{j-1}$ (since $(\delta_1, \delta_2) \in D_j$), but both possibilities contradict the minimality of i. Similarly if j is even, then $(\delta_1, \delta_3) \in D_j \cup D_{j+1}$ which is also impossible. An analogous argument works in the case that i is even. This shows that either $(\alpha, \beta) \in D_j$ for some $j \geqq 2$ or all D_j are disjoint. To exclude the second possibility, observe that $\bigcup_{j \geqq 0} D_j$ contains with each element also the whole column corresponding to that element. Therefore, by (12):

$$\left| \sum \left\{a_{\gamma\delta}: (\gamma, \delta) \in \bigcup_{j \geqq 0} D_j\right\} \right| < 8^{-1} c_n k^{-2r} .$$

Similarly $\bigcup_{j \geqq 1} D_j$ contains only complete rows, therefore by (13):

$$\left| \sum \left\{a_{\gamma\delta}: (\gamma, \delta) \in \bigcup_{j \geqq 1} D_j\right\} \right| < 8^{-1} c_n k^{-2r} .$$

But if the sets D_j are disjoint, the two sums differ exactly by $a_{\alpha\beta}$

and this contradicts $|a_{\alpha,\beta}| \geqq 4^{-1}c_n k^{-2r}$.

This shows the existence of a "loop", i.e., there exist pairwise different elements $\alpha_1, \cdots, \alpha_m \in A^{(n+1)}$, such that $(\alpha_{2i-1}, \alpha_{2i}), (\alpha_{2i+1}, \alpha_{2i}) \in C$ for all i (we put $\alpha_{m+1} = \alpha_1$); m is clearly even. Define $a = a_{\alpha_1\alpha_2}$ and $b_{\alpha_{2i-1}\alpha_{2i}} = a$, $b_{\alpha_{2i+1}\alpha_{2i}} = -a$ for all i and $b_{\alpha\beta} = 0$ in the other cases. The sums in (12) and (13) are zero for $(b_{\alpha\beta})$. If the sum in (14) would also be zero (for all choices of $\alpha', \beta' \in A^{(n)}$), then $(a_{\alpha\beta} - b_{\alpha\beta})$ would have the properties (12), (13), (14), contradicting our minimal choice of $(a_{\alpha\beta})$ (since $a_{\alpha_1\alpha_2} - b_{\alpha_1\alpha_2} = 0$). Therefore $(b_{\alpha\beta})$ has the properties (12), (13), (14) (possibly with a smaller constant in (14)) and we will assume from now on that $a_{\alpha\beta} = b_{\alpha\beta}$ for all α, β.

Now recall the definition of $A^{(n+1)}$: we have $\alpha_i = (\rho(\alpha_i), \rho_1(\alpha_i), \cdots, \rho_r(\alpha_i))$ with $\rho(\alpha_i) \in A^{(n)}$, $1 \leqq \rho_l(\alpha_i) \leqq k (l = 1, \cdots, r)$, $I^{(n+1)}_{\alpha_i} \subseteqq I^{(n)}_{\rho(\alpha_i)}$, $J^{(n+1)}_{\alpha_i} \subseteqq J^{(n)}_{\rho(\alpha_i)}$. Put $\bar{\varphi}(2i-1) = \varphi_n(\alpha_{2i-1}, \alpha_{2i})$ and $\bar{\varphi}(2i) = \varphi_n(\alpha_{2i+1}, \alpha_{2i})$. We may assume that $\alpha' = \rho(\alpha_1)$, $\beta' = \rho(\alpha_2)$ in (14), i.e.,

$$\sum \{a_{\alpha\beta}: I^{(n+1)}_{\alpha} \times J^{(n+1)}_{\beta} \subseteqq I^{(n)}_{\alpha'} \times J^{(n)}_{\beta'} \neq 0 \ .$$

Put $t = \varphi^{-}(1) = \varphi_n(\alpha_1, \alpha_2)$. It follows from the definition of $B^{(n+1)}$ that for any $\alpha \in A^{(n+1)}$, $\gamma, \delta \in A^{(n)}$ there exists at most one $\beta \in A^{(n+1)}$ such that $(\alpha, \beta) \in B^{(n+1)}$ and $I^{(n+1)}_{\alpha} \times J^{(n+1)}_{\beta} \subset I^{(n)}_{\gamma} \times J^{(n)}_{\delta}$, similarly for α, β interchanged. This shows that $\varphi^{-}(l) \neq \varphi^{-}(l+1)$ for all l. Now put $N = \{l: \varphi^{-}(l) = t\} = \{l_1, l_2, l_3, \cdots\}$ with $1 = l_1 < l_2 < l_3 \cdots$. Since $\varphi^{-}(1) = t$, we have $\rho_t(\alpha_1) + 1 = \rho_t(\alpha_2)$. If $\varphi^{-}(l) \neq t$, then $\rho_t(\alpha_{l+1}) = \rho_t(\alpha_l)$. On the other hand, if $\varphi^{-}(l) = t$ and l is even, then $\rho_t(\alpha_{l+1}) + 1 = \rho_t(\alpha_l)$ and if l is odd, then $\rho_t(\alpha_l) + 1 = \rho_t(\alpha_{l+1})$. This means that if we put $\sigma(l) = |\{j \geqq 1: l_j < l, l_j \text{ is odd}\}| - |\{j \geqq 1: l_j < l, l_j \text{ is even}\}|$ (here $|\cdot|$ stands for the cardinality of the set), then $\rho_t(\alpha_l) = \rho_t(\alpha_1) + \sigma(l)$. In particular, for $l = m+1$, we have $\alpha_{m+1} = \alpha_1$ and therefore $\sigma(m+1) = 0$. This means that $|\{j \geqq 1: l_j < m+1, l_j \text{ is odd}\}| = |\{j \geqq 1: l_j < m+1, l_j \text{ is even}\}|$. But these numbers l_j determine those indices (α, β) for which $a_{\alpha\beta} \neq 0$ and $I^{(n+1)}_{\alpha} \times J^{(n+1)}_{\beta} \subseteqq I^{(n)}_{\alpha'} \times J^{(n)}_{\beta'}$. If l_j is odd, then $a_{\alpha\beta} = +a$ and if it is even then $a_{\alpha\beta} = -a$. By summing up we would get: $\sum \{a_{\alpha\beta}: I^{(n+1)}_{\alpha} \times J^{(n+1)}_{\beta} \subseteqq I^{(n)}_{\alpha'} \times J^{(n)}_{\beta'}\} = 0$, contrary to (14). This contradiction proves that μ is extremal.

Finally since all the measures in the two disintegrations of μ are continuous, it follows that any graph or inverse graph has μ-measure zero.

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UNIVERSITÄT WIEN
STRUDLHOFGASSE 4
A-1090 WIEN, AUSTRIA