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BASIC CALCULUS OF VARIATIONS

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For the classical one-dimensional problem in the calculus of variations, a necessary condition that the integral be lower semicontinuous is that the integrand be convex as a function of the derivative. We shall see that, if the problem is properly posed, then this condition is also necessary for the k-dimensional problem. For the one-dimensional problem this condition is also sufficient. For the k-dimensional problem this condition is shown to be sufficient subject to an additional hypothesis. For the one-dimensional problem there is an existence theorem if the integrand grows sufficiently rapidly with respect to the derivative, and this result also holds for the k-dimensional problem, subject to an additional hypothesis. Some of these additional hypotheses are automatically satisfied for the one-dimensional problem.

Let G be a bounded domain in $\mathbf{R}^k$, $A = G \times \mathbf{R}^N$, Z be the space of $(N \times k)$-matrices and $F \in C(A \times Z)$. If $y\colon G \to \mathbf{R}^N$ is smooth, let $I_F(y) = \int_G F(x, y(x), y'(x))\,dx$ where $y'(x)$ is the matrix of partial derivatives of y.

If $k = N = 2$ and if $F(a, b, p) = |\det p|$ then I_F is the area integral which is lower semicontinuous though F is not convex in p for fixed (a, b). Thus the one-dimensional results do not, apparently, generalize.

There are $r = \binom{N+k}{k} - 1$ Jacobians of orders $1, \ldots, \min\{k, N\}$. Let $Y = \mathbf{R}^r$. There exists $\tau\colon Z \to Y$ such that $\tau \circ y'(x) = J(y, x)$, where $J(y, x) = [J(y)](x)$, and $J(y)$ is the collection of all Jacobians of y, whenever y is a smooth map. If $f\colon A \times Y \to \mathbf{R}$ and if $f(\theta, \tau(p)) = F(\theta, p)$ for all (θ, p), then, evidently, $I(y) = I_F(y)$ where $I(y) = \int_G f(y_*(x), J(y, x))\,dx$ and $y_*(x) = (x, y(x))$.

If $u\colon V \times W \to X$ and if $v \in V$ let $u_v(w) = u(v, w)$ for each $w \in W$.

We define a class AC of transformations y for which each component of y and each component of $J(y)$, defined in a distribution sense, is in $L = L(G)$. We consider $I(y)$ to be the basic integral, not $I_F(y)$.

Let $T = \operatorname{range} \tau$. If $k = 1$ then $T = Y$ and T can be identified with Z so that $f = F$. In general, however, setting $f_\theta \circ \tau = F_\theta$ defines f_θ on $T \subset Y$ where $T \neq Y$. Let us say that f is *T-convex* if f_θ can be extended to a function which is convex over all of Y for each $\theta \in A$. Please notice that we do *not* require that f_θ be convex. What we do require is that there exist a convex function over all of Y which extends f_θ. Then a necessary condition that I be lower semicontinuous is that f be T-convex. If the extended function is also continuous over $A \times Y$, then the condition is also sufficient.

In some applications f, rather than F, may be given initially [**1**].

If $k > 1$ then the parametric problem is not covered by the existence theorem. Even worse, the dichotomy into parametric and non-parametric problems no longer seems feasible. If $k = N = 2$ and if $F(\theta, p) = |\det p|^2$ then I is not parametric. Since it is invariant under smooth area-preserving changes of variables, it has something of the distinguishing feature of parametric integrals. Here $r = 5$ and $f_\theta(t)$ depends upon a single component of t. Thus f_θ does *not* grow with $\|t\|$.

The starting point of this paper is **[5]**. Morrey's sufficiency condition for quasiconvexity gave the idea of using f rather than F. That idea, together with the notion of the Cesari-Weierstrass integral **[2]** and the ideas used in **[7]** and **[8]** led to the sufficient condition. The compactness results are familiar **[6]**. The consistent use of quasilinear functions to approximate continuous functions, rather than Lipschitzian or smoother functions, is standard in area theory, especially in Cesari's papers.

2. If y is smooth then each component of $J(y)$ is the determinant of a submatrix of order k of y'_*, except possibly for sign. One of these submatrices is the identity. Its determinant does not correspond to any component of $J(y)$. Thus $J(y)$ has r components. Let $Y = \mathbf{R}^r$.

If $M \geq m$ let $\Lambda(M, m)$ be the collection of all strictly increasing m-termed sequences taken from $\{1,\ldots,M\}$. Let $s = \min\{k, N\}$. If $j \leq s$, if $i \in \Lambda(N, j)$ and $\alpha \in \Lambda(k, j)$, let $p^i_\alpha = \det[(p_{\alpha_m})^{i_n}]_{1\leq m,n\leq j}$ and define τ: $Z \to Y$ by $\tau(p) = \{p^i_\alpha \mid (i, \alpha) \in \bigcup_{j=1}^s(\Lambda(N, j) \times \Lambda(k, j))\}$. We may write $[{}^I_p]$ for $\tau(p)$. Similarly, if ϕ is a $(k \times k)$-matrix then the determinants of the $(k \times k)$-submatrices of $[{}^\phi_p]$ are in 1-1 correspondence with those of $[{}^I_p]$. (We delete the determinant of the top matrix, of course.)

Evidently there exists a unique linear map ϕ: $Y \to Z$ such that $\Psi \circ \tau(p) = p$ for each $p \in Z$.

If $(i, \alpha) \in \bigcup_{j=1}^s(\Lambda(M, j) \times \Lambda(k, j))$ then there exists λ, $1 \leq \lambda \leq r$, such that

$$\frac{\partial(y^{i_1},\ldots,y^{i_j})}{\partial(x^{\alpha_1},\ldots,x^{\alpha_j})} = \frac{dy^i}{dx^\alpha} = \pm\tau(y')^\lambda.$$

We can suppose that, if $N \geq k$ and $j = s = k$, then $r_0 = \binom{N+k}{k} - \binom{N}{k} \leq \lambda \leq r$.

The components of $J(y)$ are, except possibly for sign, the components of $\tau(y')$. Thus there is no loss in generality in ordering the rows of the submatrices in such a way that we can identify $J(y)$ with $\tau(y')$.

3. To obtain the necessary condition for lower semicontinuity we require some information about τ.

LEMMA 3.1. *Let* $\mu_n \in \mathbf{R}$, $n = 1,\dots,m$, *with* $\Sigma\mu_n = 1$. *If* p_n, p *and* $q \in Z$ *with* $\Sigma\mu_n\tau(p_n) = \tau(p)$ *then* $\Sigma\mu_n(p_n + q)^1 \wedge \cdots \wedge(p_n + q)^j = (p + q)^1 \wedge \cdots \wedge(p + q)^j$ *for* $j = 1,\dots,k$.

Proof. We expand and get $(p + q)^1 \wedge \cdots \wedge(p + q)^j = p^1 \wedge \cdots \wedge p^j + \Sigma_{i=1}^{j-1}\Sigma'\varepsilon_{\alpha,i}p^{\alpha_1} \wedge \cdots \wedge p^{\alpha_i}q^{\gamma_1} \wedge \cdots \wedge q^{\gamma_{j-i}} + q^1 \wedge \cdots \wedge q^j$ where Σ' is the sum over $\alpha \in \Lambda(j,i)$ and $\gamma \in (1,\dots,j) \sim \{\alpha\}$. Also, $\varepsilon_{\alpha,i} = \pm 1$. Then

$$\sum \mu_n(p_n + q)^1 \wedge \cdots \wedge (p_n + q)^j$$

$$= p^1 \wedge \cdots \wedge p^j + \sum_{n=1}^{m} \mu_n \sum_{i=1}^{j-1} \sum{}'\varepsilon_{\alpha,i}p_n^{\alpha_1} \wedge \cdots \wedge p_n^{\alpha_i} \wedge q^{\gamma_1} \wedge \cdots \wedge q^{\gamma_{j-i}}$$

$$+ q^1 \wedge \cdots \wedge q^j = (p + q)^1 \wedge \cdots \wedge (p + q)^j.$$

COROLLARY 3.2. $\tau(p + q) = \Sigma\mu_n\tau(p_n + q)$.

LEMMA 3.3. *Let* $y\colon \mathbf{R}^k \to \mathbf{R}^N$ *be quasilinear with compact support* K *and simplexes of linearity* $\delta_1,\dots,\delta_m$. *Let* $p_n = y'(x)$ *for* $x \in \operatorname{Int}\delta_n$ *and let* $\mu_n = |\delta_n|/|K|$. *Then* $\mu_n > 0$, $\Sigma\mu_n = 1$ *and* $\Sigma\mu_n\tau(p_n) = 0$.

Except for notation, this is Lemma 4.4 [**6**].

It is not hard to verify that Y is the convex hull of T.

Let us say that I is lsc if $I(y) \le \liminf I(y_n)$ whenever y_n converges uniformly to y, y_n and y satisfy a uniform Lipschitz condition (which may depend upon the sequence) and $y_n - y$ is quasilinear with support contained in a cube contained in G. (See Def. 4.4.2, [**6**].)

If $N \ge k$ and if $f(\theta, q) = f(\theta,(0,\dots,0,q^{r_0},\dots,q^r))$ for each $\theta \in A$ then we say that f *depends only upon Jacobians of maximum rank.*

LEMMA 3.4. *Let* f *depend only upon Jacobians of maximum rank and suppose that* $f_\theta \in C'$ *for each* $\theta \in A$. *If* I *is lsc then then* f *is* T-*convex.*

Proof. If $f_\theta(\tau(p)) \le \Sigma\lambda_\beta f_\theta(\tau(p_\beta))$ whenever $\theta \in A, p, p_\beta \in Z, \lambda_\beta > 0$, $\Sigma\lambda_\beta = 1$ and $\Sigma\lambda_\beta\tau(p_\beta) = \tau(p)$, then $t \mapsto \inf\{\Sigma\lambda_\beta\tau(p_\beta) \mid \Sigma\lambda_\beta\tau(p_\beta) = t\}$ is an extension of the required type. If

$$f_\theta(\tau(q)) \ge f_\theta(\tau(p)) + f_\theta'(\tau(p))\tau(q - p)$$

for all $\theta \in A, p$ and $q \in Z$, then by Corollary 3.2, $\Sigma\lambda_\beta\tau(p_\beta - p) = \tau(0)$ so $\Sigma\lambda_\beta f_\theta(\tau(p_\beta)) \ge \Sigma\lambda_\beta f_\theta(\tau(p)) + f_\theta'(\tau(p))\Sigma\lambda_\beta\tau(p_\beta - p) = \Sigma\lambda_\beta f_\theta(\tau(p)) = f_\theta(\tau(p))$.

Let $Q = \mathbf{R}^k \cap \{x \mid -\frac{1}{2} \le x^1, \ldots, x^k \le \frac{1}{2}\}$ and let $h > 0$. Let $p \in Z$. Then Q is partitioned into 3^k cells by the hyperplanes $x^\alpha = \pm h/2$, $\alpha = 1, \ldots, k$. Each of these cells, except hQ, is then subdivided into $k!$ simplexes whose vertices are contained in the set of vertices of the containing cell. Let S be the set of all these simplexes. Now we define α continuous (quasilinear) function ζ on Q into $\mathbf{R}^N$ by putting $\zeta(x) = px$ if $x \in hQ$, $\zeta(x) = 0$ if $x \in \partial Q$ and $\zeta \mid \sigma$ is linear (affine) if $\sigma \in S$. If $x \in \operatorname{Int} \sigma$ let $\zeta'(x) = p_\sigma$. Thus, by Lemma 3.3, $\tau(p)h^k + \Sigma_{\sigma \in S} \tau(p_\sigma) \mid \sigma \mid = 0$. Also, for each $\sigma \in S$ there exists $j \in \{1, \ldots, k\}$ such that j columns of p_σ are $O(h)$ and $\mid \sigma \mid = O(h^{k-j})$. By Theorem 4.4.2 [6],

$$\begin{aligned} f_\theta(\tau(0)) &\le \int_Q f_\theta(\tau(\zeta'(x)))\, dx = f_\theta(\tau(p))h^k + \sum_{\sigma \in s} f_\theta(\tau(p_\sigma)) \mid \sigma \mid \\ &= f_\theta(\tau(p))h^k + \sum_{\sigma \in s} [f_\theta(\tau(0)) + f_\theta'(\tau(0))\tau(p_\sigma) + o(\tau(p_\sigma))] \mid \sigma \mid \\ &= f_\theta(\tau(p))h^k + f_\theta(\tau(0))(1 - h^k) - f_\theta'(\tau(0))\tau(p)h^k \\ &\quad + \sum_{\sigma \in s} o(\tau(p_\sigma)) \mid \sigma \mid \end{aligned}$$

so that $f_\theta(\tau(0))h^k + f_\theta'(\tau(0))\tau(p)h^k \le f_\theta(\tau(p))h^k + \Sigma_{\sigma \in S} O(\tau(p_\sigma)) \mid \sigma \mid$. If f depends only upon Jacobians of rank k, then the last term on the right is $o(O(h^k)) = o(h^k)$ so that $f_\theta(\tau(p)) \ge f_\theta(\tau(0)) + f_\theta'(\tau(0))\tau(p)$.

Corollary 3.5. *The lemma remains valid if the differentiability condition is dropped.*

Proof. Let $F_\theta = f_\theta \circ \tau$ and suppose that $F_\theta \in C'$. Then $f_\theta = F_\theta \circ \Psi$, $f_\theta' = (F_\theta' \circ \Psi)\Psi'$ and $f_\theta \in C'$. If $F_\theta \notin C'$ we mollify. Let B be the unit sphere in Z, let $\mu \in C^\infty(Z)$ be nonnegative with support contained in B and $\int \mu(\xi)\, d\xi = 1$. If $\rho > 0$ let $\mu_\rho(\xi) = 1/\rho^{Nk}\mu(\xi/p)$.

If $y_n \to y$ then $y_n - \xi \to y - \xi$ where, because of the definition of lsc, we can suppose that $y_n - \xi$ and $y - \xi$ differ only on a compact subset of G. A routine argument shows that $y \mapsto \int_G F(y_*(x), y'(x) - \xi)\, dx$ is lsc. Thus

$$y \mapsto \int_R F_\rho\big(y_*(x), y'(x)\big)\, dx$$

is lsc where $F_\rho(\theta, p) = \int_{\rho B} F((\theta, p - \xi)\mu_\rho \mid \xi)\, d\xi$. Let $f_\rho(\theta, q) = F_\rho(\theta, \Psi q)$. Then $(f_\rho)_\theta \in C'$ since $(F_\rho)_\theta \in C'$. Thus, by the lemma, f_ρ is T-convex and the corollary follows by letting $\rho \to 0$.

Theorem 3.6. *Let I be lsc. Then f is T-convex.*

Proof. If $\theta \in A$ let $g(\theta, [{}^{\phi}_{p}]) = g_\theta([{}^{\phi}_{p}]) = f_\theta([{}^{I}_{p}])$. (See §2.) Now let

$$h\left(\theta, \begin{bmatrix} I \\ \phi \\ p \end{bmatrix}\right) = g\left(\theta, \begin{bmatrix} \phi \\ p \end{bmatrix}\right).$$

Let Z_0, Y_0 and Ψ_0 correspond to Z, Y and Ψ with $\mathbf{R}^{N+k}$ replacing $\mathbf{R}^N$. Let h_θ be defined over all of Y_0 by $h_\theta(q) = h_\theta(r)$ if $\Psi_0 q = \Psi_0 r$. By this construction $h \in C(A \times Y_0)$, h is nonnegative and h depends only upon Jacobians of maximum rank.

If $(\xi, y)\colon G \to \mathbf{R}^k \times \mathbf{R}^N$ then let

$$I_h(\xi, y) = \int_G h\left(y_*(x), \begin{bmatrix} I \\ \xi'(x) \\ y'(x) \end{bmatrix}\right) dx$$

$$= \int_G g\left(y_*(x), \begin{bmatrix} \xi'(x) \\ y'(x) \end{bmatrix}\right) dx = I(y)$$

and I_h is lsc. Thus h is T-convex. In a natural way $Y = \operatorname{dom} f_\theta \subset \operatorname{dom} h_\theta$. Furthermore, h_θ extends $f_\theta \mid T$. Thus $g_\theta = h_\theta \mid Y$ is an extension of $f_\theta \mid T$ which is convex over all of Y.

4. In this section we define a class of transformations, which we call AC, on which I is defined. This class is probably not a vector space.

Let $\mathfrak{D} = C_o^\infty(G)$, $L = L_1(G)$ and $L_p = L_p(G)$ for $p > 1$. If B is one of these spaces let $F_0 B = B$, $F_j B = 0$ if $j > k$ and, if $1 \le j \le k$, let

$$F_j B = \left\{\omega \mid \omega = \sum_{\lambda \in \Lambda(k,j)} \omega_\lambda \, dx^\lambda \text{ where each } \omega_\lambda \in B\right\}.$$

As usual, $dx^\lambda = dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_j}$.

If $\omega \in F_j L$ and if there exists $\zeta \in F_{j+1} L$ such that

$$\int \omega \wedge d\phi = (-1)^{j+1} \int \zeta \wedge \phi$$

for each $\phi \in F_{k-j-1}\mathfrak{D}$, then we say that $\omega \in \mathfrak{F}_j H$ and write $d\omega$ for ζ. If $d\omega$ exists, then $d\omega$ is unique.

By putting an appropriate norm on $\mathfrak{F}_o H$ we can identify this space with $H = H_1^1(G)$. Also, $H_o = H_{1,o}^1(G)$ is the closure, in H, of $\mathfrak{D} = \mathfrak{F}_o \mathfrak{D}$.

If $\omega_n = \Sigma \omega_{n\lambda} \, dx^\lambda$ and $\omega = \Sigma \omega_\lambda \, dx^\lambda$ are in $F_j L$ then $\omega_n \rightharpoonup \omega$ in $F_j L$ if $\omega_{n\lambda} \rightharpoonup \omega_\lambda$ in L for each λ, where $\rightharpoonup$ denotes weak convergence on compact subsets of G.

LEMMA 4.1. *If $\omega_n \rightharpoonup \omega$ in $F_j L$, if $\omega_n \in \mathfrak{F}_j H$ and if $d\omega_n \rightharpoonup \zeta$ in $F_{j+1} L$ then $\omega \in \mathfrak{F}_j H$ and $d\omega = \zeta$.*

Proof. Let $\phi \in F_{k-j-1}\mathfrak{D}$. Then

$$\int \omega \wedge d\phi = \lim \int \omega_n \wedge d\phi = (-1)^{j+1} \lim \int d\omega_n \wedge \phi = (-1)^{j+1} \int \zeta \wedge \phi.$$

LEMMA 4.2. *If* $\omega \in \mathfrak{F}_j H$ *then* $x^\alpha \omega \in \mathfrak{F}_j H$ *and*

$$d(x^\alpha \omega) = dx^\alpha \wedge \omega + x^\alpha d\omega.$$

Proof. Let $\phi \in F_{k-j-1}\mathfrak{D}$ and $\psi = x^\alpha \phi$ so that $d\psi = dx^\alpha \wedge \phi + x^\alpha d\phi$ and

$$\begin{aligned}\int x^\alpha \omega \wedge d\phi &= \int \omega \wedge [d\psi - dx^\alpha \wedge \phi] \\ &= \int \omega \wedge d\psi + (-1)^{j+1} \int dx^\alpha \wedge \omega \wedge \phi \\ &= (-1)^{j+1} \int (x^\alpha d\omega + dx^\alpha \wedge \omega) \wedge \phi.\end{aligned}$$

LEMMA 4.3. *If* $\omega \in \mathfrak{F}_j H$ *then* $d^2\omega = 0$.

Proof. Let $\zeta = d\omega$ and $\phi \in F_{k-j-2}\mathfrak{D}$. Then $\int \zeta \wedge d\phi = (-1)^j \int \omega \wedge d^2\phi = 0 = (-1)^j \int 0$ so that $d^2\omega = d\zeta = 0$.

If $z \in H$ then $dz = \sum_{\alpha \in \Lambda(k,1)} z_\alpha \, dx^\alpha$ where $\{z_\alpha\}$ is the set of distribution derivatives of z. Let M be a positive integer and $s = \min\{k, M\}$. Suppose that dz^i has been defined for $i \in \Lambda(M, j)$, $j \leq s - 1$. If $h \in \Lambda(M, j + 1)$, $m = h_1$ and $i = h \sim \{m\} \in \Lambda(M, j)$ then we define dz^h, if $z^m dz^i \in \mathfrak{F}_j H$, by $dz^h = d(z^m dz^i)$.

If dz^i is defined for $i \in \Lambda(M, j)$ and $\alpha \in \Lambda(k, j)$ then we define z^i_α by

$$dz^i = \sum_{\alpha \in \Lambda(k, j)} z^i_\alpha \, dx^\alpha$$

so that, if z is smooth, $z^i_\alpha = (\partial(z^{i_1}, \ldots, z^{i_j}) / \partial(x^{\alpha_1}, \ldots, x^{\alpha_j}))$.

Let $y \in L^N$ and suppose that dy^i is defined for each $i \in \Lambda(M, s)$, where $s = \min\{N, k\}$, and thus for each $i \in \bigcup_{j=1}^s \Lambda(M, j)$. Then we can suppose that $J(y) = \{y^i_\alpha \mid (i, \alpha) \in \bigcup_{j=1}^s (\Lambda(N, j) \times \Lambda(k, j))\}$ is an element of L^r.

If $J(y)$ is defined and if $J(y) = \tau(y')$ almost everywhere then we say that $y \in AC$. By the definition of $\mathfrak{F}_j H$, the components of $J(y)$ are functions.

The following lemmas are immediate.

LEMMA 4.4. *$y_* \in AC$ if and only if $y \in AC$ and $J(y) = \{y^i_{*\beta} \mid i \in \Lambda(k+N, j)$ and $\beta = (1,\ldots,k)\}$.*

LEMMA 4.5. *Let $j \leq s = \min\{N, k\}$ and $y \in AC$. If $(i, \alpha) \in \Lambda(N, j) \times \Lambda(k, j)$ for $1 \leq j \leq s$ then there exists $h \in \Lambda(k+N, k)$ such that, except possibly for sign, $y^h_{*\beta} = y^i_\alpha$.*

Let $y_n \in AC$ and $y \in L^N$ with $y^m_{n*} \to y^m_*$ in L for each $m \in \Lambda(k+N, 1)$. Suppose that if $j \leq k$ and $i \in \Lambda(k+N, j)$ there exists $\zeta^i \in F_jL$ such that $dy^i_{n*} \rightharpoonup \zeta^i$ in F_jL. If, in addition, $y^m_{n*}\, dy^i_{n*} \rightharpoonup y^m_*\zeta^i$ in F_jL whenever $i \in \Lambda(k+N, j)$, $j < k$, $m \in \Lambda(k+N, 1)$, and $m \notin i$ then we say that $y_n \Rightarrow y$.

THEOREM 4.6. *If $y_n \Rightarrow y$ then $y \in AC$ and $J(y_n) \rightharpoonup J(y)$ in L.*

Proof. By Lemma 4.1, $J(y)$ is defined. By Theorem 3.4.4 **[6]**, $y^m_{n*}\, dy^i_{n*} \rightharpoonup y^m_* dy^i_*$ in $L(K)$ for each compact set $K \subset G$. Hence we can suppose that $y^m_{n*}\, dy^i_{n*} \to y^m_* dy^i_*$ almost everywhere in G. We can also suppose that $i \neq (1, 2, \ldots, k)$. Hence there exists $m \in \{1,\ldots,k\}$, $m \notin i$, such that $x^m dy^i_{n*} \to x^m dy^i_*$ so that $dy^i_{n*} \to dy^i_*$ almost everywhere.

LEMMA 4.7. *If p and q are Lebesgue conjugate, if $f_n \to f$ in L_p and $g_n \rightharpoonup g$ in L_q then $f_n g_n \to fg$ in L.*

Proof. Let E be a measurable subset of a compact subset of G. Then

$$\int_E (f_n g_n - f_g)\, dx = A_n + B_n$$

where $A_n = \int_E f(g_n - g)\, dx$ and $B_n = \int_E (f_n - f) g_n\, dx$. By the weak convergence, $A_n \to 0$ and $\{\int_E |g_n(x)|^q\, dx\}^{1/q}$ is bounded independently of n. Thus $B_n \to 0$ by the Hölder inequality.

If $y \in AC$ and if $y^i_{*\beta} \in L_p$ for each $i \in \Lambda(k \times N, k)$, where $\beta = (1,\ldots,k)$, then we set $\|J(y)\|_p = \sum_{i \in \Lambda(k \times N, k)} \|y^i_{*\beta}\|_p$.

If $y_o \in AC$ let $\mathfrak{M}(y_o) = AC \cap \{y \mid y - y_o \in (H_o)^N\}$.

THEOREM 4.8. *Suppose that there exists $M > 0$ such that for each $y \in \mathfrak{M}(y_o)$ either*

(i) $\|y\|_\infty \leq M$ and $\|J(y)\|_p \leq M$ for some $p > 1$, or

(ii) $\|J(y)\|_q \leq M$ where $q = 2k/(k+1)$. Then $\mathfrak{M}(y_o)$ is $\Rightarrow$ sequentially compact.

Proof. If (i) holds then $\|y\|^1_1$ is uniformly bounded so that there exists a sequence $\{y_n\}$ in $\mathfrak{M}(y_o)$ and $\zeta \in (H_o)^N$ such that $y_n - y_o \rightharpoonup \zeta$ in $(H_o)^N$.

Thus $y_n - y_o \to \zeta$ in L. Let $y = y_o + \zeta$. By passing to a subsequence we can suppose that $y_n(x) \to y(x)$ a.e. By the bounded convergence theorem, $y_{n*} \to y_*$ in $(L_s)^N$ where $s = p/(p-1)$ is Lebesgue conjugate to p. If (ii) holds then there exists a sequence $\{y_n\}$ in $\mathfrak{M}(y_o)$ and $\zeta \in (H_{q,o})^N$ such that $y_n - y_o \to \zeta$ in $(H_{p,o})^N$. Thus, by Th. 3.5.3, **[6]**, $y_n \to y$ in L_t where $1/t = 1/q - 1/k = (k-1)/2k$ so that t is conjugate to q. The theorem follows by induction, Lemma 4.1 and Lemma 4.7.

5. We make use of a type of convexity studied by Tonelli to show that T-convexity is sufficient for lower semicontinuity.

According to Tonelli, a T-convex function f is semi-regular positive semi-normal if for each $\theta \in A$, $p, q \in Y$ with $q \neq 0$, there exists $\lambda \in \mathbf{R}$ such that $2f(\theta, p) < f(\theta, p + \lambda q) + f(\theta, p - \lambda q)$.

For the following lemma see Turner **[10]**.

LEMMA 5.1. *A necessary and sufficient condition that f be semi-regular positive semi-normal is that for each $\varepsilon > 0$ and each $(\theta, p) \in A \times Y$, there exists $\delta > 0$, $\nu > 0$, $\zeta \in Y^*$ and $\rho \in \mathbf{R}$ such that for all $\phi \in A$ with $\|\phi - \theta\| < \delta$,*

(a) *$f(\phi, q) \geq \zeta q + \rho + \nu \|q - p\|$ for each $q \in Y$ and*

(b) *$f(\phi, q) \leq \zeta q + \rho + \varepsilon$ if $\|q - p\| < \delta$.*

Let f be semi-regular positive. If $\zeta \in Y^*$ let

$$\rho_\zeta(\theta) = \inf\{f(\theta, q) - \zeta q \mid q \in Y\}$$

for each $\theta \in A$. Thus $f(\theta, p) = \sup\{\zeta p + \rho_\zeta(\theta) \mid \zeta \in Y^*\}$.

Let $\sigma_\zeta(\phi) = \liminf_{\theta \to \phi} \rho_\zeta(\theta)$ where θ and ϕ belong to A, of course. Then ρ_ζ is upper semicontinuous, σ_ζ is lower semicontinuous and $\sigma_\zeta \leq \rho_\zeta$.

THEOREM 5.2. *If f is semi-regular positive semi-normal, then $f(\theta, p) = \sup\{\zeta p + \sigma_\zeta(\theta) \mid \zeta \in Y^*\}$.*

Proof. Let $\varepsilon > 0$. By Lemma 5.1 there exist $\delta > 0$, $\nu > 0$, $\zeta \in Y^*$ and $\rho \in \mathbf{R}$ such that if $\phi \in A$ and $\|\phi - \theta\| < \delta$, then

(a) $f(\phi, q) \geq \zeta q + \rho + \|q - p\|$ for each $q \in Y$, and

(b) $f(\phi, q) \leq \zeta q + \rho + \varepsilon$ if $\|q - p\| < \delta$.

Hence $\rho_\zeta(\phi) \geq \rho$ for each $\phi \in A$ with $\|\phi - \theta\| < \delta$ so that $\sigma_\zeta(\theta) \geq \rho$ and $f(\theta, p) \leq \zeta p + \sigma_\zeta(\theta) + \varepsilon$.

We say that f is V-convex if $f(\theta, p) = \sup\{\zeta p + \sigma_\zeta(\theta) \mid \zeta \in Y^*\}$ for each $\theta \in A$. Thus f is V-convex if f is semi-regular positive semi-normal.

6. In this section we show that if $f \in C(A \times Y)$ is nonnegative and T-convex, then I is lower semicontinuous.

Let $\{e^\lambda\}$ be a dual basis for $Y^* = e^\lambda e_\mu = \delta_\mu^\lambda$ for $e_\mu \in Y$. If $\zeta \in Y^*$ there exist $\zeta_\lambda \in \mathbf{R}$ such that $\zeta = \Sigma \zeta_\lambda e^\lambda$.

Let $\mathcal{S}$ be the collection of all finite families σ of compact subsets contained in G such that if $K \in \sigma$ and $L \in \sigma$, $|K \cap L| = 0$ whenever $K \neq L$.

If $y \in AC$, $\zeta \in Y^*$ and K is a compact subset of G, let $A(\zeta, y, K) = \zeta(\int_K J(y, x)\,dx) = \int_K \zeta(J(y, x))\,dx$ and

$$B(\zeta, y, K) = \left(\inf\left\{\sigma_\zeta\left(y_*(x)\right)\right\} \mid x \in K\right) |K| .$$

Now we define $\mathcal{J}$ on AC by

$$\mathcal{J}(y) = \sup_{\sigma \in S} \sum_{K \in \sigma} \sup_{\zeta \in Y^*} \left[A(\zeta, y, K) + B(\zeta, y, K)\right].$$

LEMMA 6.1. *Let y_n and y_o belong to AC with $y_n - y_o \in (H_o)^N$. If $y_n - y_o \rightharpoonup \zeta$ in H^N and if we set $y = y_o + \zeta$ then $y - y_o \in (H_o)^N$ and $y_n \to y$ in $(L_1(K))^N$ for each compact subset K of G.*

This lemma follows from Theorems 3.2.1 and 3.4.4 [**6**].

LEMMA 6.2. *Let X be a measurable subset of G and $\{f_n\}$ be a sequence of measurable functions with $f_n(x) \to f(x)$ a.e. in X. Let $\varepsilon > 0$. Then there exists a compact set $K \subset X$ with $|X \sim K| < \varepsilon$, $f_n | K$ continuous for each n and $f_n | K \to f | K$ uniformly.*

This lemma follows from Egoroff's Theorem and Lusin's Theorem.

THEOREM 6.3. *Let f be V-convex and suppose that y_n and y are in $\mathfrak{M}(y_o)$. If $(y_n, J(y_n)) \rightharpoonup (y, J(y))$ in $L^N \times L^r$ then $\mathcal{J}(y) \leq \liminf \mathcal{J}(y_n)$.*

Proof. Let K be a compact subset of G. By Lemma 6.1 we can suppose that $y_n \to y$ in $L(K)^N$ so that (passing to a subsequence if necessary) $y_n(x) \to y(x)$ for almost all $x \in K$. Let $M > 0$, $\sigma_\zeta^M(\theta) = \min\{\sigma_\zeta(\theta), M\}$ and let $f^M(\theta, p) = \sup\{\zeta p + \sigma_\zeta^M(\theta) \mid \zeta \in Y^*\}$. It is sufficient to show that the theorem holds with f replaced by f^M. Hence we can suppose that $\sigma_\zeta(\theta) \leq M$ for all $(\theta, \zeta) \in A \times Y^*$. Let $\varepsilon > 0$. There exists $\eta \in (0, \varepsilon/M)$ such that $\int_E \zeta(J(y_*(x)))\,dx < \varepsilon$ if E is a measurable subset of K with $|E| < \eta$. By Lemma 6.2 there exists a compact set $C \subset K$ such that $|K \sim C| < \eta$, $y_n | C$ is continuous and $y_n \to y$ uniformly on C. Hence

$$\begin{aligned} B(\zeta, y, C) &= \left(\inf_{x \in C} \sigma_\zeta\left(y_*(x)\right)\right) |C| \\ &\geq \left(\inf_{x \in K} \sigma_\zeta\left(y_*(x)\right)\right) |C| \geq B(\zeta, y, K) - \varepsilon. \end{aligned}$$

Also there exist $x_n \in C$ such that $\sigma_\zeta(y_{n*}(x_n)) = \inf_{x \in C} \sigma_\zeta(y_{n*}(x))$. We can suppose that $x_n \to x \in C$. Now $y_n(x_n) \to y(x)$ so that $\sigma_\zeta(y_*(x)) \leq \liminf \sigma_\zeta(y_{n*}(x_n))$. Thus $B(\zeta, y, C) \leq \liminf B(\zeta, y_n, C)$ while $A(\zeta, y, C) = \lim A(\zeta, y_n, C)$. The theorem follows.

THEOREM 6.4. *Let f be V-convex. If $y \in AC$ then $\mathcal{I}(y) = I(y)$.*

Proof. Let K be a compact subset of G and $\zeta \in Y^*$. Then

$$\int_K f(y_*(x), J(y, x))\, dx$$
$$\geq \int_K \left[\zeta(J(y, x)) + \sigma_\zeta(y_*(x))\right] dx \geq A(\zeta, y, K) + B(\zeta, y, K)$$

so that $I(y) \geq \mathcal{I}(y)$ and we can suppose that $\mathcal{I}(y) < \infty$. If L is an interval contained in G let $\mathcal{S}_L = \mathcal{S} \cap \{\sigma \mid \bigcup_{K \in \sigma} K \subset L\}$ and let

$$\Phi(L) = \sup_{\sigma \in \mathcal{S}_L} \sum_{K \in \sigma} \sup_{\zeta \in Y^*} [A(\zeta, y, K) + B(\zeta, y, K)].$$

Then Φ is nonnegative, superadditive and of bounded variation. Let $D\Phi$ be the Lebesgue derivative of Φ with respect to cubes. Then $D\Phi(x) \geq \zeta(J(y, x)) + \sigma_\zeta(y_*(x))$ so that $D\Phi(x) \geq f(y_*(x), J(y, x))$ almost everywhere in G. Evidently $\mathcal{I}(y) \geq \sup_{\sigma \in \mathcal{S}'} \Sigma_{L \in \sigma} \Phi(E)$ where $\mathcal{S}' = \mathcal{S} \cap \{\sigma \mid \sigma$ is a family of finitely many non-overlapping intervals$\}$. Thus $\mathcal{I}(y) \geq \sup_{\sigma \in \mathcal{S}'} \Sigma_{L \in \sigma} \int_L f(y_*(x), J(y, x))\, dx = I(y)$.

COROLLARY 6.5. *The theorem holds if $f \in C(A \times Y)$ and f_θ is convex for each $\theta \in A$. Thus I is lsc if f is continuous and T-convex.*

Proof. Let $\varepsilon > 0$ and $g(\theta, q) = f(\theta, q) + \varepsilon \|q\|$ for each $(\theta, q) \in A \times Y$. Let $I_g(y) = \int_G g(y_*(x), J(y, x))\, dx$. If $J(y_n) \rightharpoonup J(y)$ in L^r then there exists $m > 0$ such that $\|J(y_n)\| < m$ for each n. Hence $I(y) \leq I_g(y) \leq \liminf I_g(y_n) = \liminf[I(y_n) + \varepsilon \|J(y_n)\|] \leq \liminf I(y_n) + m\varepsilon$ since g is semi-regular positive semi-normal and hence V-convex.

The construction in Theorem 3.5 can be used to show that not only is T-convexity a necessary condition that I be lower semi-continuous with respect to the convergence of that theorem, but also with respect to the convergence of Corollary 6.5.

The gap between the necessary and sufficient conditions for lower semi-continuity can now be described by the fact that f can be T-convex without being continuous (but see the paragraph preceding Corollary 7.3).

Since $\Rightarrow$ is stronger than $\rightharpoonup$, the following corollary is immediate.

COROLLARY 6.6. *If $y_n \Rightarrow y$ in $\mathfrak{M}(y_o)$ then $I(y) \leq \liminf I(y_n)$.*

7. We conclude with an existence theorem and some minor generalizations.

THEOREM 7.1. *Let $f \in C(A \times Y)$ be nonnegative and f_θ be convex for each $\theta \in A$. If $\mathfrak{M}(y_o)$ is $\Rightarrow$ compact and if* $\inf\{I(y) \mid y \in \mathfrak{M}(y_o)\} < \infty$ *then I attains its minimum on $\mathfrak{M}(y_o)$.*

This result follows from Corollary 6.6.

COROLLARY 7.2. *Suppose that there exists $m > 0$ such that for each $(\theta, s) \in A \times Y$ either*

(i) *There exists $M > 0$ and $p > 1$ such that $\|y\|_\infty < M$ and $f(\theta, s) \geq m\|s\|^p$, or*

(ii) *$f(\theta, s) \geq m\|s\|^q$ where $q = 2k/(k+1)$. If* $\inf\{I(y) \mid y \in \mathfrak{M}(y_o)\} < \infty$ *then I attains its minimum on $\mathfrak{M}(y_o)$.*

The corollary follows from Theorem 4.8.

Let Y_1 be a compact convex subset of Y. If $y_o \in AC$ and if $J(y_o, x) \in Y_1$ for almost all $x \in G$, then let

$$\mathfrak{M}_1(y_o) = \mathfrak{M}(y_o) \cap \{y \mid J(y, x) \in Y_1 \text{ for almost all } x \in G\}.$$

Let $f \in C(A \times Y_1)$. If I is lower semicontinuous on $\mathfrak{M}_1(y_o)$ then, as before, f must be T-convex, i.e., there exists g_θ: $Y_1 \to \mathbf{R}$ where g_θ is convex and extends f_θ for each $\theta \in A$. Since Y_1 is compact, it follows that g is continuous so, for this case, a necessary and sufficient condition that I be lower semicontinuous is that f be T-convex. Thus the next corollary follows from the preceding one.

COROLLARY 7.3. *Let Y_1 be a compact convex subset of Y and $f \in C(A \times Y_1)$ be T-convex. If, in addition, f satisfies* (i) *or* (ii) *and* $\inf\{I(y) \mid y \in \mathfrak{M}_1(y_o)\} < \infty$ *then I attains its minimum on $\mathfrak{M}_1(y_o)$.*

Let Y_2 be a compact subset of Y and $f \in C(A \times Y_2)$. Let Y_1 be the convex hull of Y_2 and let g be defined on $A \times Y_1$ by

$$g(\theta, q) = \inf\left\{ \sum_{i=1}^{n} \lambda_i f(\theta, p_i) \mid p_i \in Y_2, \right.$$
$$\left. \lambda_i > 0,\ \sum \lambda_i = 1, \text{ and } \sum \lambda_i p_i = q \right\}.$$

If $g \in C(A \times Y_1)$ is T-convex and if

$$\inf\{I_g(y) \mid y \in \mathfrak{M}_1(y_o)\} < \infty,$$

where $I_g(y) = \int_G g(y_*(x), J(y, x))\,dx$, then, by Corollary 7.3, there exists $z \in \mathfrak{M}_1(y_o)$ such that $g(z) = \min\{I_g(y) \mid y \in \mathfrak{M}_1(y_o)\}$. Then z is called a *relaxed minimizer for f on Y_2.*

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