PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

THE JACOBSON DESCENT THEOREM

DAVID J. WINTER

A direct proof of the Jacobson Descent Theorem is given and used to prove the Jacobson-Bourbaki Correspondence Theorem.

The purpose of this paper is to give a proof of the *Jacobson Descent Theorem*, Theorem 1, which is direct in that it does not assume that $A = \mathrm{Hom}_{K^A} K$. This is then used to prove the *Jacobson-Bourbaki Correspondence Theorem*, Theorem 2. The approach simplifies earlier proofs.

A variation of a theme of Hochschild appearing in Jacobson [2] and Winter [3] recurs here in the concentrated form of the dual bases x_i, R_j which thread their way through both proofs. Thus, this paper underlines the importance of this natural duality.

Throughout the paper, K denotes a field, End K denotes the ring of endomorphisms of K as additive group, A denotes a subring of End K containing the K-span KI of the identity I of End K and V denotes a vector space over K of finite or infinite dimension $V: K$. Regard A as left K-vector space in the obvious way.

DEFINITION 1. An *A-product* on V is a mapping $A \times V \to V$, denoted $(T, v) \to T(v)$, such that V is an A-module and

$$(xT)(v) = x(T(v)) \quad (x \in K, T \in A, v \in V). \qquad \square$$

Clearly $T(v)$ $(T \in A, v \in K)$ is an A-product for K.

Suppose henceforth that $T(v)$ $(T \in A, v \in V)$ is an A-product for V, and $V^A = \{v \in V \mid T(xv) = T(x)v \text{ for } T \in A, x \in K\}$. In particular, we have then defined K^A.

DEFINITION 2. For k a subfield of K, a *k-form* of V is a k-subspace V' of V whose k-bases are K-bases of V. $\square$

THEOREM 1 (*Jacobson* [1]). *Let* $A: K < \infty$, *then* V^A *is a* K^A-*form of* V.

Proof. $\hat{K} = \{\hat{x} \mid x \in K\}$ separates A and therefore contains a basis $\hat{x}_1, \ldots, \hat{x}_n$ for the K-dual space $\mathrm{Hom}_K(A, K)$ of A where $\hat{x} \in \mathrm{Hom}_K(A, K)$ is defined for $x \in K$ by $\hat{x}(T) = T(x)$ $(T \in A)$. Letting $R_1, \ldots, R_n$ be a dual basis for A, so that $R_i(x_j) = \delta_{ij}$ $(1 \le i, j \le n)$, we have $T(xR_i)(x_j) = T(x\delta_{ij}) = T(x)\delta_{ij} = (T(x)R_i)(x_j)$ $(1 \le i, j \le n)$ so that $T(xR_i) = T(x)R_i$ $(1 \le i \le n)$ for all T, since the x_j separate A.

Letting $v \in V$, we therefore have $T(xR_i(v)) = (T(xR_i))(v) = T(x)R_i(v)$ for all $T \in A$, $x \in V$, so that $R_i(V) \subset V^A$ and, in particular, $R_i(K) \subset K^A$ $(1 \leq i \leq n)$.

It follows that the K-span KV^A of V^A is V. For we have $I = \sum_1^n y_i R_i$ for suitable $y_i \in K$, so that $v = \sum_1^n y_i R_i v \in KV^A$ for all $v \in V$.

Finally, let v_i $(i \in I)$ be a K^A-basis for V^A. Suppose that $\sum_{i \in I} y_i v_i = 0$ with the y_i in K. Then $0 = \sum_{i \in I} R_j(y_i) v_i$ with the $R_j(y_i) \in K^A$, so $R_j(y_i) = 0$ $(1 \leq i, j \leq n)$ and $y_i = 0$ $(1 \leq i \leq n)$. □

THEOREM 2 (*Jacobson* [**2**]). *Let* $A: K < \infty$. *Then* $A = \operatorname{Hom}_{K^A} K$.

Proof. A as left A-module satisfies $(xS)T = x(ST)$ $(x \in K, S, T \in A)$, so that A^A is a K^A form of A and A^A contains a basis $R_1, \ldots, R_n$ for A over K. Choosing $x_i \in K$ so that $I = x_1 R_1 + \cdots + x_n R_n$, we have $x = \sum_{i=1}^n x_i R_i(x)$, $R_i(x) \in K^A$, for $x \in K$, so that $K: K^A \leq A: K \leq \operatorname{Hom}_{K^A} K: K \leq K: K^A$ and $A = \operatorname{Hom}_{K^A} K$. □

It is clear, in retrospect, that the above $x_1, \ldots, x_n$ form a basis for K over $K^A \overset{\text{def}}{=} k$ and that A^A is the dual space $\operatorname{Hom}_k(K, k) = K^*$ of K over k. The equations $x_j = \sum_{i=1}^n x_i R_i(x_j)$ show that $R_i(x_j) = \delta_{ij}$, that is, $R_1, \ldots, R_n$ is a dual basis for K^*. Finally, $I = x_1 R_1 + \cdots + x_n R_n$ shows that $T = \sum_1^n x_j R_j T$ and $T(x_i) = \sum_1^n x_j R_j T(x_i) = \sum_1^n (R_j \hat{x}_i(T)) x_j$. Thus, the $X_{ij} = R_j \hat{x}_i$ (composite) $(1 \leq i, j \leq n)$ are the coordinate functions on the $T \in A$ relative to the basis x_i. They form a basis for the k-dual space A^* of $A = \operatorname{Hom}_k K$. Since we may identify $\hat{K} = \{\hat{x} \mid x \in K\}$ with K, it follows that the k-dual space A^* of $A = \operatorname{Hom}_k K = KK^*$ can be identified with A whereby X_{ij} corresponds to $X_i R_j$—that is, we have a nondegenerate bilinear k-pairing $\langle \, , \rangle$ on $A \times A$ such that $\langle x_i R_j, T \rangle = R_j T(x_i)$. This pairing is also characterized by the condition $\langle xR, yS \rangle = R(y)S(x)$ $(x, y \in K, R, S \in K^*)$. Since $(x_i R_j)(x_r R_s) = x_i (R_j x_r R_s) = x_i R_j(x_r) R_s = x_i \delta_{jr} R_s$, the $E_{ij} = x_i R_j$ form a system of matrix units for A. We have $\langle E_{ij}, E_{rs} \rangle = \langle x_i R_j, x_r R_s \rangle = R_j(x_r) R_s(x_i) = \delta_{jr} \delta_{is} = \operatorname{Trace}(E_{ij} E_{rs})$. It follows that $\langle S, T \rangle = \operatorname{Trace} ST (S, T \in A)$.

REFERENCES

[1] Nathan Jacobson, *Forms of algebras*, Yeshiva Sci. Confs., 7 (1966), 41–71.

[2] _____, *Lectures in Abstract Algebra*, Vol. III Van Nostrand, Princeton (1964).

[3] David J. Winter, *Structures of Fields*, Graduate Texts in Mathematics No. 16, Springer-Verlag (1974).

Received May 8, 1981. This research was supported in part by the National Science Foundation.

UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109