PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 1, 1982

QUASI-ISOMETRIC DILATIONS OF OPERATOR-VALUED MEASURES AND GROTHENDIECK'S INEQUALITY

MILTON ROSENBERG

Let $M(\cdot)$ be a strongly countably-additive (s.c.a.) (continuous linear) operator-valued measure on an arbitrary σ-algebra $\mathscr{B}$ of subsets of an arbitrary set Ω from a Hilbert space W to a Hilbert space $\mathscr{H}$. Is there a Hilbert space $\mathscr{K} \supseteqq \mathscr{H}$ and a s.c.a. quasi-isometric measure $\tilde{M}(\cdot)$ (cf. Masani, BAMS 76 (1970), 427–528) on $\mathscr{B}$ from W to $\mathscr{K}$ such that $M(\cdot) = P \circ \tilde{M}(\cdot)$ where P is the projection on $\mathscr{K}$ onto $\mathscr{H}$? In other words, has such an $M(\cdot)$ a "quasi-isometric dilation $\tilde{M}(\cdot)$"? We show that when W or $\mathscr{H}$ is finite-dimensional the answer is affirmative, and that when W is finite-dimensional there is a unique (up to isomorphism) quasi-isometric dilation $\tilde{M}(\cdot)$ of $M(\cdot)$ such that trace$(\tilde{M}(\Omega)^*\tilde{M}(\Omega))$ is a minimum. This generalizes results of Miamee and Salehi, and Niemi. Our results depend on Grothendieck's inequality.

CONTENTS

1. Introduction 135
2. Definitions and the Equivalence Theorem 137
3. Existence of 2-majorants for Hilbert space-valued measures....142
4. The problem of the existence and uniqueness of 2-majorants for operator-valued measures 145
5. Appendix..154

1. Introduction. In 1977 Niemi [15] proved that a countably-additive (c.a.)[1] measure $\xi(\cdot)$ on the Borel family $\mathscr{B}$ of a locally compact Hausdorff space Ω with values in a Hilbert space $\mathscr{H}$ over $\boldsymbol{F}$,[2] is the projection of a countably-additive orthogonally-scattered (c.a.o.s.) measure $\tilde{\xi}(\cdot)$ on $\mathscr{B}$ with values in a larger Hilbert space $\mathscr{K}$. More fully, $\xi(B) = P\{\tilde{\xi}(B)\}$, $B \in \mathscr{B}$, where P is the projection on $\mathscr{K}$ onto $\mathscr{H}$. Stated differently, $\xi(\cdot)$ has an "orthogonally-scattered dilation to $\tilde{\xi}(\cdot)$".

Niemi was influenced by Abreu's 1976 paper [2] in which he gave a sufficient condition [2, Th. 3] for an $\mathscr{H}$-valued measure to be the projection of a c.a.o.s. measure with values in a larger space $\mathscr{K}$. However, Niemi interpreted vector-valued measures not as set-functions but as linear operators on spaces of continuous functions which vanish at infinity. As early as 1970 Abreu [1] had shown

[1] We shall abbreviate "finitely additive", "countably additive," "weakly countably additive," "strongly countably additive", repectively, as "f.a.", "c.a.", "w.c.a.", "s c.a.".

[2] Throughout this paper $\boldsymbol{F}$ will stand for the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$.

that every process harmonizable in the sense of Cramér is the projection of a stationary process. In 1978 Miamee and Salehi [14] guided by the work of Niemi, in the course of generalizing Abreu's theorem for processes harmonizable in the sense of Rozanov ([14, Main Th. 5]), derived Niemi's theorem for the case $\Omega = \boldsymbol{R}$, cf [14, Cor. 6].

To understand the relation of our work with the preceding, we must recall the definitions of an orthogonally-scattered measure and of a quasi-isometric measure, cf. Masani [11], [12]. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}$ be a σ-algebra over a set Ω. An $\mathscr{H}$-valued set function $\xi(\cdot)$ on $\mathscr{B}$ is said to be *countably-additive orthogonally-scattered* (c.a.o.s.) if and only if

$$(\xi(A), \xi(B))_{\mathscr{H}} = \mu(A \cap B)\ , \qquad A, B \in \mathscr{B}\ ,$$

where μ is a c.a. nonnegative real-valued measure on $\mathscr{B}$.[3] Now let W and $\mathscr{H}$ be Hilbert spaces and let $M(\cdot)$ be a W-to-$\mathscr{H}$ (continuous linear) operator-valued set function on $\mathscr{B}$. Then $M(\cdot)$ is said to be *strongly countably-additive quasi-isometric* (c.a.q.i.) if and only if

$$M(B)^* M(A) = H(A \cap B)\ , \qquad A, B \in \mathscr{B}\ ,$$

where $H(\cdot)$ is a s.c.a. W-to-W nonnegative hermitian operator-valued measure on $\mathscr{B}$.[4]

It is natural to ask if, in analogy to the result of Niemi, every s.c.a. W-to-$\mathscr{H}$ operator-valued measure $M(\cdot)$ on $\mathscr{B}$ is obtainable by projection from a W-to-$\mathscr{K}$ c.a.q.i. measure $\tilde{M}(\cdot)$ on $\mathscr{B}$, where the Hilbert space $\mathscr{K}$ is larger than $\mathscr{H}$; specifically if

$$M(B) = P \circ \tilde{M}(B)\ , \qquad B \in \mathscr{B}\ ,$$

where P is the projection on $\mathscr{K}$ onto $\mathscr{H}$. Stated differently, the question is whether such an $M(\cdot)$ has a "quasi-isometric dilation $\tilde{M}(\cdot)$".

This paper is addressed to the operator-valued question just described. In it the fundamental concept of a 2-majorizable measure due to Persson and Pietsch [17] plays a fundamental role, as it does in the papers of Niemi and of Miamee and Salehi. However in our paper this concept, defined so far for vector-valued measures, has to be defined for operator-valued measures. In §2, in our main Theorem 2.9 we give a set of equivalent conditions pertaining to dilatability, 2-majorizability, and the positive definiteness of certain kernels (2.8). In this theorem and in the rest of this paper, we interpret dilatability in terms of injections into Hilbert spaces rather

[3] In [**12**; 2.1] it is indicated that such a $\xi(\cdot)$ is necessarily c.a.

[4] In [**12**; 8.6(e)] it is shown that such an $M(\cdot)$ is s.c.a.

than imbeddings into Hilbert spaces, cf. [13; §1]. In light of Theorem 2.9 the central question is whether every W-to-$\mathscr{H}$ s.c.a. operator-valued measure $M(\cdot)$ is 2-majorizable? In the case of a vector-valued measure with Ω locally-compact Hausdorff, an affirmative answer was given by Niemi [15, Th. 4] on the basis of earlier work by Pietsch [18] and Rogge [20]. In §3 for the purpose of proving a generalization of this result for operator-valued measures, we give a new proof of the vector result (3.9), with $\mathscr{B}$ an arbitrary σ-algebra over an arbitrary set Ω, in which a central role is played by Grothendieck's inequality (3.2). We also give a new proof of the uniqueness of a minimum 2-majorant (3.10) valid for any Ω, originally due to Pietsch, for compact Hausdorff spaces [18, Satz 2].

In §4 we turn to the question of the 2-majorizability of any W-to-$\mathscr{H}$ s.c.a. measure $M(\cdot)$. We are able to give an affirmative answer only in the case where either W or $\mathscr{H}$ is finite-dimensional (4.1), (4.3), unfortunately. We also show for finite-dimensional W the existence and uniqueness of a minimum trace 2-majorant (4.7 and 4.14). We exhibit the explicit form of the minimum trace 2-majorant in the case where Ω consists of 2 points (Example 4.15).

We refer the reader to [22] for facts on the generalized inverse $A^{\#}$ of an operator A. In general, for an operator A we let $\mathscr{R}(A) =$ range A, $A^* =$ adjoint of A, $\tau A =$ trace of A, $|A| =$ Banach norm of A, $|A|_E =$ euclidean norm of $A = \sqrt{(\tau A^* A)}$. We denote $P_{\mathscr{M}}$ as the orthogonal projection with range $\mathscr{M}$.

2. Definitions and the equivalence theorem. In this section

$$(2.1) \qquad \begin{cases} (\text{i}) & \mathscr{B} \text{ is a } \sigma\text{-algebra over an arbitrary set } \Omega\,; \\ (\text{ii}) & W,\ \mathscr{H}, \text{ and } \mathscr{K} \text{ are Hilbert spaces over } \boldsymbol{F}\,. \end{cases}$$

DEFINITION 2.2. Let Ω, $\mathscr{B}$, W, $\mathscr{K}$ be as above.

(a) A W-to-$\mathscr{K}$ (continuous linear) operator-valued set function $\tilde{M}(\cdot)$ on $\mathscr{B}$ is said to be a *strongly countably additive quasi-isometric* (c.a.q.i.) measure iff

$$\tilde{M}(B)^* \tilde{M}(A) = H(A \cap B)\,, \qquad A, B \in \mathscr{B}\,,$$

where $H(\cdot)$ is a s.c.a. W-to-W nonnegative hermitian operator-valued measure on $\mathscr{B}$.[5] $H(\cdot)$ is called the *control measure* of $\tilde{M}(\cdot)$.

(b) A $\mathscr{K}$-to-$\mathscr{K}$ operator-valued set function $E(\cdot)$ on $\mathscr{B}$ is said to be a *spectral measure* iff $E(\cdot)$ is s.c.a. on $\mathscr{B}$, $E(B)$ is an orthogonal projection for each $B \in \mathscr{B}$ and $E(B)E(A) = E(A \cap B)$, $A, B \in \mathscr{B}$.[6]

With the notation of (2.1) we assume

[5] In [**12**; 8.6(e)] it is shown that $\tilde{M}(\cdot)$ is s.c.a.

[6] Note, we do not stipulate that $E(\Omega) = I$.

$$(2.3)\quad \begin{cases} \text{(i)} & M(\cdot) \text{ is a s.c.a. } W\text{-to-}\mathscr{H} \text{ operator-valued measure on } \mathscr{B}; \\ \text{(ii)} & H(\cdot) \text{ is a s.c.a. } W\text{-to-}W \text{ nonnegative hermitian operator-valued measure on } \mathscr{B}. \end{cases}$$

DEFINITION 2.4. Let $M(\cdot)$ and $H(\cdot)$ be as in (2.3). We say that $M(\cdot)$ is *2-majorizable with respect to* $H(\cdot)$ or that $H(\cdot)$ *is a 2-majorant of* $M(\cdot)$ iff for all $n \geqq 1$ and all $B_1, \cdots, B_n \in \mathscr{B}$ and all $w_1, \cdots, w_n \in W$

$$\left| \sum_{i=1}^{n} M(B_i)w_i \right|_{\mathscr{H}}^{2} \leqq \sum_{i=1}^{n} \sum_{j=1}^{n} (H(B_i \cap B_j)w_i, w_j)_W \,.$$

DEFINITION 2.5. Let $M(\cdot)$ be as in (2.3). We say that

(a) $M(\cdot)$ has a quasi-isometric dilation $\tilde{M}(\cdot)$ iff $\tilde{M}(\cdot)$ is a W-to-$\mathscr{K}$ c.a.q.i. measure on $\mathscr{B}$ where $\mathscr{K}$ is a Hilbert space, and $\exists$ an isometry J on $\mathscr{H}$ to $\mathscr{K}$ such that

$$M(B) = J^* \tilde{M}(B) \,, \qquad B \in \mathscr{B} \,,$$

(b) $M(\cdot)$ has a *spectral dilation* $E(\cdot)$ iff $E(\cdot)$ is a $\mathscr{K}$-to-$\mathscr{K}$ spectral measure on $\mathscr{B}$ where $\mathscr{K}$ is a Hilbert space, and $\exists$ continuous linear operators S on W to $\mathscr{K}$ and T on $\mathscr{K}$ to $\mathscr{H}$ such that

$$M(\cdot) = TE(\cdot)S \,.$$

In the vector case (i.e., $W = \boldsymbol{F}$) the above definitions assume the known forms which we now state.

DEFINITION 2.6. Let Ω, $\mathscr{B}$, $\mathscr{H}$ be as in (2.1). Let $\xi(\cdot)$ be an $\mathscr{H}$-valued c.a. vector measure on $\mathscr{B}$ and let $\mu(\cdot)$ be a nonnegative real-valued c.a. measure on $\mathscr{B}$. We say that $\xi(\cdot)$ is *2-majorizable with respect to* $\mu(\cdot)$ or that $\mu(\cdot)$ *is a 2-majorant of* $\xi(\cdot)$ iff for all $n \geqq 1$, and all $B_1, \cdots, B_n \in \mathscr{B}$ and all $a_1, \cdots, a_n \in \boldsymbol{F}$

$$\left| \sum_{i=1}^{n} a_i \xi(B_1) \right|_{\mathscr{H}}^{2} \leqq \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j \mu(B_i \cap B_j) \,.$$

DEFINITION 2.7. Let $\xi(\cdot)$ be as in 2.6. We say that

(a) $\xi(\cdot)$ has a c.a.o.s. *dilation* $\tilde{\xi}(\cdot)$ iff $\tilde{\xi}(\cdot)$ is a $\mathscr{K}$-valued c.a.o.s. measure on $\mathscr{B}$ where $\mathscr{K}$ is a Hilbert space, and $\exists$ an isometry J on $\mathscr{H}$ to $\mathscr{K}$ such that

$$\xi(B) = J^* \tilde{\xi}(B) \,, \qquad B \in \mathscr{B} \,,$$

(b) $\xi(\cdot)$ has a spectral dilation $E(\cdot)$ iff $E(\cdot)$ is a $\mathscr{K}$-to-$\mathscr{K}$ spectral measure on $\mathscr{B}$ where $\mathscr{K}$ is a Hilbert space, and $\exists$ a continuous linear operator T on $\mathscr{K}$ to $\mathscr{H}$ and a vector $x_0 \in \mathscr{K}$

such that

$$\xi(\cdot) = TE(\cdot)x_0 .$$

LEMMA 2.8. *Let* $M(\cdot)$ *and* $H(\cdot)$ *be as in* (2.3), *and let*

$$K(A, B) = H(A \cap B) - M(B)^* M(A) , \qquad A, B \in \mathscr{B} .$$

Then (a) $\forall B_1, \cdots, B_n \in \mathscr{B}$ *and* $\forall w_1, \cdots, w_n \in W$

$$(*) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} (K(B_i, B_j)w_i, w_j)_W = \sum_{i=1}^{n}\sum_{j=1}^{n} (H(B_i \cap B_j)w_i, w_j)_W - \left|\sum_{i=1}^{n} M(B_i)w_i\right|^2_{\mathscr{H}} .$$

(b) $H(\cdot)$ *is a 2-majorant of* $M(\cdot)$ *iff the kernel* $K(\cdot, \cdot)$ *in* (a) *is positive definite on* $\mathscr{B} \times \mathscr{B}$, i.e., L.H.S. (*) *is always* $\geqq 0$ *and* $K(A, B) = K(B, A)^*$.

Proof. (a) Just expand the L.H.S. (*) after making the substitution $K(B_i, B_j) = H(B_i \cap B_j) - M(B_j)^* M(B_i)$.

(b) Immediate from Definition 2.4. □

2.9. *The Equivalence Theorem.* Let $M(\cdot)$ be a s.c.a. W-to-$\mathscr{H}$ operator-valued measure on $\mathscr{B}$ where $\mathscr{B}$, W, and $\mathscr{H}$ are as in (2.1). Then (a) the following conditions are equivalent:

(α) $M(\cdot)$ has a 2-majorant $H(\cdot)$,

(β) $M(\cdot)$ has a quasi-isometric dilation $\tilde{M}(\cdot)$,

(γ) $M(\cdot)$ has a spectral dilation $E(\cdot)$;

(b) $H(\cdot)$ is a 2-majorant of $M(\cdot) \Leftrightarrow M(\cdot)$ has a quasi-isometric dilation $\tilde{M}(\cdot)$ with control measure $H(\cdot)$.

Proof. (a) $(\alpha) \Rightarrow (\beta)$:[7] Note ($\alpha$) implies that the kernel $K(\cdot, \cdot)$ defined in 2.8 is positive definite, cf. 2.8(b). By the Kernel theorem (Masani [13; p. 421]) ∃ a Hilbert space $\mathscr{H}_1$, and a function $X(\cdot)$ on $\mathscr{B}$ such that $X(B)$ is a continuous linear operator on W to $\mathscr{H}_1$ and

$$K(A, B) = X(B)^* X(A) , \qquad A, B \in \mathscr{B} .$$

Now define $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}_1 = \{(x; x') : x \in \mathscr{H}, x' \in \mathscr{H}_1\}$ and for $B \in \mathscr{B}$ define $\tilde{M}(B) : W \to \mathscr{K}$ by $\tilde{M}(B)w = M(B)w \oplus X(B)w$. We shall show that

$$(1) \qquad \tilde{M}(B)^* \tilde{M}(A) = H(A \cap B) , \qquad A, B \in \mathscr{B} .$$

Note for $A, B \in \mathscr{B}$ and $w, w' \in W$

[7] An alternative more direct proof of "$(\alpha) \Rightarrow (\beta)$" is in the Appendix, cf. A. 8.

$$(2)\quad \begin{aligned}(\tilde{M}(B)^*\tilde{M}(A)w, w')_W &= (\tilde{M}(A)w, \tilde{M}(B)w')_W \\ &= (M(A)w, M(B)w')_{\mathscr{H}} + (X(A)w, X(B)w')_{\mathscr{H}_1} \\ &= (M(B)^*M(A)w, w')_W + (X(B)^*X(A)w, w')_W \\ &= (\{M(B)^*M(A) + X(B)^*X(A)\}w, w')_W = (H(A \cap B)w, w')_W .\end{aligned}$$

So by Definition 2.2(a) $\tilde{M}(\cdot)$ is c.a.q.i. Finally let $\forall x \in \mathscr{H}$, $J(x) = (x; 0) \in \mathscr{K}$. Then J is an isometry on $\mathscr{H}$ to $\mathscr{K}$ and therefore $J^* = J^{-1}P_{\mathscr{R}(J)}$. So $J^*\tilde{M}(B) = M(B)$, $B \in \mathscr{B}$.

$(\beta) \Rightarrow (\gamma)$: For each $B \in \mathscr{B}$, let $\mathscr{M}_B$ be the subspace spanned by $\{\tilde{M}(A)(w): A \in \mathscr{B}, A \subseteq B, w \in W\}$, and let $E(B)$ be the projection on $\mathscr{K}$ onto $\mathscr{M}_B$. Then $E(\cdot)$ is a spectral measure on $\mathscr{B}$ for $\mathscr{K}$ such that $\forall B \in \mathscr{B}$

$$(3)\qquad \tilde{M}(B) = E(B)\tilde{M}(\Omega) \qquad \text{(cf. [13; (5.8)–(5.11)])} .$$

Hence $M(\cdot) = J^*\tilde{M}(\cdot) = J^*E(\cdot)\tilde{M}(\Omega)$.

$(\gamma) \Rightarrow (\alpha)$: Let $M(\cdot) = TE(\cdot)S$, cf. 2.5 (b); and let $B_1, \cdots, B_n \in \mathscr{B}$ and $w_1, \cdots, w_n \in W$. Then

$$(4)\quad \begin{aligned}\left|\sum_{i=1}^n M(B_i)w_i\right|^2_{\mathscr{H}} &= \left|T\sum_{i=1}^n E(B_i)Sw_i\right|^2_{\mathscr{H}} \leqq |T|^2\left|\sum_{i=1}^n E(B_i)Sw_i\right|^2_{\mathscr{K}} \\ &= |T|^2\sum_{i=1}^n\sum_{j=1}^n (S^*E(B_i \cap B_j)Sw_i, w_j)_W .\end{aligned}$$

So $H(\cdot)$ defined for $B \in \mathscr{B}$ by $H(B) = |T|^2S^*E(B)S$ is a 2-majorant of $M(\cdot)$.

(b) The forward implication "$\Rightarrow$" has been shown in the proof that $(\alpha) \Rightarrow (\beta)$, cf. (1). To prove the converse "$\Leftarrow$", note that for $B_1, \cdots, B_n \in \mathscr{B}$ and $w_1, \cdots, w_n \in W$

$$(5)\quad \begin{aligned}\left|\sum_{i=i}^n M(B_i)w_i\right|^2_{\mathscr{H}} &= \left|J^*\sum_{i=1}^n \tilde{M}(B_i)w_i\right|^2_{\mathscr{H}} \leqq \left|\sum_{i=1}^n \tilde{M}(B_i)w_i\right|^2_{\mathscr{K}} \\ &= \sum_{i=1}^n\sum_{j=1}^n (H(B_i \cap B_j)w_i, w_j)_W .\end{aligned}$$

So $H(\cdot)$ is a 2-majorant of $M(\cdot)$. □

In the case that $\mathscr{H} = W$ and the values of $M(\cdot)$ are hermitian operators on W to W the Equivalence theorem can be augmented as follows.

COROLLARY 2.10. *Let $M(\cdot)$ be a* s.c.a. *W-to-W hermitian operator-valued measure on $\mathscr{B}$. Then each of the conditions* (α), (β), (γ) *of* 2.9 *is equivalent to a "Jordan decomposition":*

(δ) $\exists$ *two* s.c.a. *W-to-W nonnegative hermitian operator-valued measures $M_1(\cdot)$ and $M_2(\cdot)$ such that $M(\cdot) = M_1(\cdot) - M_2(\cdot)$.*

Proof. $(\gamma)\Rightarrow(\delta)$: By hypothesis we have $\forall B\in\mathscr{B}$, $TE(B)S = M(B) = M(B)^* = S^*E(B)T^*$. Thus $\forall B\in\mathscr{B}$,

$$\begin{aligned} M(B) &= [TE(B)S + S^*E(B)T^*]/2 \\ &= [(T+S^*)E(B)(T^*+S)]/2 - [S^*E(B)S + TE(B)T^*]/2 \qquad (1) \\ &= M_1(B) - M_2(B). \end{aligned}$$

$(\delta)\Rightarrow(\gamma)$: By the Naimark Dilation theorem, cf. [13; 5.12], $M_1(\cdot) = T_1^*E_1(\cdot)T_1$, $M_2(\cdot) = T_2^*E_2(\cdot)T_2$ where $E_1(\cdot)$ and $E_2(\cdot)$ are spectral measures on $\mathscr{B}$, respectively for two Hilbert spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ and where T_1 and T_2 are continuous linear operators, respectively on W to $\mathscr{K}_1$ and to $\mathscr{K}_2$. Thus $\forall B\in\mathscr{B}$, we may write

$$M(B) = [T_1^*, -T_2^*]\begin{bmatrix} E_1(B) & 0 \\ 0 & E_2(B)\end{bmatrix}\begin{bmatrix} T_1 \\ T_2\end{bmatrix} = TE(B)S\ , \qquad (2)$$

where $E(\cdot)$ is spectral measure on $\mathscr{B}$ for $\mathscr{K}_1\oplus\mathscr{K}_2$, $S = \begin{bmatrix} T_1 \\ T_2\end{bmatrix}$ is a continuous operator on W to $\mathscr{K}_1\oplus\mathscr{K}_2$ and $T = [T_1^*, -T_2^*]$ is a continuous operator on $\mathscr{K}_1\oplus\mathscr{K}_2$ to W. □

It should be noted that upon taking $W = \boldsymbol{F}$ that Lemma 2.8 and the Equivalence Theorem 2.9 assume the following form for vector-valued measures.

LEMMA 2.11. *Let $\xi(\cdot)$ and $\mu(\cdot)$ be as in Definition 2.6, and let*

$$k(A, B) = \mu(A\cap B) - (\xi(A), \xi(B))\ , \qquad A, B\in\mathscr{B}\ .$$

Then (a) $\forall B_1, \cdots, B_n\in\mathscr{B}$ and $\forall a_1, \cdots, a_n\in\boldsymbol{F}$

$$\sum_{i=1}^n\sum_{j=1}^n k(B_i, B_j)a_i\bar{a}_j = \sum_{i=1}^n\sum_{j=1}^n \mu(B_i\cap B_j)a_i\bar{a}_j - \left|\sum_{i=1}^n a_i\xi(B_i)\right|_{\mathscr{H}}^2 . \qquad (*)$$

(b) $\mu(\cdot)$ is a 2-majorant of $\xi(\cdot)$ iff the kernel $k(\cdot,\cdot)$ in (a) is *positive definite* on $\mathscr{B}\times\mathscr{B}$, i.e., L.H.S. (*) is always $\geqq 0$ and $k(A, B) = \overline{k(B, A)}$.

THEOREM 2.12. *Let $\xi(\cdot)$ be a* c.a. *$\mathscr{H}$-valued measure on $\mathscr{B}$. Then* (a) *the following conditions are equivalent:*

(α) *$\xi(\cdot)$ has a 2-majorant $\mu(\cdot)$,*

(β) *$\xi(\cdot)$ has a* c.a.o.s. *dilation $\tilde{\xi}(\cdot)$,*

(γ) *$\xi(\cdot)$ has a spectral dilation $E(\cdot)$;*

(b) *$\mu(\cdot)$ is a 2-majorant of $\xi(\cdot)\Leftrightarrow\xi(\cdot)$ has a* c.a.o.s. *dilation $\tilde{\xi}(\cdot)$ whose control measure is $\mu(\cdot)$* [i.e., $(\tilde{\xi}(A), \tilde{\xi}(B))_{\mathscr{K}} = \mu(A\cap B)$].

Result 2.12 (b) is found in Niemi's paper [15; Th. 12].

3. Existence of 2-majorants for Hilbert space-valued measures.

The proof of our main theorem in this section depends heavily on a remarkable inequality of Grothendieck to discuss which we first introduce the Grothendieck norms:

DEFINITION 3.1. For an $n \times n$ matrix $A = [a_{ij}]$ with entries in $\boldsymbol{F}$ and $\mathscr{K}$ an arbitrary Hilbert space over $\boldsymbol{F}$, define $|A|_{\mathscr{K}}$ by

$$|A|_{\mathscr{K}} = \sup\left\{\left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i, y_j)_{\mathscr{K}}\right| : x_i, y_j \in \mathscr{K} \text{ and } |x_i|_{\mathscr{K}}, |y_j|_{\mathscr{K}} \leqq 1\right\}.$$

(Note this definition also holds for $\mathscr{K} = \boldsymbol{F}$ in which case $(x_i, y_j)_{\boldsymbol{F}} = x_i\bar{y}_j$.)

LEMMA 3.2. (*Grothendieck's inequality*, cf. [10; p. 68], [19]). $\exists$ *a positive constant* $\gamma > 1$ *such that for all* $n \geqq 1$, *all* $n \times n$ *matrices* $A = [a_{ij}]$ *with entries in* $\boldsymbol{F}$ *and all Hilbert spaces* $\mathscr{K}$ *over* $\boldsymbol{F}$

$$(*) \qquad |A|_{\mathscr{K}} \leqq \gamma |A|_{\boldsymbol{F}}.$$

A more useful formulation of the condition (*) reads as follows: For all $x_1, \cdots, x_n, y_1, \cdots, y_n$ in $\mathscr{K}$

$$(3.3) \qquad \left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_i, y_j)_{\mathscr{K}}\right| \leqq \gamma |A|_{\boldsymbol{F}} \max_{1\leqq i\leqq n} |x_i|_{\mathscr{K}} \cdot \max_{1\leqq j\leqq n} |y_j|_{\mathscr{K}}.$$

We now stipulate that:

$$(3.4) \qquad \begin{cases} \mathscr{B} \text{ and } \mathscr{H} \text{ are as in (2.1) and} \\ \xi(\cdot) \text{ is an } \mathscr{H}\text{-valued c.a. measure on } \mathscr{B}. \end{cases}$$

We consider $\boldsymbol{F}$-valued $\mathscr{B}$-measurable *simple functions* ψ and their *integrals* $E_\xi(\psi)$ with respect to $\xi(\cdot)$:

$$(3.5) \qquad \begin{cases} \text{(a)} \quad \psi = \sum_{i=1}^{n} b_i \chi_{B_i}, \qquad B_i \in \mathscr{B}, \qquad b_i \in \boldsymbol{F} \\ \text{(b)} \quad E_\xi(\psi) = \int_\Omega \psi(\omega)\xi(d\omega) = \sum_{i=1}^{n} b_i\xi(B_i) \in \mathscr{H}. \end{cases}$$

It is easily shown that the definition of $E_\xi(\psi)$ is independent of the representation of ψ. We shall denote the set of $\boldsymbol{F}$-valued $\mathscr{B}$-measurable simple functions by $S(\boldsymbol{F}) = S(\mathscr{B}, \boldsymbol{F})$.

It readily follows, cf. [6 (I), p. 323], that for each $\psi \in S(\boldsymbol{F})$

$$(3.6) \qquad |E_\xi(\psi)|_{\mathscr{H}} \leqq \|\xi\|(\Omega) \max_{\omega\in\Omega} |\psi(\omega)|,$$

where $\|\xi\|(\Omega)$ is the *semivariation of* $\xi(\cdot)$ [6 (I), p. 320]. It is known that $\|\xi\|(\Omega)$ is $< \infty$, cf. [6, p. 320, 4(b)].

It is easy to see that for $\phi = \sum_{i=1}^m a_i \chi_{A_i}$, $\psi = \sum_{j=1}^n b_j \chi_{B_j} \in S(F)$ we have

$$(3.7)\quad \begin{cases} \text{(a)} \quad (E_\xi(\phi), E_\xi(\psi))_{\mathscr{K}} = \sum_{i=1}^m \sum_{j=1}^n a_i \bar{b}_j (\xi(A_i), \xi(B_j))_{\mathscr{K}}\,, \\ \text{(b)} \quad |E_\xi(\phi)|_{\mathscr{K}}^2 = \sum_{i=1}^m \sum_{j=1}^m a_i \bar{a}_j (\xi(A_i), \xi(A_j))_{\mathscr{K}}\,. \end{cases}$$

We next prove the key lemma needed for our main Theorem 3.9.

LEMMA 3.8. *$\exists$ a real number $K > 0$ such that for all positive integers m and all $\phi_1, \cdots, \phi_m \in S(F)$*

$$(*)\qquad \sum_{k=1}^m |E_\xi(\phi_k)|_{\mathscr{K}}^2 \leqq K \cdot \max_{\omega \in \Omega} \sum_{k=1}^m |\phi_k(\omega)|^2\,.$$

Proof. Let $\phi = \sum_{i=1}^n a_i \chi_{B_i}$, $\psi = \sum_{i=1}^n b_i \chi_{B_i} \in S(F)$ where $(B_i)_1^n$ is a disjoint sequence. Then by (3.7)(a) and (3.6) we have

$$(1)\qquad \begin{aligned} \left|\sum_{i=1}^n \sum_{j=1}^n (\xi(B_i), \xi(B_j))_{\mathscr{K}} a_i \bar{b}_j\right| &= |(E_\xi(\phi), E_\xi(\psi))_{\mathscr{K}}| \\ &\leqq \|\xi\|(\Omega)^2 \cdot \max_{1 \leqq i \leqq n} |a_i| \cdot \max_{1 \leqq j \leqq n} |b_j|\,. \end{aligned}$$

Without loss of generality, we may assume that each ϕ_k in (*) is of the form $\phi_k = \sum_{i=1}^n b_{ki} \chi_{B_i}$, with the same disjoint sequence $(B_i)_1^n$ for each k. Then

$$(2)\qquad \begin{aligned} \sum_{k=1}^m |E_\xi(\phi_k)|_{\mathscr{K}}^2 &= \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n (\xi(B_i), \xi(B_j))_{\mathscr{K}} b_{ki} \bar{b}_{kj} \\ &= \sum_{i=1}^n \sum_{j=1}^n (\xi(B_i), \xi(B_j))_{\mathscr{K}} \sum_{k=1}^m b_{ki} \bar{b}_{kj} \end{aligned}$$

But then letting $x_i = (b_{ki})_{k=1}^m \in \mathscr{K} = F^m$ and noting that $(x_i, x_j)_{\mathscr{K}} = \sum_{k=1}^m b_{ki} \bar{b}_{kj}$, it follows from (3.3), and equation (1), on taking $[a_{ij}] = [(\xi(B_i), \xi(B_j))_{\mathscr{K}}]$, that

$$(3)\qquad \begin{aligned} \text{R.H.S. (2)} &\leqq \gamma \cdot \|\xi\|(\Omega)^2 \max_{1 \leqq i \leqq n} |x_i|_{\mathscr{K}}^2 \\ &= \gamma \cdot \|\xi\|(\Omega)^2 \max_{\omega \in \Omega} \left(\sum_{k=1}^m |\phi_k(\omega)|^2\right)\,. \end{aligned}$$

Thus (*) is true with $K = \gamma \cdot \|\xi\|(\Omega)^2$. □

THEOREM 3.9 (*Existence*). *Corresponding to every $\mathscr{H}$-valued* c.a. *measure $\xi(\cdot)$ on a σ-algebra $\mathscr{B}$ over Ω, $\exists$ a* c.a. *nonnegative real-valued measure $\mu(\cdot)$ on $\mathscr{B}$ with respect to which $\xi(\cdot)$ is 2-majoriza-*

ble, cf. *Def.* 2.6.

Proof. Taking K as in Lemma 3.8, let for all $\psi \in S(\boldsymbol{R})$

$$(1) \qquad S(\psi) = \inf_{\substack{\{\phi_i\}_{i=1}^m \subseteq S(F) \\ m \geqq 1}} \left\{ \max_{\omega \in \Omega} \left[K\psi(\omega) + K \sum_{k=1}^m |\phi_k(\omega)|^2 - \sum_{k=1}^m |E_\xi(\phi_k)|_{\mathscr{H}}^2 \right] \right\}$$

cf. Pietsch [18]. Then, by elementary considerations, it may be shown that $S(\cdot)$ is a positive homogeneous subadditive functional on $S(\boldsymbol{R})$ such that $K \cdot \min_{\omega \in \Omega} \psi(\omega) \leqq S(\psi) \leqq K \cdot \max_{\omega \in \Omega} \psi(\omega)$. Thus by the Hahn-Banach theorem, cf. [25; Cor., p. 103], there exists a linear functional T on $S(\boldsymbol{R})$ such that $T(\psi) \leqq S(\psi)$, from which it readily follows that for $\psi \in S(\boldsymbol{R})$

$$(2) \qquad K \cdot \min_{\omega \in \Omega} \psi(\omega) \leqq -S(-\psi) \leqq T(\psi) \leqq S(\psi) \leqq K \cdot \max_{\omega \in \Omega} \psi(\omega) .$$

Moreover from (1), it follows that for $\phi \in S(\boldsymbol{F})$,

$$(3) \qquad \begin{aligned} S(-|\phi|^2) &\leqq \max_{\omega \in \Omega} [K\{-|\phi(\omega)|^2\} + K|\phi(\omega)|^2 - |E_\xi(\phi)|_{\mathscr{H}}^2] \\ &= -|E_\xi(\phi)|_{\mathscr{H}}^2 . \end{aligned}$$

Thus since $-T(|\phi|^2) = T(-|\phi|^2) \leqq S(-|\phi|^2)$, it follows that for $\phi \in S(\boldsymbol{F})$

$$(4) \qquad |E_\xi(\phi)|_{\mathscr{H}}^2 \leqq T(|\phi|^2) .$$

Define ν on $\mathscr{B}$ by $\nu(B) = T(\chi_B)$. Note by (2) that $T(\chi_\Omega) = K$. Then $\nu(\cdot)$ is a finitely-additive (f.a.) nonnegative real measure on $\mathscr{B}$. To complete the proof we need to replace $\nu(\cdot)$ by a countably additive measure. Let $\mu(\cdot)$ be the c.a. measure defined from $\nu(\cdot)$ as in Lemma A.1 (see Appendix). We shall show that for each $\phi = \sum_{i=1}^n b_i \chi_{B_i} \in S(\boldsymbol{F})$ with $(B_i)_1^n$ disjoint, that

$$(5) \qquad |E_\xi(\phi)|_{\mathscr{H}}^2 \leqq \sum_{i=1}^n |b_i|^2 \mu(B_i) = \int_\Omega |\phi|^2 d\mu ,$$

i.e., by the sentence following (3.5) (b), that $\mu(\cdot)$ is a 2-majorant of $\xi(\cdot)$:

Let for $1 \leqq i \leqq n$ and $m \geqq 1$ $(B_{ik}^m)_{k=1}^\infty$ be a disjoint sequence in $\mathscr{B}$ such that $B_i = \bigcup_{k=1}^\infty B_{ik}^m$ and $\mu(B_i) = \lim\downarrow_{m\to\infty} \sum_{k=1}^\infty \nu(B_{ik}^m)$,[8] cf. A.1. Then for each m, $\phi_N^m = \sum_{i=1}^n b_i \{\sum_{k=1}^N \chi_{B_{ik}^m}\}$ converges to ϕ as $N \to \infty$ and by (4) for each m and N

$$|E_\xi(\phi_N^m)|_{\mathscr{H}}^2 \leqq \sum_{i=1}^n |b_i|^2 \left\{ \sum_{k=1}^N \nu(B_{ik}^m) \right\} .$$

So on letting $N \to \infty$, we obtain for each m

[8] lim↓ means "nonincreasing limit", i.e., limit of nonincreasing values.

$$(6) \qquad |E_\xi(\phi)|^2_{\mathscr{H}} \leqq \sum_{i=1}^{n} |b_i|^2 \left\{\sum_{k=1}^{\infty} \nu(B_{ik}^m)\right\} .$$

Thus, by the definition of $\mu(\cdot)$, on letting $m \to \infty$, we obtain (5) from (6). □

THEOREM 3.10 (*Uniqueness of minimal 2-majorant*). *Given an $\mathscr{H}$-valued* c.a. *measure $\xi(\cdot)$ on $\mathscr{B}$ there exists one and only one 2-majorant $\mu_0(\cdot)$ of $\xi(\cdot)$ such that*

$$\mu_0(\Omega) = \inf\{\mu(\Omega)\colon \mu(\cdot) \text{ is a 2-majorant of } \xi(\cdot)\} .$$

Proof. The proof which depends on Pietsch's inequality: $4(a^{-1/2} + b^{-1/2})^{-2} \leqq (a + b)/2$ for $a, b > 0$ with equality only if $a = b$, is subsumed in the proof we shall give in §4 of Theorem 4.14.

4. The problem of the existence and uniqueness of 2-majorants for operator-valued measures. Let Ω, $\mathscr{B}$ be as in (2.1).

THEOREM 4.1. *Let W be a q-dimensional Hilbert space over $\boldsymbol{F}$ and $\mathscr{H}$ be an arbitrary Hilbert space over $\boldsymbol{F}$. Then corresponding to every* s.c.a. *W-to-$\mathscr{H}$ operator-valued measure $M(\cdot)$ on $\mathscr{B}$, $\exists$ a* s.c.a. *W-to-W nonnegative hermitian operator-valued measure $H(\cdot)$ on $\mathscr{B}$ with respect to which $M(\cdot)$ is 2-majorizable,* cf. *Def.* 2.4. *Moreover $H(\cdot)$ may be chosen to be of the form*

$$H(\cdot) = \sum_{j=1}^{q} \mu_j(\cdot) P_j ,$$

where $P_1, \cdots, P_q$ are rank 1 *orthogonal projections on W to W such that $\sum_{j=1}^{q} P_j = I$, $P_i P_j = 0$ for $i \neq j$, and $\mu_1(\cdot), \cdots, \mu_q(\cdot)$ are* c.a. *nonnegative real-valued measures on $\mathscr{B}$.*

Proof. Let $\beta_1, \cdots, \beta_q$ be an o.n. basis of W. Then $\xi_k(\cdot) = M(\cdot)\beta_k$ is c.a. on $\mathscr{B}$ to $\mathscr{H}$ and therefore by 3.9 has a 2-majorant $\nu_k(\cdot)$. We shall show that we may take as a 2-majorant of $M(\cdot)$

$$(1) \qquad H(\cdot) = q \sum_{k=1}^{q} \nu_k(\cdot) P_k ,$$

where P_k is the projection onto the space spanned by β_k: For $B_1, \cdots, B_n \in \mathscr{B}$, $w_1, \cdots, w_n \in W$ we have on representing $w_i = \sum_{k=1}^{q} c_{ik}\beta_k$

$$\left|\sum_{i=1}^{n} M(B_i) w_i\right|^2_{\mathscr{H}} = \left|\sum_{i=1}^{n} M(B_i)\Big(\sum_{k=1}^{q} c_{ik}\beta_k\Big)\right|^2_{\mathscr{H}}$$

$$= \left|\sum_{k=1}^{q} \Big(\sum_{i=1}^{n} c_{ik}\xi_k(B_i)\Big)\right|^2_{\mathscr{H}}$$

$$\leqq q \sum_{k=1}^{q} \left| \sum_{i=1}^{n} c_{ik} \xi_k(B_i) \right|_{\mathscr{H}}^2 \quad \text{by the Schwarz ineq .}$$

$$\leqq q \sum_{k=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ik} \bar{c}_{jk} \nu_k(B_i \cap B_j) \quad \text{by Def. 2.6 ,}$$

since $\nu_k(\cdot)$ is a 2-majorant of $\xi_k(\cdot)$. Thus

$$(2) \qquad \left| \sum_{i=1}^{n} M(B_i) w_i \right|_{\mathscr{H}}^2 \leqq q \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{q} c_{ik} \bar{c}_{jk} \nu_k(B_i \cap B_j) .$$

On the other hand it is easily checked from (1)

$$(3) \qquad (H(B_i \cap B_j) w_i, w_j)_W = q \sum_{k=1}^{q} \nu_k(B_i \cap B_j) c_{ik} \bar{c}_{jk} .$$

Combining (2) and (3) we get the inequality of Def. 2.4. □

Unfortunately we are as yet unable to prove the last theorem for infinite-dimensional W. To point out some other aspects of the existence problem for finite-dimensional W, we need the following lemma, part (b) of which is an adjunct to the Equivalence Theorem 2.9.

LEMMA 4.2. *Let* $M(\cdot)$ *be a* s.c.a. W*-to-*$\mathscr{H}$ *operator-valued measure on* $\mathscr{B}$. *Then* (a) $M(\cdot)^*$ *is a* s.c.a. $\mathscr{H}$*-to-*W *operator-valued measure on* $\mathscr{B}$.

(b) $M(\cdot)$ *has a 2-majorant* $\Leftrightarrow M(\cdot)^*$ *has a 2-majorant.*

Proof. (a) We have the sequence of implications: $M(\cdot)$ is s.c.a. $\Rightarrow M(\cdot)$ is w.c.a. $\Rightarrow M(\cdot)^*$ is w.c.a. $\Rightarrow M(\cdot)^*$ is s.c.a., where the last implication follows from [9; Th. 3.6.2].

(b) By 2.9, we have the following sequence of equivalences: $M(\cdot)$ has a 2-majorant $\Leftrightarrow M(\cdot) = TE(\cdot)S \Leftrightarrow M(\cdot)^* = S^*E(\cdot)T^* \Leftrightarrow M(\cdot)^*$ has a 2-majorant. □

COROLLARY 4.3. *Let* W *be an arbitrary Hilbert space over* $\boldsymbol{F}$ *and* $\mathscr{H}$ *be a* q*-dimensional Hilbert space over* $\boldsymbol{F}$. *Then corresponding to every* s.c.a. W*-to-*$\mathscr{H}$ *operator-valued measure* $M(\cdot)$ *on* $\mathscr{B}$ $\exists$ *a* s.c.a. W*-to-*W *nonnegative hermitian operator-valued measure* $H(\cdot)$ *on* $\mathscr{B}$ *with respect to which* $M(\cdot)$ *is 2 majorizable.*

Proof. By 4.2 (a) and 4.1, $M(\cdot)^*$ is s.c.a. and 2-majorizable. Thus by 4.2(b) $M(\cdot)$ is 2-majorizable. □

Let Ω, $\mathscr{B}$, W, $\mathscr{H}$ be as in (2.1). Let $M(\cdot)$ be a s.c.a. W-to-$\mathscr{H}$ operator-valued measure on $\mathscr{B}$ and let $H(\cdot)$ be a s.c.a. W-to-W nonnegative hermitian operator-valued measure on $\mathscr{B}$. We now introduce the concepts of W-valued $\mathscr{B}$-measurable *simple functions* f, g, etc., and the *integral* $E_M(f)$ of f with respect to $M(\cdot)$ and the *integral* $\int_\Omega (dHf, g)$ of the (ordered) pair $\{f, g\}$ with respect to $H(\cdot)$:

$$(4.4)\quad \begin{cases} \text{(a)}\quad f = \sum_{i=1}^{m} w_i \chi_{B_i}\,, \qquad g = \sum_{j=1}^{n} w'_j \chi_{C_j} \quad (B_i, C_j \in \mathscr{B}, w_i, w'_j \in W) \\ \text{(b)}\quad E_M(f) = \int_\Omega M(d\omega) f(\omega) = \sum_{i=1}^{m} M(B_i) w_i \in \mathscr{H} \\ \text{(c)}\quad \int_\Omega (dHf, g)_W = \int_\Omega (H(d\omega)f(\omega), g(\omega))_W \\ \qquad\qquad = \sum_{i=1}^{m} \sum_{j=1}^{n} (H(B_i \cap C_j) w_i, w'_j)_W\,. \end{cases}$$

It is readily shown that the two integrals defined in (b) and (c) are independent of the representations of f and g and that when the B_i are disjoint we have

$$(4.5)\qquad \int_\Omega (dHf, f)_W = \sum_{i=1}^{m} (H(B_i) w_i, w_i)_W\,.$$

We shall denote the set of *W-valued $\mathscr{B}$-measurable simple functions* by $S(W) = S(\mathscr{B}, W)$. We note by (4.4) and Def. 2.4, that

$$(4.6)\qquad \begin{cases} H(\cdot) \text{ is a 2-majorant of } M(\cdot) \text{ iff} \\ |E_M(f)|^2_{\mathscr{H}} \leqq \int_\Omega (dHf, f)_W \quad \forall f \in S(W)\,. \end{cases}$$

THEOREM 4.7 (*Existence of a minimum trace 2-majorant*). *Let W be a q-dimensional Hilbert space over* $\boldsymbol{F}$ *and* $\mathscr{H}$ *be an arbitrary Hilbert space over* $\boldsymbol{F}$. *Given a* s.c.a. *W-to-$\mathscr{H}$ operator-valued measure $M(\cdot)$ on $\mathscr{B}$, there exists a* s.c.a. *2-majorant $H_0(\cdot)$ of $M(\cdot)$ such that*

$$\textit{trace } H_0(\Omega) = \inf\{\textit{trace } H(\Omega):\ H(\cdot) \textit{ is a 2-majorant of } M(\cdot)\}\,.$$

Proof. By 4.1 the class of 2-majorants of $M(\cdot)$ is not empty. Let

$$(1)\qquad K = \inf\{\tau H(\Omega):\ H(\cdot) \text{ is a 2-majorant of } M(\cdot)\}$$

and let $(H_n(\cdot))_{n=1}^\infty$ be a sequence of 2-majorants of $M(\cdot)$ such that $\tau H_n(\Omega) \searrow K$. To prove the theorem we introduce the following space and linear functional.

Let ℓ_∞ be the linear space of bounded real functions $\phi(\cdot)$ on $\boldsymbol{N}_+$.[9] Define the functional S on ℓ_∞ by $S(\phi) = \overline{\lim}_{n\to\infty} \phi(n)$ and observe that

[9] $\boldsymbol{N}_+$ and $\boldsymbol{R}_{0+}$ denote, respectively, the set of positive integers and the set of non-negative real numbers.

it is positively homogeneous, and subadditive. Hence by the Hahn-Banach theorem there exists a linear functional T on ℓ_∞ such that $T(\phi) \leqq S(\phi)$ and therefore for each $\phi(\cdot) \in \ell_\infty$

$$\varliminf_{n\to\infty} \phi(n) = -S(-\phi) \leqq T(\phi) \leqq S(\phi) = \varlimsup_{n\to\infty} \phi(n) . \tag{2}$$

So T is nonnegative and continuous with respect to the sup norm on ℓ_∞.

Now let $B \in \mathscr{B}$ be fixed and define for $w \in W$, $g_w(n) = g_w^B(n)$ by

$$g_w(n) = H_n(B)^{1/2} w , \qquad n \geqq 1 . \tag{3}$$

Since $|H_n(B)| \leqq \tau H_n(B) \leqq \tau H_n(\Omega) \leqq \tau H_1(\Omega)$ $\forall n \geqq 1$, it follows

$$|g_w(n)|_W^2 = (H_n(B)w, w)_W \leqq \tau H_1(\Omega) |w|_W^2 , \qquad \forall n \geqq 1 . \tag{4}$$

Thus $|g_w(\cdot)|_W^2 \in \ell_\infty$. Define for each $w \in W$

$$R_B(w) = T(|g_w(\cdot)|_W^2) . \tag{5}$$

Then by (2) and (4) it follows for $\forall w \in W$

$$0 \leqq R_B(w) \leqq \tau H_1(\Omega) |w|_W^2 . \tag{6}$$

We now proceed to show $R_B(\cdot)^{1/2}$ is a seminorm on W satisfying the parallelogram law. We do this by exhibiting the connection between $R_B(\cdot)^{1/2}$ and an ℓ_2-norm. Since $T \in \ell_\infty'$ (the dual of ℓ_∞), we have for all $\psi(\cdot) \in \ell_\infty$

$$T(\psi) = \int_{N_+} \psi(n)\alpha(dn) , \tag{7}$$

where α is a finitely-additive measure on 2^{N_+} to $\boldsymbol{R}_{0+}$,[9] cf. [6; p. 296, Th. 16]. Now consider the space $\ell_2 = \ell_2\ (N_+, 2^{N_+}, \alpha; W)$ of W-valued functions on N_+ which are square-integrable with respect to α. This is a pre-Hilbert space under the usual ℓ_2-norm, $|\cdot|_2$, cf. [6; p. 120, Lemma 3(b)]. By (7) and (5)

$$\begin{aligned}(|g_w(\cdot)|_2)^2 &= \int_{N_+} |g_w(n)|_W^2 \alpha(dn) \\ &= T(|g_w(\cdot)|_W^2) = R_B(w) < \infty .\end{aligned}$$

Thus $\forall w \in W$

$$g_w(\cdot) \in \ell_2 \quad \text{and} \quad R_B(w)^{1/2} = |g_w(\cdot)|_2 . \tag{8}$$

From (3) it is obvious that for $c \in \boldsymbol{R}$ and $w, w_1, w_2 \in W$

$$g_{cw}(\cdot) = c g_w(\cdot) \quad \text{and} \quad g_{w_1+w_2}(\cdot) = g_{w_1}(\cdot) + g_{w_2}(\cdot) . \tag{9}$$

Since $|\cdot|_2$ is a norm satisfying the parallelogram law, it follows readily from (8) and (9) that $R_B(\cdot)^{1/2}$ is a seminorm satisfying the parallelogram law. Since by (6) $R_B(\cdot)$ is bounded, it follows from the *J-VN* Lemma A.2 (in the Appendix) that $R_B(\cdot)$ comes from a bounded nonnegative hermitian sesquilinear functional on $W \times W$ to $\boldsymbol{F}$, and thus from a continuous nonnegative hermitian linear operator $N(B)$ on W so that

$$R_B(w) = (N(B)w, w)_W \qquad \forall w \in W .$$

Thus using (8) and (3) we see that for $\forall w \in W$

$$(10) \qquad \begin{aligned} (N(B)w, w)_W &= R_B(w) = |g_w(\cdot)|_2^2 \\ &= \int_{N_+} |g_w(\cdot)|_W^2 \alpha(dn) = \int_{N_+} (H_n(B)w, w)_W \alpha(dn) . \end{aligned}$$

This shows that $N(\cdot)$ is a finitely additive measure on $\mathscr{B}$, from the finite-additivity of $H_n(\cdot)$, $n \geqq 1$.

Let $H(\cdot)$ correspond to this $N(\cdot)$ as in Lemma A.3. Then

$$(11) \qquad \begin{cases} H(\cdot) \text{ is a s.c.a. } W\text{-to-}W \text{ operator-valued measure on } \mathscr{B}, \text{ and} \\ 0 \lesssim H(B) \lesssim N(B) \lesssim N(\Omega) . \end{cases}$$

We claim that the $H(\cdot)$ just obtained is the desired $H_0(\cdot)$. We first show that this $H(\cdot)$ is a 2-majorant of $M(\cdot)$.

Let $f = \sum_{i=1}^m w_i \chi_{B_i} \in S(W)$ with B_i's disjoint, then by definitions (4.4)(b)(c) and since each $H_n(\cdot)$ is a 2-majorant of $M(\cdot)$ we have

$$(12) \qquad \begin{aligned} |E_M(f)|_{\mathscr{H}}^2 &\leqq \int_\Omega (dH_n f, f)_W = \sum_{i=1}^m (H_n(B_i)w_i, w_i)_W \\ &= \sum_{i=1}^m |g_{w_i}^{B_i}(n)|_W^2 \qquad \forall n \geqq 1 , \end{aligned}$$

where the last equality follows by (3) and (4). But thus by (2), (5) and (10)

$$\begin{aligned} |E_M(f)|_{\mathscr{H}}^2 &\leqq \varliminf_{n\to\infty} \sum_{i=1}^m |g_{w_i}^{B_i}(n)|_W^2 \leqq T\left(\sum_{i=1}^m |g_{w_i}^{B_i}(\cdot)|_W^2\right) \\ &= \sum_{i=1}^m T(|g_{w_i}^{B_i}(\cdot)|_W^2) = \sum_{i=1}^m R_{B_i}(w_i) \\ &= \sum_{i=1}^m (N(B_i)w_i, w_i)_W , \end{aligned}$$

i.e., $|\sum_{i=1}^m M(B_i)w_i|_{\mathscr{H}}^2 \leqq \sum_{i=1}^m (N(B_i)w_i, w_i)_W$. But then for $H(\cdot)$ corresponding to $N(\cdot)$ as in A.3 it readily follows, as with Theorem 3.9 (5), that

$$(13) \qquad |E_M(f)|_{\mathscr{H}}^2 \leqq \sum_{i=1}^m (H(B_i)w_i, w_i)_W = \int_\Omega (dHf, f)_W ,$$

i.e., cf. (4.6) and the statement after (4.4), $H(\cdot)$ is a 2-majorant of $M(\cdot)$.

We denote by (11) that

$$\tau H(\Omega) \leqq \tau N(\Omega) . \tag{14}$$

Next, on noting $g_w^\Omega(n) = g_w(n)$ and letting $\beta_1, \cdots, \beta_q$ be an o.n. basis of W we obtain

$$\begin{aligned}\tau N(\Omega) &= \sum_{k=1}^{q} (N(\Omega)\beta_k, \beta_k)_W = \sum_{k=1}^{q} R_\Omega(\beta_k) \\ &= \sum_{k=1}^{q} T(|g_{\beta_k}(\cdot)|_W^2) = T\Big(\sum_{k=1}^{q} |g_{\beta_k}(\cdot)|_W^2\Big) .\end{aligned} \tag{15}$$

But by (3) and (4)

$$\begin{aligned}\sum_{k=1}^{q} |g_{\beta_k}(n)|_W^2 &= \sum_{k=1}^{q} (H_n(\Omega)\beta_k, \beta_k)_W \\ &= \tau H_n(\Omega) \searrow K \quad \text{as} \quad n \longrightarrow \infty ,\end{aligned} \tag{16}$$

cf. (1). Hence by (2), R.H.S. (15) $= K$. Thus

$$\tau N(\Omega) = K . \tag{17}$$

Therefore by (14) $\tau H(\Omega) \leqq K$. But since K is the infimum of the traces of 2-majorizing measures, it follows that $\tau H(\Omega) = K$. So existence is established. □

To prove uniqueness of the minimum trace 2-majorant we need to introduce further results on integrals.

Throughout the rest of this section we assume W is a finite-dimensional Hilbert space with o.n. basis $(\beta_i)_{i=1}^q$. For a s.c.a. W-to-$\mathscr{H}$ operator-valued measure $M(\cdot)$ on $\mathscr{B}$ we define the *semivariation* $\|M\|(\cdot)$ of $M(\cdot)$ by

$$\|M\|(B) = \sup \left| \sum_{i=1}^{m} M(B_i)w_i \right|_{\mathscr{H}} \tag{4.8}$$

where the supremum is taken over all finite partitions $(B_i)_1^m$ of B and all $w_i \in W$ with $|w_i|_W \leqq 1$.

Since each $w_i = \sum_{j=1}^q a_{ij}\beta_j$, with $1 \geqq |w_i|_W^2 = \sum_{j=1}^q |a_{ij}|^2$, it follows that $\|M\|(\Omega) \leqq \sum_{j=1}^q \|M(\cdot)\beta_j\|(\Omega) < \infty$, cf. [6 (I); p. 320].

We shall call a function $f(\cdot)$ on Ω to W *$\mathscr{B}$-measurable* iff for each open sphere $S(x, r) = \{y: |y - x|_W < r\}$ we have $f^{-1}(S(x, r)) \in \mathscr{B}$. We shall denote the set of *bounded $\mathscr{B}$-measurable W-valued functions on Ω* by $\boldsymbol{B}(W) = \boldsymbol{B}(\mathscr{B}, W)$. Since W is finite-dimensional it is easily proven that $\boldsymbol{B}(W)$ is the closure of the linear space of simple functions $\boldsymbol{S}(W)$ under the sup norm.

DEFINITION 4.9. For $f \in \boldsymbol{B}(W)$ we define

$$E_M(f) = \int_\Omega M(d\omega)f(\omega) = \lim_{n\to\infty} E_M(f_n) ,$$

where $(f_n)_1^\infty$ is any sequence of simple functions converging uniformly to f, cf. (4.4) (a, b).

Since for a simple function f we have

$$|E_M(f)|_{\mathscr{H}} \leqq \|M\|(\Omega) \cdot \sup_\omega |f(\omega)|_W , \tag{4.10}$$

it follows that the integral in 4.9 exists, is well-defined, and also satisfies (4.10).

Next we recall some facts on s.c.a. W-to-W nonnegative hermitian measures $H(\cdot)$ on $\mathscr{B}$ when W is a finite-dimensional Hilbert space, cf. [21; §2] and [22 (I); §2 and p. 207 (1)]. The symbol ν shall denote a σ-finite nonnegative real measure on $\mathscr{B}$. We say $H(\cdot)$ *is absolutely continuous with respect to* ν $[H \ll \nu]$ iff $\nu(B) = 0 \Rightarrow H(B) = 0$. Because W is finite-dimensional it follows that $H(\cdot)$ is c.a. in the euclidean norm $|\ \ |_E$ and has a finite total variation measure $|H|_E(B) = \sup(\sum_{i=1}^n |H(B_i)|_E)$ (taken over all finite partitions $(B_i)_1^n$ of B). Further

THEOREM 4.11. *Let* $H \ll \nu$. *Then* (a) $\exists$ *a unique* a.e. (ν) $\mathscr{B}$-*measurable*[10] *W-to-W operator-valued function* $H'_\nu(\cdot)$ *on* Ω *such that*

$$H(B) = \int_B H'_\nu(\omega)d\nu \qquad \forall B \in \mathscr{B} ;$$

where the last is a Bochner integral, and

$$|H|_E(B) = \int_B |H'_\nu(\omega)|_E d\nu \qquad \forall B \in \mathscr{B} . \tag{b}$$

Moreover, (c) $0 \lesssim H'_\nu(\omega)$ a.e. (ν) [*We may always take a version of* $H'_\nu(\cdot)$ *which is* $\gtrsim 0$ *everywhere.*]

The use of 4.11 (a) allows us to adopt the definition

$$\int_\Omega (dHf, g)_W = \int_\Omega (H'_\nu(\omega)f(\omega), g(\omega))_W d\nu \qquad \forall f, g \in \boldsymbol{B}(W) \tag{4.12}$$

where ν is any measure such that $H \ll \nu$. It is readily shown: for $f = \sum_{i=1}^n w_i \chi_{B_i}$, $g = \sum_{j=1}^m w'_i \chi_{C_j} \in \boldsymbol{S}(W)$

$$\int_\Omega (dHf, g)_W = \sum_{i=1}^n \sum_{j=1}^m H(B_i \cap C_j)w_i, w'_j)_W ;$$

[10] I.e, with respect to the Borel σ-algebra in the Banach space of W-to-W bounded operators under the norm $|\cdot|_E$.

and that if $f, g \in \boldsymbol{B}(W)$ are uniform limits of sequences of simple functions $(f_n)_1^\infty$, $(g_n)_1^\infty$, then $\int_\Omega (dHf, g)_W = \lim_{n\to\infty} \int_\Omega (dHf_n, g_n)_W$ (which limit exists).

From (4.6) and Definitions 4.9 and (4.12) it readily follows that when W is finite-dimensional

$$(4.13) \qquad \begin{cases} H(\cdot) \text{ is a 2-majorant of } M(\cdot) \text{ iff} \\ |E_M(f)|_{\mathscr{H}}^2 \leqq \int_\Omega (dHf, f)_W \qquad \forall f \in \boldsymbol{B}(W)\,. \end{cases}$$

In the subsequent discussion, we shall be concerned with $\mathscr{B}$-measurable W-valued functions and $\mathscr{B}$-measurable W-to-W operator-valued functions and with various functions formed from these by various operations. In all cases the new functions are again $\mathscr{B}$-measurable by virtue of the finite-dimensionality of W.

THEOREM 4.14 (*Uniqueness of the minimum trace* 2-*majorant*). *Let W be a q-dimensional Hilbert space over $\boldsymbol{F}$ and $\mathscr{H}$ be an arbitrary Hilbert space over $\boldsymbol{F}$. Given a* s.c.a. *W-to-$\mathscr{H}$ operator-valued measure $M(\cdot)$ on $\mathscr{B}$, there exists one and only one* s.c.a. 2-*majorant $H_0(\cdot)$ of $M(\cdot)$ such that*

$$\textit{trace } H_0(\Omega) = \inf\{\textit{trace } H(\cdot)\colon H(\Omega) \textit{ is a 2-majorant of } M(\cdot)\}\,.$$

Proof. By Theorem 4.7 existence is assured. Now suppose that $H_1(\cdot)$ and $H_2(\cdot)$ are both 2-majorants of $M(\cdot)$ such that $\tau H_1(\Omega) = K = \tau H_2(\Omega)$, where K is as in Theorem 4.7 (1). Let $m(\cdot)$ be the measure $\tau H_1(\cdot) + \tau H_2(\cdot)$ and let, for brevity, $G_1(\cdot) = H'_{1,m}(\cdot)$, $G_2(\cdot) = H'_{2,m}(\cdot)$ as in 4.11. So, by (4.13), for $f \in \boldsymbol{B}(W)$

$$|E_M(f)|_{\mathscr{H}}^2 \leqq \int_\Omega (G_i(\omega)f(\omega), f(\omega))_W dm\,, \qquad i = 1, 2\,.$$

Let $P_i(\omega)$ = projection onto range of $G_i(\omega)$ for $i = 1, 2$. The first step in our proof is to prove $P_1(\omega) = P_2(\omega)$ a.e. (m): It is readily shown that $E_M(P_1 f) = E_M(f) = E_M(P_2 f)$[11] for all $f \in \boldsymbol{B}(W)$ and thus for $P(\omega)$ = projection onto $\mathscr{R}(G_1(\omega)) \cap \mathscr{R}(G_2(\omega))$[12] it readily follows $E_M(f) = E_M(Pf)$ and thus for all $f \in \boldsymbol{B}(W)$

$$|E_M(f)|_{\mathscr{H}}^2 = |E_M(Pf)|_{\mathscr{H}}^2 \leqq \int_\Omega (G_i(\omega)P(\omega)f(\omega), P(\omega)f(\omega))_W dm\,,$$

[11] We use the convention that $P_1 f$ is the function defined by $(P_1 f)(\omega) = P_1(\omega)\{f(\omega)\}$, etc.

[12] For proving $\mathscr{B}$-measurability, note that by [**3**; Th. 8]

$$P(\omega) = 2P_1(\omega)(P_1(\omega) + P_2(\omega))^{\#}P_2(\omega)\,.$$

$i = 1, 2$. So we must have $P_1 = P = P_2$ a.e. (m) $\Big($for otherwise since $\tau H(\Omega) = \int_\Omega \tau H'_\nu(\omega)d\nu$, there is a 2-majorant with trace smaller than K, which is a contradiction$\Big)$. Thus $\mathscr{R}(G_1(\omega)) = \mathscr{R}(G_2(\omega))$ a.e. (m).

The proof of uniqueness shall now be accomplished by showing $G_1(\omega) = G_2(\omega)$ a.e. (m): Let $F(\omega) = (G_1(\omega)^{1/2\#} + G_2(\omega)^{1/2\#})^\#$ and note that $P(\omega) = (G_1(\omega)^{1/2\#} + G_2(\omega)^{1/2\#})F(\omega)$ a.e. (m). Let

$$B_n = \{\omega : |G_1(\omega)^{1/2\#}F(\omega)|_E \leqq n,\ |G_2(\omega)^{1/2\#}F(\omega)|_E \leqq n\} .$$

So $B_n \nearrow \Omega$ as $n \to \infty$. Then for $f \in S(W)$

$$\begin{aligned}|E_M(\chi_{B_n}f)|_{\mathscr{K}} &= |E_M(\chi_{B_n}Pf)|_{\mathscr{K}} \leqq |E_M(\chi_{B_n}G_1^{1/2\#}Ff)|_{\mathscr{K}} + |E_M(\chi_{B_n}G_2^{1/2\#}Ff)|_{\mathscr{K}} \\ &\leqq \left(\int_{B_n} (G_1(G_1^{1/2\#}F)f,\ G_1^{1/2\#}Ff)_W dm\right)^{1/2} \\ (1)\qquad &\quad + \left(\int_{B_n} (G_2(G_2^{1/2\#}F)f,\ G_2^{1/2\#}Ff)_W dm\right)^{1/2} \\ &= 2\left(\int_{B_n} (F^2 f,\ f)_W dm\right)^{1/2} .\end{aligned}$$

Thus

$$\begin{aligned}(2)\qquad |E_M(\chi_{B_n}f)|^2_{\mathscr{K}} &\leqq \int_{B_n} (4F^2 f,\ f)_W dm \\ &= \int_{B_n} (4(G_1(\omega)^{1/2\#} + G_2(\omega)^{1/2\#})^{2\#} f(\omega),\ f(\omega))_W dm\end{aligned}$$

where clearly by A.7 (*) in the Appendix, we may replace B_n by Ω (on letting $n \to \infty$). So by (2) $H_3(\cdot)$ defined by $H_3(B) = \int_B 4F(\omega)^2 dm$ is a 2-majorant of $M(\cdot)$.

Let $C = \{\omega : G_1(\omega) \neq G_2(\omega)\}$. Since W is separable it follows $C = \bigcup_{i=1}^\infty \{\omega : ((G_1(\omega) - G_2(\omega))w_i,\ w_i)_W \neq 0\} \in \mathscr{B}$, where $\{w_i\}_1^\infty$ is a dense subset of W. Then by A.7 (*) $4\tau(F(\omega)^2) \leqq (1/2)\tau(G_1(\omega) + G_2(\omega))$ with strict inequality on the set C. But thus if $m(C) > 0$, it follows that

$$\begin{aligned}\tau H_3(\Omega) &= \left(\int_C + \int_{\Omega - C}\right) 4\tau(F(\omega)^2)dm < \int_\Omega (1/2)\tau(G_1(\omega) + G_2(\omega))dm \\ &= (1/2)(\tau H_1(\Omega) + \tau H_2(\Omega)) = K ,\end{aligned}$$

i.e., $\tau H_3(\Omega) < K$, which contradicts 4.7 (1). So we must have $m(C) = 0$, i.e., $G_1(\omega) = G_2(\omega)$ a.e. (m). □

We conclude this section with the following

EXAMPLE 4.15. *Explicit form of the minimum trace 2-majorant in 4.14 when $\Omega = \{\omega_1, \omega_2\}$ and $\mathscr{B} = 2^\Omega$*: Denote $M_1 = M\{\omega_1\}$, $M_2 =$

$M\{\omega_2\}$. Then the explicit $H(\cdot)$ of 4.14 is

$$H_1 = M_1^* M_1 + (M_1^* M_2 M_2^* M_1)^{1/2}$$
$$H_2 = M_2^* M_2 + (M_2^* M_1 M_1^* M_2)^{1/2}$$

such that

$$\begin{bmatrix} M_1^* M_1 & M_1^* M_2 \\ M_2^* M_1 & M_2^* M_2 \end{bmatrix} \leqq \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{on} \quad W \oplus W ,$$

and trace $(H_1 + H_2)$ is a minimum. [The proof which is lengthy, is presented elsewhere. See "Added in proof" section.]

APPENDIX. We begin by proving two lemmas on nonnegative real-valued and on nonnegative hermitian operator-valued measures on a σ-algebra $\mathscr{B}$ over a set Ω.

LEMMA A.1. *Let* $\nu(\cdot)$ *be a* f.a. *nonnegative real-valued measure on* $\mathscr{B}$. *Define for each* $B \in \mathscr{B}$

$$\mu(B) = \inf \left\{ \sum_{i=1}^{\infty} \nu(B_i) \right\} ,$$

where the infimum is taken over all countable partitions $(B_i)_1^\infty$ *of* B *into sets in* $\mathscr{B}$. *Then* $\mu(\cdot)$ *is a* c.a. *nonnegative real measure on* $\mathscr{B}$ *such that for each* $B \in \mathscr{B}$

$$0 \leqq \mu(B) \leqq \nu(B) \leqq \nu(\Omega) < \infty .$$

Proof. We leave it to the reader to show $\mu(\cdot)$ is f.a.

We now show that $\mu(\cdot)$ is *continuous from above at* $\emptyset$: Let $A_n \searrow \emptyset$. We shall show $\mu(A_n) \searrow 0$. [Proof. Note for each $n \geqq 1$, $A_n = \bigcup_{i=n}^{\infty} (A_i - A_{i+1})$ and that since ν is f.a., we have $\sum_{i=1}^{\infty} \nu(A_i - A_{i+1}) \leqq \nu(A_1) < \infty$. Thus

$$0 \leqq \mu(A_n) \leqq \sum_{i=n}^{\infty} \nu(A_i - A_{i+1}) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty .$$

Hence, cf. Halmos [8; p. 39, Th. F], $\mu(\cdot)$ is c.a. □

To generalize Lemma A.1 we need the following form of the Jordan-Von Neumann theorem, which is further used in the proof of Theorem 4.7.

A.2. *Jordan-Von Neumann Lemma.* Let $\mathscr{H}$ be a Hilbert space over $\boldsymbol{F}$ and let $R(\cdot)$ be a function from $\mathscr{H}$ to $\boldsymbol{R}_{0+}$ such that

(i) $R(\cdot)^{1/2}$ is a seminorm,

(ii) $R(\cdot)^{1/2}$ satisfies the parallelogram law, i.e., $R(x-y)+R(x-y)=2R(x)+2R(y)$, each $x, y \in \mathscr{H}$,

(iii) there exists $K>0$ such that $R(x) \leqq K|x|^2_{\mathscr{H}}$ $\forall x \in \mathscr{H}$.

Then (a) $R(\cdot)$ can be recovered from a unique bounded nonnegative hermitian sesquilinear functional $T(\cdot, \cdot)$[13] on $\mathscr{H} \times \mathscr{H}$ to $\boldsymbol{F}$, i.e.,

$$(*) \qquad R(x) = T(x, x) \quad \text{and} \quad |T(x, y)| \leqq K|x|_{\mathscr{H}} \cdot |y|_{\mathscr{H}} ,$$

(b) $\exists$ a unique bounded nonnegative hermitian linear operator A on $\mathscr{H}$ to $\mathscr{H}$ such that $|A| \leqq K$, $T(x, y) = (Ax, y)_{\mathscr{H}}$ and necessarily $R(x) = (Ax, x)_{\mathscr{H}}$.

Proof. (a) Follow the proof of J-VN Theorem in [24, p. 124, Th. 1] and obtain the second part of (*) by use of the Schwarz inequality.

(b) Cf. [4, Th. 21.1]. □

We now generalize Lemma A.1.

LEMMA A.3. *Let $\mathscr{H}$ be a Hilbert space over $\boldsymbol{F}$ and let $N(\cdot)$ be a* f.a. *nonnegative $\mathscr{H}$-to-$\mathscr{H}$ operator-valued measure on $\mathscr{B}$. Let for each $x \in \mathscr{H}$, $\nu_x(\cdot) = (N(\cdot)x, x)$ [which is obviously a* f.a. *nonnegative real-valued measure on $\mathscr{B}$] and let for each $x \in \mathscr{H}$, $\mu_x(\cdot)$ be the nonnegative real-valued measure on $\mathscr{B}$ corresponding to $\nu_x(\cdot)$ as in Lemma* A.1. *Then*

(a) *for each $B \in \mathscr{B}$ $\exists$ a unique bounded $\mathscr{H}$-to-$\mathscr{H}$ nonnegative hermitian operator $H(B)$ such that*

$$(H(B)x, x)_{\mathscr{H}} = \mu_x(B) , \qquad \text{each} \quad x \in \mathscr{H} ,$$

(b) *the set function $H(\cdot)$ is* s.c.a. *on $\mathscr{B}$ such that*

$$0 \lesssim H(B) \lesssim N(B) \lesssim N(\Omega) , \qquad \text{each} \quad B \in \mathscr{B} .$$

Proof. (a) Let B be a fixed set in $\mathscr{B}$. We shall prove that the function $R_B(\cdot)$ defined on $\mathscr{H}$ by $R_B(x) = \mu_x(B)$ satisfies the conditions (i), (ii), (iii) in the J-VN Lemma A.2. To carry out the proof we introduce some notation. Let $\mathscr{P} = (B_i)_1^\infty$ denote a partition of B into subsets in $\mathscr{B}$. Define for $x \in \mathscr{H}$

$$(1) \qquad \nu_x(\mathscr{P}) = \sum_{i=1}^{\infty} \nu_x(B_i) .$$

[13] A functional $T(\cdot, \cdot)$ on $\mathscr{H} \times \mathscr{H}$ to $\boldsymbol{F}$ is called *bounded*; *nonnegative*; *hermitian*; *sesquilinear* iff, respectively, $\exists C > 0$ such that $|T(x, y)| \leqq C|x|_{\mathscr{H}} \cdot |y|_{\mathscr{H}}$ $\forall x, y \in \mathscr{H}$; $T(x, x) \geqq 0$ $\forall x \in \mathscr{H}$; $T(x, y) = \overline{T(y, x)}$ $\forall x, y \in \mathscr{H}$; $T(ax+by, z) = aT(x, z) + bT(y, z)$, $T(z, ax+by) = \bar{a}T(z, x) + \bar{b}T(z, y)$ $\forall a, b \in \boldsymbol{F}$, $\forall x, y, z \in \mathscr{H}$. In [**4**], the words "positive" and "symmetric" are used respectively for "nonnegative" and "hermitian".

Then, by definition, cf. A.1,

$$\mu_x(B) = \inf\{\nu_x(\mathscr{P}) : \mathscr{P} \text{ is a partition of } B\} . \tag{2}$$

From (2) it easily follows that for each $x \in \mathscr{H}$ and $c \in \boldsymbol{F}$ that $\mu_x(B) \geqq 0$ and $\mu_{cx}(B) = |c|^2\mu_x(B)$.

By the *superposition* $\mathscr{P}_1 \circ \mathscr{P}_2$ of two partitions $\mathscr{P}_1$, $\mathscr{P}_2$ of B we mean the partition of B composed of all intersections of pairs of sets, one from $\mathscr{P}_1$, one from $\mathscr{P}_2$. It follows similarly as in A.1 that

$$\nu_x(\mathscr{P}_1 \circ \mathscr{P}_2) \leqq \nu_x(\mathscr{P}_1),\ \nu_x(\mathscr{P}_2) . \tag{3}$$

We now prove condition (i) holds: It only remains to show that as a function of $x \in \mathscr{H}$, $\sqrt{\mu_x(B)}$ satisfies the triangle inequality

$$\sqrt{\mu_{x+y}(B)} \leqq \sqrt{\mu_x(B)} + \sqrt{\mu_y(B)} \qquad \text{for} \quad x, y \in \mathscr{H} . \tag{4}$$

Note that since in (2) $\mu_x(B)$ is the infimum of a set of nonnegative real numbers, there exists a sequence of partitions $(\mathscr{P}_n^x)$ of B such that $\mu_x(B) = \lim\downarrow_{n\to\infty} \nu_x(\mathscr{P}_n^x)$. Similarly there is a sequence of partitions $(\mathscr{P}_n^y)$ such that $\mu_y(B) = \lim\downarrow_{n\to\infty} \nu_y(\mathscr{P}_n^y)$. Let for each n, $\mathscr{P}_n = \mathscr{P}_n^x \circ \mathscr{P}_n^y$. Then by (3) it follows that

$$\mu_x(B) = \lim_{n\to\infty} \nu_x(\mathscr{P}_n) , \qquad \mu_y(B) = \lim_{n\to\infty} \nu_y(\mathscr{P}_n) . \tag{5}$$

Now let $\mathscr{P} = (B_i)_1^\infty$ be an arbitrary partition of B. Then

$$\begin{aligned} \sqrt{\nu_{x+y}(\mathscr{P})} &= \sqrt{\left(\sum_{i=1}^{\infty} \nu_{x+y}(B_i)\right)} \\ &= \sqrt{\left(\sum_{i=1}^{\infty} |N(B_i)^{1/2}(x+y)|_{\mathscr{H}}^2\right)} \\ &= \sqrt{\left(\sum_{i=1}^{\infty} |N(B_i)^{1/2}x + N(B_i)^{1/2}y|_{\mathscr{H}}^2\right)} , \end{aligned} \tag{6}$$

where each of the sequences $(N(B_i)^{1/2}x)_{i=1}^\infty$, $(N(B_i)^{1/2}y)_{i=1}$ is in the Hilbert space $\ell_2(N_+, 2^{N_+}, \mu; \mathscr{H})$, where $\mu(\cdot)$ is the "counting measure". But thus

$$\begin{aligned} \text{R.H.S. (6)} &\leqq \sqrt{\left(\sum_{i=1}^{\infty} |N(B_i)^{1/2}x|_{\mathscr{H}}^2\right)} + \sqrt{\left(\sum_{i=1}^{\infty} |N(B_i)^{1/2}y|_{\mathscr{H}}^2\right)} \\ &= \sqrt{\nu_x(\mathscr{P})} + \sqrt{\nu_y(\mathscr{P})} , \end{aligned} \tag{7}$$

i.e., $\sqrt{\nu_{x+y}(\mathscr{P})} \leqq \sqrt{\nu_x(\mathscr{P})} + \sqrt{\nu_y(\mathscr{P})}$. Therefore, using the sequence $(\mathscr{P}_n)$ of (5) we have

$$\begin{aligned} \sqrt{\mu_{x+y}(B)} &\leqq \underline{\lim}_{n\to\infty} \sqrt{\nu_{x+y}(\mathscr{P}_n)} \\ &\leqq \underline{\lim}_{n\to\infty} \left(\sqrt{\nu_x(\mathscr{P}_n)} + \sqrt{\nu_y(\mathscr{P}_n)}\right) \end{aligned} \tag{8}$$

$$= \sqrt{\mu_x(B)} + \sqrt{\mu_y(B)} \quad \text{by (5)} .$$

So (4) holds.

We now prove condition (ii) holds, i.e., that for $x, y \in \mathscr{H}$,

$$(9) \qquad \mu_{x+y}(B) + \mu_{x-y}(B) = 2\mu_x(B) = 2\mu_y(B) .$$

Let $(\mathscr{P}_n^{x+y})$, $(\mathscr{P}_n^{x-y})$, $(\mathscr{P}_n^{x})$, $(\mathscr{P}_n^{y})$ be sequences of partitions of B such that $\mu_z(B) = \lim\downarrow_{n\to\infty} \nu_z(\mathscr{P}_n^z)$for $z = x + y$, $x - y$, x, and y. Let for each n, $\mathscr{P}_n = \mathscr{P}_n^{x+y} \circ \mathscr{P}_n^{x-y} \circ \mathscr{P}_n^{x} \circ \mathscr{P}_n^{y}$. Then $\mu_z(B) = \lim_{n\to\infty} \nu_z(\mathscr{P}_n)$ for $z = x + y$, $x - y$, x, and y. So (9) readily follows. Moreover for each n, $\nu_x(\mathscr{P}_n) \leqq (N(B)x, x)_{\mathscr{H}}$. So $\mu_x(B) \leqq (N(B)x, x)_{\mathscr{H}}$, and thus condition (iii) holds. Therefore by A.2 (b) there exists $H(B)$ such that (a) holds.

(b) Note that for $x, y \in \mathscr{H}$

$$(10) \qquad (H(B)x, y)_{\mathscr{H}} = \{\mu_{x+y}(B) - \mu_{x-y}(B) + i\mu_{x+iy}(B) - i\mu_{x-iy}(B)\}/4 ,$$

[since $\mu_z(B) = (H(B)z, z)_{\mathscr{H}}$ for $z \in \mathscr{H}$]. But by A.1 each $\mu_z(\cdot)$ is c.a. Thus $H(\cdot)$ is w.c.a. and therefore by [9, Th. 3.6.2] $H(\cdot)$ is s.c.a. The inequality $H(B) \lesssim N(B)$ follows immediately from $\mu_x(B) \leqq (N(B)x, x)_{\mathscr{H}}$. □

Our goal now is to obtain a matrix generalization of the following inequality.

LEMMA A.4. *For 2 real numbers $a, b > 0$*

$$(a^2 + b^2)/2 \geqq 4(a^{-1} + b^{-1})^{-2}$$

with equality holding $\Leftrightarrow a = b$.

Proof. This is equivalent to the fact that for $x, y > 0$ $(x + y)/2 \geqq 4xy(x^{1/2} + y^{1/2})^{-2}$ with equality holding $\Leftrightarrow x = y$, whose proof is contained in [18]. □

In the next two results we deal with linear operators on an n-dimensional Hilbert space over $\boldsymbol{F}$. We shall switch from an operator to its matrix representation (with respect to a given o.n. basis) as appropriate.

LEMMA A.5. *Let $A = [a_{ij}] > 0$ and let $A^{-1} = [c_{ij}]$. Then $c_{ii} \geqq 1/a_{ii}$ for $i = 1, \cdots, n$ with equality holding simultaneously for all $i \Leftrightarrow A =$* diag (a_{ii}). *Equivalently, $a_{ii} \geqq 1/c_{ii}$ for $i = 1, \cdots, n$ with equality for all $i \Leftrightarrow A =$* diag (a_{ii}).

Proof. We show the case $i = 1$: Let A be partitioned as

$$A = \left[\begin{array}{c|c} a_{11} & a_{12} \cdots a_{1n} \\ \hline a_{21} & \\ \vdots & D \\ a_{n1} & \end{array}\right] = \left[\begin{array}{c|c} a_{11} & a_1 \\ \hline a_1^* & D \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & a_1 D^{-1} \\ \hline 0 & I \end{array}\right] \cdot \left[\begin{array}{c|c} a_{11} - a_1 D^{-1} a_1^* & 0 \\ \hline 0 & D \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline D^{-1} a^* & I \end{array}\right] .$$

Then $\det A = \det D \cdot \det (a_{11} - a_1 D^{-1} a_1^*) = (\det D)(a_{11} - a_1 D^{-1} a_1^*)$, and cofactor $(a_{11}) = \det D$. Thus

$$c_{11} = \operatorname{cof}(a_{11})/\det A = 1/(a_{11} - a_1 D^{-1} a_1^*) \geqq 1/a_{11}$$

with equality holding $\Leftrightarrow a_1 = 0$, i.e., $\Leftrightarrow a_{12} = a_{13} = \cdots = a_{1n} = 0$. □

We now prove the generalization of A.4.

THEOREM A.6. *Let A, B be linear operators on an n-dimensional Hilbert space over F such that A, B are > 0. Then*

$$(1) \qquad \tau\{(A^2 + B^2)/2\} \geqq 4\tau\{(A^{-1} + B^{-1})^{-2}\}$$

with equality holding $\Leftrightarrow A = B$.

Proof. We first show that if $A = [a_{ij}]$ with respect to a given o.n. basis, then

$$(2) \qquad \tau(A^2) \geqq \sum_{n=1}^{n} a_{ii}^2 \text{ with equality holding} \Longleftrightarrow A = \operatorname{diag}(a_{ii}) .$$

[*Proof*: $\tau(A^2) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} a_{ji} = \sum_{i,j=1}^n |a_{ij}|^2 \geqq \sum_{i=1}^n a_{ii}^2$.]

Now choose an o.n. basis such that $A^{-1} + B^{-1}$ is diagonal, i.e., $A^{-1} = [c_{ij}]$, $B^{-1} = [d_{ij}]$, $A^{-1} + B^{-1} = \operatorname{diag}(c_{ii} + d_{ii})$, $c_{ij} = -d_{ij}$ for $i \neq j$. Then by (2) and Lemmas A.5 and A.4,

$$(3) \qquad \begin{aligned} \tau((A^2 + B^2)/2) &\geqq \sum_{i=1}^{n} (a_{ii}^2 + b_{ii}^2)/2 \geqq \sum_{i=1}^{n} ((c_{ii}^{-1})^2 + (d_{ii}^{-1})^2)/2 \\ &\geqq 4 \sum_{i=1}^{n} (c_{ii} + d_{ii})^{-2} = 4\tau\{(A^{-1} + B^{-1})^{-2}\} \end{aligned}$$

where the first 2 inequalities are each equality $\Leftrightarrow A = \operatorname{diag}(a_{ii})$, $B = \operatorname{diag}(b_{ii})$. The last inequality is equality $\Leftrightarrow c_{ii}^{-1} = d_{ii}^{-1}$, all i. We note that if $A = \operatorname{diag}(a_{ii})$ and $B = \operatorname{diag}(b_{ii})$, then $c_{ii}^{-1} = a_{ii}$ and $d_{ii}^{-1} = b_{ii}$. So equality holds in (1) $\Leftrightarrow A = \operatorname{diag}(a_{ii})$, $B = \operatorname{diag}(b_{ii})$ and $a_{ii} = b_{ii}$ for all i. Thus equality holds in (1) $\Rightarrow A = B$. Conversely, $A = B \Rightarrow$ equality holds in (1). □

The following inequality is crucial for proving uniqueness in Theorem 4.14.

COROLLARY A.7. *Let A, B be linear operators on an n-dimensional Hilbert space over $\boldsymbol{F}$ such that A, B are $\gtrsim 0$ and $\mathscr{R}(A) = \mathscr{R}(B)$. Then*

$$(*) \quad 4\tau\{(A^{1/2\#} + B^{1/2\#})^{2\#}\} \leqq (1/2)\tau(A + B) \quad (\# \text{ denotes generalized inverse})$$

with equality holding if and only if $A = B$.

Proof. This is an easy consequence of A.6. □

We conclude the appendix by giving the alternate proof of "$(\alpha) \Rightarrow (\beta)$" in Theorem 2.9 promised in footnote 7.

THEOREM A.8. $(\alpha) \Rightarrow (\beta)$ *in Theorem* 2.9.

Proof. Cf. (4.4) and (4.6). Define an inner product $(\ ,\ )_1$ on $S(W)$ by

$$(f, g)_1 = \int_\Omega (dHf, g)_W - (E_M(f), E_M(g))_{\mathscr{H}} ,$$

where we identify f and g if $|f - g|_1 = 0$. Denote by $\mathscr{H}_1$ the completion of $S(W)$ under $|\ \ |_1$ and define $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}_1$. Define $\forall B \in \mathscr{B}$, $\tilde{M}(B)$: $W \rightarrow \mathscr{K}$ by

$$\tilde{M}(B)w = M(B)w \oplus \chi_B w \in \mathscr{K} .$$

Then for $B, C \in \mathscr{B}$ and $w_1, w_2 \in W$ it follows that

$$(1) \quad \begin{cases} (\tilde{M}(B)w_1, \tilde{M}(C)w_2)_{\mathscr{K}} = (M(B)w_1 \oplus \chi_B w_1, M(C)w_2 \oplus \chi_C w_2)_{\mathscr{K}} \\ \quad = (M(B)w_1, M(C)w_2)_{\mathscr{H}} + \left\{\int (dH\chi_B w_1, \chi_C w_2)_W \right. \\ \qquad \left. - (E_M(\chi_B w_1), E_M(\chi_C w_2))_{\mathscr{H}}\right\} \\ \quad = (H(B \cap C)w_1, w_2)_W , \quad \text{i.e.,} \quad \tilde{M}(C)^*\tilde{M}(B) = H(B \cap C) , \end{cases}$$

i.e., by Definition 2.2 (a) $\tilde{M}(\cdot)$ is c.a.q.i. Clearly $J^*\tilde{M}(\cdot) = M(\cdot)$ for J defined as earlier after 2.9 (2). □

ACKNOWLEDGMENT. The author thanks the referee for his suggestions.

Added in proof. (1) The proof of Example 4.15 will appear as the article "Explicit structure of the 2-majorant of an operator-valued measure in a special case." To appear in the book "Prediction and Harmonic Analysis. The Pesi Masani Volume," edited by V. Mandrekar and H. Salehi, North-Holland Pub. Co.

(2) In Theorem 4.14, the proof that $E_M(f) = E_M(Pf)$ follows readily from John Von Neumann's Alternating Projection Theorem, which is Theorem 13.7 in his monograph "Functional Operators II," Annals of Math. Studies, No. 22, Princeton Univ. Press, Princeton, 1950.

(3) After submitting this paper, we found that fragments of the Equivalence Theorem are proved in the paper of Jose L. Abreu, "Transformation-valued measures," Advances in Math. 27 (1978), 1–11. Inadvertently, we also left out some relevant references to Niemi, which are stated in [14].

REFERENCES

1. J. L. Abreu, *A note on harmonizable and stationary sequences*, Bol. Soc. Mat. Mexicana, **15** (1970), 48–51.
2. J. L. Abreu, *H-valued generalized functions and orthogonally scattered measures*, Advances in Math., **19** (1976), 382–412.
3. W. N. Anderson, Jr., and R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl., **26** (1969), 576–594.
4. G. Bachman and L. Narici, *Functional Analysis*, Academic Press, 1966, New York.
5. H. Cramer, *On some classes of nonstationary stochastic processes*, Proc. IV Berkeley Symp. Math. Stat. Prob., **2** (1962), 57–78, Univ. Calif. Press.
6. N. Dunford and J. T. Schwartz, *Linear Operators, I, II*, Interscience-Wiley, 1958, 1963, New York.
7. A. Grothendieck, *Resume de la theorie metrique des produits tensoriels topologiques*, Bol. Soc. Matem. Sao Paulo, **8** (1956), 1–79.
8. P. R. Halmos, *Measure Theory*, Van Nostrand, 1950, Princeton.
9. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, A. M. S., (1956), Providence, R. I.
10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I*, Springer-Verlag, 1977, New York.
11. P. R. Masani, *Orthogonally-scattered measures*, Advances in Math., **2** (1968), 61–117.
12. ———, *Quasi-isometric measures and their applications*, Bull. Amer. Math. Soc., **76** (1970), 427–528.
13. ———, *Dilations as propagators of Hilbertian varieties*, SIAM J. Anal., **9** (1978), 414–456.
14. A. G. Miamee and H. Salehi, *Harmonizability, v-boundedness and stationary dilation of stochastic processes*, Indiana Univ. Math. J., **27** (1978), 37–50.
15. H. Niemi, *On orthogonally scattered dilations of bounded vector measures*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **3** (1977), 43–52.
16. ———, *On vector measures related to deterministic nonstationary processes*, Math. Ann., **229** (1977), 13–23.
17. A. Persson and A. Pietsch, *p-nukleare und p-integrale Abbildungen in Banachräumen*, Studia Math., **33** (1969), 19–62.
18. A. Pietsch, *p-majorisierbare vektorwertige Masse*, Wiss. Z. Friedrich-Schiller-Univ. Jena Math. Naturwiss. Reihe, **18** (1969), 243–247.
19. R. E. Rietz, *A proof of the Grothendieck inequality*, Israel J. Math., **19** (1974), 271–276.
20. R. Rogge, *Masse mit Werten in einem Hilbertraum*, Wiss. Z. Friedrich-Schiller-

Univ. Jena Math. Naturwiss. Reihe, **18** (1969), 253–257.
21. M. Rosenberg, *The square-integrability of matrix-valued functions with respect to a nonnegative hermitian measure*, Duke Math. J., **31** (1964), 291–298.
22. ———, *Spectral integrals of operator-valued functions from the study of multivariate stationary stochastic processes, I, II*, J. Multivariate Anal., **4** (1974), 166–209; **6** (1976), 538–571.
23. Yu. A. Rozanov, *Spectral analysis of abstract functions*, Theor. Prob. Appl., **4** (1959), 271–287.
24. A. Wilansky, *Functional Analysis*, Blaisdell Pub. Co., 1964, New York.
25. K. Yoshida, *Functional Analysis*, Academic Press, 1965, New York.

Received March 9, 1981 and in revised form July 1, 1981.

ST. JOHN'S UNIVERSITY
JAMAICA, NY 11439