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p-ADIC ANALOG OF HEINE'S HYPERGEOMETRIC q-SERIES

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Generalized complex analytic special functions of various types, depending on a parameter $0<q<1$, have recently been studied by R. Askey, G. E. Andrews and others [1-3]. The purpose of this paper is to discuss a p-adic analytic construction which is analogous to the classical theory of E. Heine's [7] q-extension of the hypergeometric function.

In the theory of hypergeometric series one denotes $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$. The corresponding notation for q-series is $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$. This "extends" the ordinary $(\alpha)_k$ in the sense that for $a=q^\alpha$, $b=q^\beta$ we have $\lim_{q\to 1}(a;q)_k/(b;q)_k = (\alpha)_k/(\beta)_k$.

In the complex analytic theory of q-extensions one takes $0 < q < 1$ and defines the q-gamma function as

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \tag{1}$$

and the q-hypergeometric functions as

$${}_m\phi_n\left(\begin{matrix}a_1\cdots a_m\\ b_1\cdots b_n\end{matrix};q,x\right) = \sum_{j=0}^{\infty}\frac{(a_1;q)_j\cdots(a_m;q)_j}{(q;q)_j(b_1;q)_j\cdots(b_n;q)_j}((1-q)^j q^{j(j-1)/2})^{n+1-m}x^j.$$

The functions Γ_q and ${}_m\phi_n$ satisfy many relations which generalize well-known identities for the ordinary gamma and hypergeometric functions, and as $q\to 1^-$ we have $\Gamma_q(x)\to\Gamma(x)$ and

$${}_m\phi_n\left(\begin{matrix}a_1\cdots a_m\\ b_1\cdots b_n\end{matrix};q,x\right)\longrightarrow {}_mF_n\left(\begin{matrix}\alpha_1\cdots \alpha_m\\ \beta_1\cdots \beta_n\end{matrix};x\right)\quad\text{if}\quad a_i=q^{\alpha_i},\ b_i=q^{\beta_i}\,.$$

These q-identities, many of which go back to Euler, Jacobi, Heine, Rogers, and Ramanujan, have applications to combinatorics, Lie algebras, orthogonal polynomials, modular functions, and other areas.

We shall be especially interested in one identity, the following variant of Heine's transformation rule for ${}_2\phi_1$ [8]:

$${}_2\phi_1\left(\begin{matrix}a\ \ b\\ c\end{matrix};q,c/ax\right) = \frac{(c/a;q)_\infty(c/x;q)_\infty}{(c;q)_\infty(c/ax;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}a\ \ b/x\\ c/x\end{matrix};q,c/a\right). \tag{2}$$

If we set $x=b$, then the ${}_2\phi_1$ on the right becomes 1; and if we set

$a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, and use (1) we obtain

$$(3)\qquad {}_2\phi_1\begin{pmatrix} q^\alpha & q^\beta \\ q^\gamma & \end{pmatrix}; q, q^{\gamma-\alpha-\beta}\Big) = \frac{\Gamma_q(\gamma)\Gamma_q(\gamma-\alpha-\beta)}{\Gamma_q(\gamma-\alpha)\Gamma_q(\gamma-\beta)},$$

which extends the well-known relation ${}_2F_1(\alpha, \beta, \gamma; 1) = \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)$.

In the p-adic analytic theory we work in the p-adic completion Ω_p of the algebraic closure of $\boldsymbol{Q}_p$, and we take $q = 1 + t$, $t \in \Omega_p$, $|t|_p < 1$. We shall often assume that $|t|_p < p^{-1/(p-1)}$, in which case $q^\gamma = \exp(\gamma \log q)$ is well-defined for $\gamma \in \Omega_p$, $|\gamma|_p < p^{-1/(p-1)}/|t|_p$, i.e., on a disc strictly larger than the unit disc. In any case q^γ is well-defined for $\gamma \in \boldsymbol{Z}_p$ whenever $|t|_p < 1$.

In defining the p-adic analog of ${}_2\phi_1\left(\begin{matrix} a & b \\ & c \end{matrix}; q, x\right)$, we shall take a, b, $c \in \Omega_p$ and suppose that $a = q^\alpha$, $b = q^\beta$ with $-\alpha = a_0 + a_1 p + \cdots \in \boldsymbol{Z}_p$, $-\beta = b_0 + b_1 p + \cdots \in \boldsymbol{Z}_p$. We shall usually further suppose that c is *not* in the compact set $q^{\boldsymbol{Z}_p}$, and let $\varepsilon = \operatorname{dist}(c, q^{\boldsymbol{Z}_p}) = \min_{j \in \boldsymbol{Z}}(|cq^j - 1|_p) > 0$.

In this paper we shall prove identities analogous to (2) and (3) for our p-adic ${}_2\phi_1$. In particular, we relate the p-adic hypergeometric q-series to the q-extension $\Gamma_{p,q}$ [10] of Y. Morita's p-adic gamma function Γ_p [12] and also to J. Diamond's p-adic log gamma function G_p [4]. Then, in the special case $c = q$, when p-adic convergence of the series for ${}_2\phi_1$ becomes subtler, we introduce a q-extension of Dwork's modified hypergeometric function [6], prove convergence and a formula analogous to (3) under certain conditions, and formulate a conjecture on the validity of these results without the "nonsupersingularity" conditions.

1. In [10] we defined a p-adic analog of Γ_q by setting

$$(4)\qquad \Gamma_{p,q}(\alpha) = \lim_{n\to\alpha}(-1)^n \prod_{j<n,\, p\nmid j} \frac{1-q^j}{1-q}.$$

$\Gamma_{p,q}$ satisfies a functional equation, reflection formula and multiplication formula analogous to the formulas satisfied by the ordinary gamma function, and $\Gamma_{p,q}$ approaches Morita's function Γ_p [12] as $q \to 1$.

We now define a p-adic q-gamma function depending on $a = q^\alpha$, $b = q^\beta$, and $c \notin q^{\boldsymbol{Z}_p}$ by setting

$$(5)\qquad \Gamma_p\begin{pmatrix} c & c/ab \\ c/a & c/b \end{pmatrix}; q\Big) = \lim_{n\to-\alpha} \frac{(c/b; q)_n}{(c; q)_n}, \qquad a, b \in q^{\boldsymbol{Z}_p},\ c \notin q^{\boldsymbol{Z}_p}.$$

THEOREM 1. *The limit* (5) *exists, is symmetric in* a, b, *and is continuous in* a, b, c.

Proof. Let $A_n = (c/b; q)_n/(c; q)_n$. Then

$$A_n = \lim_{m\to-\beta} (cq^m; q)_n/(c; q)_n = \lim_{m\to-\beta} (c; q)_{n+m}/(c; q)_m(c; q)_n ,$$

which shows that the definition (5) is symmetric in a, b. Next,

$$\frac{A_{n+kp^N}}{A_n} = \lim_{m\to-\beta} \frac{(cq^{m+n}; q)_{kp^N}}{(cq^n; q)_{kp^N}} = \lim_{m\to-\beta} \prod_{n\leqq j<n+m} \frac{1-cq^{j+kp^N}}{1-cq^j} .$$

Since cq^j is bounded away from 1 and $q^{kp^N}\to 1$ as $N\to\infty$, the last product approaches 1 as $N\to\infty$ uniformly in m and n. This shows existence of the limit and its continuity as a function of a, b, c.

It will also be useful to have a version of (5) which makes sense when $c\in q^{Z_p}$. Now suppose $\varepsilon = \operatorname{dist}(c, q^{Z_p}) < |t|_p$ (where we allow $\varepsilon = 0$, i.e., $c\in q^{Z_p}$). Since $\varepsilon < |t|_p$, there is a unique $0\leqq j_0<p$ such that $|cq^{j_0}-1|_p < |t|_p$. For $a = q^\alpha$, $-\alpha = a_0 + a_1p + \cdots$, we define the modified symbol $(\ \ )_k^*$ by $(c/a; q)_k^* = \prod(1-(c/a)q^j)$, where the product is over $0\leqq j<k$, $p\nmid a_0+j-j_0$. We then define

$$(6) \qquad \Gamma_p^*\begin{pmatrix} c & c/ab \\ c/a & c/b \end{pmatrix}; q\Bigg) = \lim_{n\to\to\alpha} \frac{(c/b; q)_n^*}{(c; q)_n^*},\ a, b\in q^{Z_p},\ \operatorname{dist}(c, q^{Z_p}) < |t|_p .$$

THEOREM 2. *The limit* (6) *exists, is symmetric in a, b, and is continuous in a, b, c. If $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma\in q^{Z_p}$, then*

$$(7) \qquad \Gamma_p^*\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right) = \frac{\Gamma_{p,q}(\gamma)\Gamma_{p,q}(\gamma-\alpha-\beta)}{\Gamma_{p,q}(\gamma-\alpha)\Gamma_{p,q}(\gamma-\beta)} ,$$

where $\Gamma_{p,q}$ is the function (4). *Now suppose $c\notin q^{Z_p}$ but still $|cq^{j_0}-1|_p<|t|_p$. Set $a' = q^{a_0}a$, $b' = q^{b_0}b$, $c' = q^{j_0}c$, where $-\alpha = a_0+a_1p+\cdots$, $-\beta = b_0+b_1p+\cdots$. For simplicity suppose $j_0\geqq a_0$, $j_0\geqq b_0$. Then*

$$(8) \qquad \Gamma_p^*\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right) = \frac{1}{\varepsilon(a, b)}\cdot\frac{\Gamma_p\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right)}{\Gamma_p\left(\begin{matrix} c' & c'/a'b' \\ c'/a' & c'/b' \end{matrix}; q^p\right)} ,$$

where

$$\varepsilon(a, b) = \begin{cases} 1 & \text{if } a_0+b_0\leqq j_0 ; \\ 1-c'/a'b' & \text{if } a_0+b_0>j_0 . \end{cases}$$

Proof. Existence, symmetry and continuity are proved just as in Theorem 1. By continuity, it suffices to prove (7) when $\alpha=-n$, $\beta=-m$ and $\gamma=l$. Let $\prod_j'$ denote $\prod_{j,p\nmid j}$. Then the left side of (7) is

$$\frac{\prod'_{l+m\leqq j<l+m+n}(1-q^j)}{\prod'_{l\leqq j<l+n}(1-q^j)}$$

by (6), and the right side is (see (4))

$$\frac{\prod'_{j<l}(1-q^j)\prod'_{j<l+m+n}(1-q^j)}{\prod'_{j<l+n}(1-q^j)\prod'_{j<l+m}(1-q^j)}=\frac{\prod'_{l+m\leqq j<l+m+n}(1-q^j)}{\prod'_{l\leqq j<l+n}(1-q^j)}\,.$$

To prove (8), by continuity it suffices to take $-\alpha=n=a_0+pn'$, $-\beta=m=b_0+pm'$. The left side equals

$$\frac{\prod\limits_{\substack{0\leqq j<n\\ p\nmid b_0+j-j_0}}(1-cq^{j+m})}{\prod\limits_{\substack{0\leqq j<n\\ p\nmid j-j_0}}(1-cq^j)}=\frac{\Gamma_p\begin{pmatrix}c & c/ab\\ c/a & c/b\end{pmatrix};q\Big)}{\left(\dfrac{\prod\limits_{\substack{0\leqq j<n' \text{ if } a_0+b_0\leqq j_0\\ 0\leqq j<n'+1 \text{ if } a_0+b_0>j_0}}(1-c'q^{p(j+m')})}{\prod\limits_{0\leqq j<n'}(1-c'q^{pj})}\right)}$$

$$=\frac{\Gamma_p\left(\begin{matrix}c & c/ab\\ c/a & c/b\end{matrix};q\right)}{\varepsilon(a,b)\Gamma_p\left(\begin{matrix}c' & c'/a'b'\\ c'/a' & c'/b'\end{matrix};q^p\right)}\,.$$

(Note: if we had $j_0<a_0$ or $j_0<b_0$, then $\varepsilon(a,b)$ would be a slightly more complicated expression; in any case, we shall later be interested in c for which $j_0=p-1$.)

This completes the proof of Theorem 2.

2. We now proceed to p-adic hypergeometric q-series. Let $q=1+t$, $a=q^\alpha$, $b=q^\beta$ be as before. Suppose $c\notin q^{Z_p}$. We define

$${}_2\phi_{1,p}\left(\begin{matrix}a & b\\ & c\end{matrix};q,x\right)=\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}x^k$$

whenever the sum converges.

LEMMA. *If* $|t|_p<p^{-1/(p-1)}$, *then*

$$\frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k}c^k\longrightarrow 0 \text{ as } k\longrightarrow\infty\,,$$

uniformly in a,b.

Proof. Since for any $n\geqq 1$ we have

$$\left|\prod_{0\leqq j<k}\frac{1-q^{n+j}}{1-q^{j+1}}\right|_p=\left|\prod_{0\leqq j<k}\frac{n+j}{j+1}\right|_p=\left|\binom{n+k-1}{k}\right|_p\leqq 1\,,$$

it follows (passing to the limit as $n \to \alpha$) that $|(a;q)_k/(q;q)_k|_p \leqq 1$, and similarly $|(b;q)_k/(q;q)_k|_p \leqq 1$. Hence it suffices to show that $(q;q)_k c^k/(c;q)_k \to 0$. If $|c|_p > 1$, then $|(c;q)_k|_p = |c|_p^k$, and the assertion follows because $(q;q)_k$ clearly approaches 0 as $k \to \infty$. Suppose $|c|_p \leqq 1$. We show that $(q;q)_k/(c;q)_k \to 0$. Let $\varepsilon = \operatorname{dist}(c, q^{Z_p}) > 0$.

Case (i). $\varepsilon \geqq |t|_p$.
Since

$$|1 - q^{j+1}|_p = |(j+1)t|_p \leqq \begin{cases} \varepsilon & \text{if } p \nmid j+1\,, \\ \varepsilon/p & \text{if } p \mid j+1\,, \end{cases}$$

while $|1 - cq^j|_p \geqq \varepsilon$, it follows that

$$\prod_{0 \leqq j < k} \frac{1 - q^{j+1}}{1 - cq^j} \longrightarrow 0\,.$$

Case (ii). $\varepsilon < |t|_p$.
Choose $k_0 \geqq 0$ so that $|1 - cq^{k_0}|_p = \varepsilon$. We set

$$C = \prod_{0 \leqq j < k_0} \left| \frac{1}{1 - cq^j} \right|_p\,,$$

and we use the fact that

$$|1 - cq^{k_0+j}|_p = |1 - cq^{k_0} + cq^{k_0}(1 - q^j)|_p = \max(\varepsilon, |1 - q^j|_p)$$

(here equality holds in the non-archimedean triangle inequality because strict inequality would mean that $\varepsilon = |1 - q^j|_p > |1 - cq^{k_0+j}|_p$, contradicting the definition of ε). Thus,

$$\left| \prod_{0 \leqq j < k} \frac{1 - q^{j+1}}{1 - cq^j} \right|_p \leqq C \prod_{0 < j < k - k_0} \left| \frac{1 - q^j}{1 - cq^{k_0+j}} \right|_p = C \prod_{\substack{0 < j < k - k_0 \\ |1 - q^j|_p < \varepsilon}} \frac{|1 - q^j|_p}{\varepsilon}\,,$$

which approaches 0 as $k \to \infty$. This completes the proof.

THEOREM 3. *Let* $q = 1 + t$, $|t|_p < p^{-1/(p-1)}$. *Then* ${}_2\phi_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, x\right)$ *converges and is continuous for* $a, b \in q^{Z_p}$, $c \notin q^{Z_p}$, $|x|_p \leqq |c|_p$. ${}_2\phi_{1,p}$ *satisfies the following transformation rule for* $x \in q^{Z_p}$:

$$(9) \qquad {}_2\phi_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, c/ax\right) = \Gamma_p\left(\begin{matrix} c & c/ax \\ c/a & c/x \end{matrix}; q\right) {}_2\phi_{1,p}\left(\begin{matrix} a & b/x \\ & c/x \end{matrix}; q, c/a\right)\,.$$

In particular, for $x = b$ *this gives*

$$(10) \qquad {}_2\phi_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, c/ab\right) = \Gamma_p\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right)\,.$$

Proof. Since $|c/ax|_p = |c|_p$ for $a, x \in q^{Z_p}$, the lemma ensures convergence and continuity of each of the series in (9). Theorem 1 ensures convergence and continuity of $\Gamma_p\begin{pmatrix} c & c/ax \\ c/a & c/x \end{pmatrix}; q\Big)$. By continuity, it suffices to prove (9) for $a = q^{-n}$, $b = q^{-m}$, $x = q^{-l}$, i.e., to prove that

$$\begin{aligned}\sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{-m}; q)_k}{(c; q)_k (q; q)_k} (cq^{n+l})^k &= \prod_{0 \leq k < n} \frac{1 - cq^{l+k}}{1 - cq^k} \\ &\times \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{l-m}; q)_k}{(cq^l; q)_k (q; q)_k} (cq^n)^k .\end{aligned} \tag{11}$$

But these are finite sums and finite products, and the formal identity in $\boldsymbol{Q}(q, c)$ follows from Heine's classical identity (2), which becomes the same as (11) when we set $a = q^{-n}$, $b = q^{-m}$, $x = q^{-l}$. (Of course, this identity is initially over the complex numbers, but it gives an identity of elements of $\boldsymbol{Q}(q, c)$.) This completes the proof.

REMARK. If $a = q^\alpha$, $b = q^\beta$, $\alpha, \beta \in \boldsymbol{Z}_p$, and $c = q^\gamma$, $\gamma \notin \boldsymbol{Z}_p$, then it is easy to verify that

$$\begin{aligned}\lim_{q \to 1} \log_p \Gamma_p\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right) &= G_p(\gamma) + G_p(\gamma - \alpha - \beta) \\ &\quad - G_p(\gamma - \alpha) - G_p(\gamma - \beta) ,\end{aligned}$$

where $G_p\colon \Omega_p \setminus \boldsymbol{Z}_p \to \Omega_p$ is J. Diamond's p-adic log gamma function [3]. As a corollary of Theorem 3, we then obtain the following relation of Diamond [5]:

$$\begin{aligned}\log_p F_p(\alpha, \beta, \gamma; 1) &= G_p(\gamma) + G_p(\gamma - \alpha - \beta) - G_p(\gamma - \alpha) \\ &\quad - G_p(\gamma - \beta),\ \gamma \in \Omega_p \setminus \boldsymbol{Z}_p,\ \alpha, \beta \in \boldsymbol{Z}_p .\end{aligned}$$

Here

$$F_p(\alpha, \beta, \gamma; x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j}{(\gamma)_j j!} x^j .$$

3. We now want to extend the definition of ${}_2\phi_{1,p}$ to certain a, b, c with $c \in q^{Z_p}$, in particular with $c = q$. The case $c = q$ will be the q-extension of Dwork's [6] p-adic analytic continuation $\Theta(\alpha, \beta; x)$ of the series

$$F_p(\alpha, \beta, 1; x) = \sum_{j=0}^{\infty} \frac{(\alpha)_j (\beta)_j}{j!^2} x^j .$$

Suppose that $a = q^\alpha$, $b = q^\beta$, $-\alpha = a_0 + a_1 p + \cdots \in \boldsymbol{Z}_p$, $-\beta = b_0 + b_1 p + \cdots \in \boldsymbol{Z}_p$, $|cq^{j_0} - 1|_p < |t|_p$, $0 \leqq j_0 < p$, and $a' = q^{a_0} a$, $b' = q^{b_0} b$, $c' = q^{j_0} c$. Note that in the definition that follows we make a shift in argument $x \mapsto cx/ab$ so that Theorem 3 involves evaluation at $x = 1$ rather than at $x = c/ab$.

DEFINITION.

$$\phi_N\begin{pmatrix} a & b \\ & c \end{pmatrix}; q, x\Big) = \sum_{0 \leq j < p^N} \frac{(a;q)_j(b;q)_j}{(c;q)_j(q;q)_j}\left(\frac{cx}{ab}\right)^j$$

$$(12) \qquad {}_2\phi^*_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, x\right) = \lim_{N\to\infty} \frac{\phi_{N+1}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, x\right)}{\phi_N\left(\begin{matrix} a' & b' \\ & c' \end{matrix}; q^p, x^p\right)}$$

if the limit exists.

Note that if $c \notin q^{Z_p}$ and $|x|_p \leqq 1$, then the limit (12) exists, and

$$(13) \qquad {}_2\phi^*_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, x\right) = {}_2\phi_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, \frac{cx}{ab}\right)\Big/ {}_2\phi_{1,p}\left(\begin{matrix} a' & b' \\ & c' \end{matrix}; q^p, \frac{c'x^p}{a'b'}\right).$$

The above definition of ${}_2\phi^*_{1,p}$ is a natural q-extension of Dwork's hypergeometric functions in [6].

THEOREM 4. *Let* $|t|_p < p^{-1/(p-1)}$, *and let* $\mathscr{D} \subset q^{Z_p} \times q^{Z_p} \times D$, *where* $D = \{c \mid |c - q|_p < |t|_p\}$, *be the largest set on which the limit* (12) *exists and is continuous in* a, b, c. *Then for* $a, b, c \in \mathscr{D}$

$$(14) \qquad {}_2\phi^*_{1,p}\left(\begin{matrix} a & b \\ & c \end{matrix}; q, 1\right) = \varepsilon(a, b)\Gamma^*_p\left(\begin{matrix} c & c/ab \\ c/a & c/b \end{matrix}; q\right),$$

where Γ^*_p *is defined in* (6) *and* $\varepsilon(a, b)$ *is defined in Theorem* 2.

Proof. Note that $j_0 = p - 1$ for $c \in D$. If $a, b, c \in q^{Z_p} \times q^{Z_p} \times (D \setminus q^{Z_p})$, then we use (13) with $x = 1$ together with (10) and (8) to obtain (14). Since $q^{Z_p} \times q^{Z_p} \times (D \setminus q^{Z_p}) \subset \mathscr{D}$ is dense, the theorem follows.

We now look more closely at the case $c = q$.

THEOREM 5 (Dwork [6]). *Let*

$$F^{(i)}(X) = \sum_{j=0}^{\infty} B^{(i)}(j)X^j \in \Omega_p[[X]], \; i \geqq 0,$$

and let

$$F^{(i)}_N(X) = \sum_{0 \leqq j < p^N} B^{(i)}(j)X^j.$$

Suppose that

(1) $B^{(i)}(0) = 1, \; i \geqq 0;$

(2) $|B^{(i)}(j)/B^{(i+1)}([j/p])|_p \leqq 1,\ i, j \geqq 0;$

(3) $B^{(i)}(j+lp^N)/B^{(i+1)}([j/p]+lp^{N-1}) \equiv B^{(i)}(j)/B^{(i+1)}([j/p]) \bmod p^N$, $i, j, l \geqq 0$.

Further suppose that the $B^{(i)}(j)$ *depend continuously on parameters* $a_1, \cdots, a_m \in \Omega_p^m$ *and satisfy* (1)–(3) *for* $a_1, \cdots, a_m \in S \subset \Omega_p^m$. *Let* $R \subset S \times \{x \in \Omega_p \mid |x|_p \leqq 1\}$ *be the subset on which*

$$|F_1^{(i)}(x^{p^k})|_p = 1 \text{ for all } i, k \geqq 0 \tag{15}$$

("nonsupersingularity condition"). Then

$$f(x) = \lim_{N\to\infty} \frac{F_{N+1}^{(0)}(x)}{F_N^{(1)}(x^p)}$$

exists and is continuous on R.

REMARKS. 1. If, as in our case below, we have $|B^{(i)}(j) - l_0|_p < 1$ for some $0 \leqq l_0 < p$, i.e., if the $B^{(i)}(j)$ have residue classes in the prime field, then (15) need only be verified for $k = 0$.

2. This formulation of Theorem 5 is somewhat different from Dwork's. Dwork further assumes that for some fixed r: $B^{(i+r)}(j) = B^{(i)}(j)$ for all i, j. In that case (15) is only a finite set of conditions, the set of x satisfying (15) (the "nonsupersingular" x) is quasi-connected, and Dwork shows that $f(x)$ is analytic there. We do not want the periodicity condition, but we do want the continuous dependence on parameters. An examination of Dwork's proof in [6] shows that the same proof applies without any changes at all under our assumptions in Theorem 5.

THEOREM 6. *Suppose that* $|q-1|_p < p^{-1(p-1)}$, *and set*

$$B^{(0)}(j) = B^{(0)}(j; a, b; q) = \frac{(a;q)_j(b;q)_j}{(q;q)_j^2}\left(\frac{q}{ab}\right)^j$$

for $a = q^\alpha$, $b = q^\beta$, $-\alpha = a_0 + a_1p + \cdots \in \boldsymbol{Z}_p$, $-\beta = b_0 + b_1p + \cdots \in \boldsymbol{Z}_p$. *Define* $\alpha^{(i)}$ *and* $\beta^{(i)}$ *by* $-\alpha^{(i)} = a_i + a_{i+1}p + \cdots$, $-\beta^{(i)} = b_i + b_{i+1}p + \cdots$, *and let* $a^{(i)} = q^{p^i\alpha^{(i)}}$, $b^{(i)} = q^{p^i\beta^{(i)}}$. *Let* $B^{(i)}(j) = B^{(0)}(j; a^{(i)}, b^{(i)}; q^{p^i})$. *Then* $B^{(i)}(j)$ *satisfies conditions* (1)–(3) *of Theorem* 5. *Suppose* $|x-1|_p < 1$. *Then condition* (15) *holds if and only if* $a_i + b_i < p$ *for all* i, i.e., *if and only if there is no carrying when* $-\alpha$ *and* $-\beta$ *are added.*

Proof. Condition (1) is trivial. It suffices to prove conditions (2) and (3) for $i = 0$; then the conditions for i will follow by replacing a, b, q by $a^{(i)}, b^{(i)}, q^{p^i}$. Setting $j = j_0 + pj_1$, $0 \leqq j_0 < p$, so that $[j/p] = j_1$, we have

$$\frac{(a;q)_j}{(a';q^p)_{j_1}} = \begin{cases} (a;q)_j^* & \text{if } j_0 \leqq a_0 ; \\ (a;q)_j^*(1-a'q^{pj_1}) & \text{if } j_0 > a_0 , \end{cases}$$

where we recall that

$$(a;q)_j^* = \prod_{0 \leqq k < j, p \nmid k - a_0} (1 - aq^k) .$$

Since $|q/ab|_p = 1$ and $|1 - aq^k|_p = |k + \alpha|_p \cdot |t|_p$, it follows that

$$|B^{(0)}(j)/B^{(1)}(j_1)|_p = \begin{cases} 1 & \text{if } j_0 \leqq a_0, j_0 \leqq b_0 ; \\ |pj_1 + p\alpha'|_p & \text{if } a_0 < j_0 \leqq b_0 ; \\ |pj_1 + p\beta'|_p & \text{if } b_0 < j_0 \leqq a_0 ; \\ |pj_1 + p\alpha'|_p \cdot |pj_1 + p\beta'|_p & \text{if } j > a_0, j > b_0 . \end{cases}$$

This proves (2).

To prove (3) it clearly suffices to take $l = 1$. For simplicity we further assume that $j_0 \leqq a_0$, $j_0 \leqq b_0$; the other cases are treated similarly. Then, since both sides of (3) are p-adic units, it suffices to prove that

$$\frac{(a;q)_{j+p^N}^*(b;q)_{j+p^N}^*(q;q)_j^{*2}}{(a;q)_j^*(b;q)_j^*(q;q)_{j+p^N}^{*2}} \cdot \frac{(a'b')^{p^{N-1}}}{(ab)^{p^N}} \equiv 1 \bmod p^N . \tag{16}$$

By continuity, we may suppose that $a = q^{-n}$, $b = q^{-m}$. Now

$$\frac{(a;q)_{j+p^N}^*(q;q)_j^*}{(a;q)_j^*(q;q)_{j+p^N}^*} = \frac{\prod_{j \leqq k < j+p^N, p \nmid k - a_0} (1 - aq^k)}{\prod_{j \leqq k < j+p^N, p \nmid k+1} (1 - q^{k+1})}$$
$$= \prod_{j-n \leqq k \leqq j, p \nmid k} \left(\frac{1 - q^k}{1 - q^{k+p^N}} \right) .$$

But $(1-q^k)/(1-q^{k+p^N}) \equiv 1 \bmod p^N$ if $p \nmid k$. Since also $(a'b')^{p^{N-1}}/(ab)^{p^N}$ is of the form $(q^z)^{p^N}$ for some p-adic integer z (namely, $z = -\alpha - \beta + \alpha' + \beta'$), it follows that the left side of (16) is a product of terms which are all congruent to 1 mod p^N, as desired.

Finally, suppose $|x - 1|_p < 1$, and let P be the maximal ideal (open unit disc) in Ω_p. We have

$$F_1^{(i)}(x) = \sum_{0 \leqq j < p} \frac{(a^{(i)}; q^{p^i})_j (b^{(i)}; q^{p^i})_j}{(q^{p^i}; q^{p^i})_j^2} \left(\frac{q^{p^i}}{a^{(i)} b^{(i)}} \right)^j$$
$$\equiv \sum_{0 \leqq j < p} \frac{(\alpha^{(i)})_j (\beta^{(i)})_j}{j!^2} \bmod P$$
$$\equiv \sum_{0 \leqq j < p} \binom{a_i}{j} \binom{b_i}{j} \bmod P$$
$$= \binom{a_i + b_i}{a_i} .$$

Hence $|F_1^{(i)}(x)|_p = 1$ if and only if $a_i + b_i < p$.

THEOREM 7. *Suppose that the conditions of Theorem* 6 *hold with* $|x-1|_p < 1$ *and* $a_i + b_i < p$ *for all* i. *Then the limit* (12) *exists and*

$$(17) \qquad {}_2\phi_{1,p}^{*}\begin{pmatrix} a \ b \\ q \end{pmatrix}; q,\ 1\Bigg) = \Gamma_p^{*}\begin{pmatrix} q & q/ab \\ q/a & q/b \end{pmatrix}; q\Bigg),$$

where Γ_p^* *is defined in* (6).

Proof. Existence and continuity in a, b of the left side follow from Theorems 5 and 6. It then suffices to verify (17) for $a = q^{-n}$, $b = q^{-m}$. In that case both sides involve finite sums and products, and the proof is very similar to that of Theorems 2 and 3.

REMARK. Theorem 7 is a q-extension of Theorem 2 in [11].

Conjecture. Theorem 7 *holds without the condition that* $a_i + b_i < p$ *for all* i. *If* $a_0 + b_0 \geqq p$, *then the factor* $\varepsilon(a, b)$ *defined in Theorem* 2 *must be inserted on the right. If* $\alpha + \beta$ *is a nonpositive integer, we require that both* α *and* β *be nonpositive integers* (*otherwise the limit* (12) *would give* 0/0).

REMARKS. 1. The proof of Theorem 7 shows that the conjecture holds whenever one of α or β is a nonpositive integer (and the other can be any p-adic integer).

2. Using Diamond's method in [5], one can prove the conjecture under a fairly weak assumption: that the p-adic absolute value of the partial sums $\left|\phi_N\begin{pmatrix} a' \ b' \\ q^p \end{pmatrix}; q^p,\ 1\Bigg)\right|_p$ grows strictly slower than p^N. In addition, Theorem 7 and the conjecture can be generalized to ${}_2\phi_{1,p}^{*}\begin{pmatrix} a \ b \\ c \end{pmatrix}; q, 1\Bigg)$ for $c \neq q$. In our context, Diamond's method involves letting $z \notin q^{Z_p}$ approach $c \in q^{Z_p}$ and estimating the difference between the ratio on the right in (12) (with $x = 1$) and the same ratio with c replaced by z.

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