PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

FREE PRODUCTS IN THE CLASS OF ABELIAN l-GROUPS

WAYNE B. POWELL AND CONSTANTINE TSINAKIS

The main objective of this paper is to present several constructions of free products in the class of abelian l-groups which are sufficiently concrete to allow for an in depth examination of their structure. Some applications of these constructions are discussed, and it is shown that abelian l-group free products satisfy the subalgebra property. Further, some questions on free l-groups over group free products are considered for a variety of l-groups which is either abelian or contains the representable l-groups. Finally, a general observation is made about countable chains and countable disjoint sets in free algebras.

1. Introduction. Let $\mathcal{U}$ be a class of l-groups (lattice ordered groups) and $(G_i \mid i \in \mathcal{I})$ a family of members of $\mathcal{U}$. The $\mathcal{U}$-free product of this family is an l-group $G \in \mathcal{U}$, denoted by ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} G_i$, together with a family of l-monomorphisms $(\alpha_i\colon G_i \to G \mid i \in \mathcal{I})$ such that

(i) $\bigcup_{i\in\mathcal{I}} \alpha_i(G_i)$ generates G as an l-group;

(ii) for every $H \in \mathcal{U}$ and every family of l-homomorphisms $(\beta_i\colon G_i \to H \mid i \in \mathcal{I})$, there exists a (necessarily) unique l-homomorphism $\beta\colon G \to H$ satisfying $\beta_i = \beta\alpha_i$ for all $i \in \mathcal{I}$.

Following the usual practice we shall speak of ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} G_i$ as the $\mathcal{U}$-free product of $(G_i \mid i \in \mathcal{I})$. To simplify our notation, we use the "internal" definition of a $\mathcal{U}$-free product, that is, we identify each free factor G_i with its image $\alpha_i(G_i)$ in ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} G_i$, and thus we think of each G_i as an l-subgroup of ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} G_i$. As a consequence of general existence theorems (See Grätzer [**13**, p. 186] or Pierce [**25**, p. 107]), $\mathcal{U}$-free products always exist in any class of l-groups closed under products and l-subgroups.

In this paper we concentrate on the class $\mathcal{A}$ of abelian l-groups, although many of our results also hold in the important class of vector lattices. Our main goal is to develop a reasonable representation theory for $\mathcal{A}$-free products. This is done in §2 where we give several methods of constructing these products, among the most useful of which represents ${}^{\mathcal{A}}\bigsqcup_{i\in\mathcal{I}} G_i$ $(G_i \in \mathcal{A})$ as a subdirect product of totally ordered abelian groups each determined by the primes of the individual G_i's. We also show here how the $\mathcal{A}$-free products relate to the free abelian l-groups over partially ordered abelian groups.

The third and fourth sections of the paper are devoted to considering several different properties for free products of l-groups. In particular using the representation theory established in §2 we show that the subalgebra property is satisfied for $\mathcal{A}$-free products.

In the fifth and final section further applications of the representation as well as open questions are discussed. In addition an observation is made about countable chains and countable disjoint sets in the free algebras for classes of general ordered algebraic systems.

Free products in classes of lattice ordered groups have been considered by Franchello [**10**], Holland and Scrimger [**16**], and Martinez [**22**] and [**23**]. However, apart from some special cases, no reasonable representation has previously been found, and thus very limited information is known about these objects. In [**31**] and [**32**] Weinberg presented a nice construction of the free abelian l-groups and consequently was able to prove several important results. His work was generalized to the classes of vector lattices (Topping [**29**]), all l-groups (Conrad [**9**]), torsion free f-modules (Bigard [**5**]), and all f-modules (Powell [**26**]). As the free object on n generators is the free product of n copies of the free object on one generator, the above mentioned results give very specialized information about free products. Our construction of $\mathcal{Q}$-free products is similar in spirit to all the aforementioned constructions.

Background information on the theory of l-groups can be found in Bigard et al. [**6**], Conrad [**8**], and Fuchs [**11**]. General references related to free products are Grätzer [**13**] and Pierce [**25**].

Throughout this paper we use $\bigsqcup_{i\in\mathcal{I}} G_i$ to signify ${}^{\mathcal{Q}}\bigsqcup_{i\in\mathcal{I}} G_i$. The symbol $\oplus$ refers to the group theoretic direct sum while $\boxplus$ denotes the cardinal sum of l-groups; that is, their direct sum endowed with the componentwise order. Finally, $A \Subset B$ means that A is a finite non-empty subset of B.

2. Representations for free products of abelian l-groups.

Let $(G_i \mid i \in \mathcal{I})$ be a family of abelian l-groups, $G = \bigsqcup_{i\in\mathcal{I}} G_i$, and $H = \bigoplus_{i\in\mathcal{I}} G_i$. In this section we give several representations for G. The first of these describes G as a sublattice generated by H inside a particular l-group (Theorem 2.4). It is then shown that this l-group containing G can be modified so that it is easily determined by H (Theorems 2.6 and 2.7). Finally, a representation is given for G as a sublattice of a product of abelian l-groups which are free over certain partially ordered groups (Theorem 2.8).

Before we state the simple proposition below, we introduce the following terminology. A group homomorphism φ of H into an l-group H' is said to be *admissible* if the restriction of φ on each individual G_i is an l-homomorphism.

PROPOSITION 2.1. *Let L be an abelian l-group containing each G_i as an l-subgroup. If $H = \bigoplus_{i\in\mathcal{I}} G_i$ is a subgroup of L and generates L as a lattice, then the following conditions are equivalent.*

(i) $L = G$

(ii) *For each abelian l-group L' and each admissible homomorphism φ: $H \to L'$, there exists a (necessarily) unique l-homomorphism φ': $L \to L'$ extending φ.*

Let us pause for a moment to recall the concept of a free l-group over a *po*-group (partially ordered group). This is a specialization in the setting of l-groups of what is called in universal algebra the free algebra (extension) over a partial algebra. Consider a class $\mathcal{U}$ of l-groups. Let $F \in \mathcal{U}$ and G be an o-subgroup of F; that is, G is a subgroup of F and the partial order of G is the restriction to G of the order of F. Then F is the $\mathcal{U}$-free l-group over G if:

(i) G generates F as an l-group;

(ii) If f is an o-homomorphism (an order preserving group homomorphism) of G into an l-group $H \in \mathcal{U}$, then there exists a (necessarily) unique l-homomorphism g: $F \to H$ extending f.

We use $F_{\mathcal{U}}(G)$ to denote F, and we simply write $F(G)$ instead of $F_{\mathcal{Q}}(G)$. The uniqueness of $F_{\mathcal{U}}(G)$ is clear; however its existence, like that of free products, is not guaranteed. See Pierce [**25**] for general existence theorems and Weinberg [**31**] and Conrad [**8**] or [**9**] for corresponding theorems in classes of l-groups.

Proposition 2.1 implies the following result (See Martinez [**22**] and Holland and Scrimger [**16**]).

PROPOSITION 2.2. $\bigsqcup_{i\in\mathcal{I}} G_i = F(\boxplus_{i\in\mathcal{I}} G_i)$ *if and only if each G_i is totally ordered.*

We shall make use of the following elementary lemma (See Bigard et al. [**6**, p. 298]).

LEMMA 2.3. *Let L and L' be l-groups and let M be a subgroup of L which generates L as a lattice. Let φ: $M \to L'$ be a group homomorphism such that for each $(x_{jk} \mid j \in \mathcal{J}, k \in \mathcal{K}) \Subset M$,*

$$\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} x_{jk} = 0 \textit{ implies } \bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \varphi(x_{jk}) = 0.$$

Then φ can be extended to an l-homomorphism φ': $L \to L'$.

Note that φ' is defined in an obvious fashion. If

$$x = \bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} x_{jk} \left((x_{jk} \mid j \in \mathcal{J}, k \in \mathcal{K}) \Subset M\right)$$

is in L, then

$$\varphi'(x) = \bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \varphi(x_{jk}).$$

Now, let

$$\Lambda = \Big\{ (H/K, T) \mid K \text{ is a subgroup of } H, T \text{ is the positive cone of a total order on } H/K, \text{ and the map } \sum_{i \in \mathcal{I}} g_i \mapsto \sum_{i \in \mathcal{I}} (g_i + K) \colon H \to (H/K, T) \text{ is admissible} \Big\}.$$

Note that if $(H/K, T) \in \Lambda$, then K is a convex o-subgroup of H and T extends the positive cone of the inherited quotient order on H/K.

A subset Δ of Λ is said to *separate points* (for H) if:

(1) $\cap \{K \mid (H/K, T) \in \Delta\} = \{0\}$.

(2) For every totally ordered abelian group L, for every admissible group homomorphism φ: $H \to L$, and for every $(h_j \mid j \in \mathcal{J}) \Subset H$ such that $\bigwedge_{j \in \mathcal{J}} \varphi(h_j) > 0$ in L, there exists $(H/K, T) \in \Delta$ such that $\bigwedge_{j \in \mathcal{J}} (h_j + k) > 0$ in $(H/K, T)$.

Now let Δ be a non-void subset of Λ and consider the group homomorphism $h \mapsto \langle h \rangle = (\ldots, h + K, \ldots)$ of H into $\Pi_\Delta (H/K, T)$. Denote by $H(\Delta)$ the sublattice of $\Pi_\Delta (H/K, T)$ generated by $\{\langle h \rangle \mid h \in H\}$. That is,

$$H(\Delta) = \Big\{ \bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \langle h_{jk} \rangle \mid (h_{jk} \mid j \in \mathcal{J}, k \in \mathcal{K}) \Subset H \Big\}$$

Using this notation we have the following representation theorem for G.

THEOREM 2.4. *For a family of abelian l-groups $(G_i \mid i \in \mathcal{I})$ consider the unique l-homomorphism Ψ: $G = \bigsqcup_{i \in \mathcal{I}} G_i \to H(\Delta)$ extending the maps $g_i \mapsto \langle g_i \rangle$: $G_i \to H(\Delta)$. G is isomorphic to $H(\Delta)$ under Ψ if and only if Δ separates points.*

Proof. Suppose first that Δ separates points. Let $G_i' = \{\langle g_i \rangle \mid g_i \in G_i\}$ and $H' = \{\langle h \rangle \mid h \in H\}$, so that $H(\Delta)$ is the sublattice of $\Pi_\Delta (H/K, T)$ generated by H'. As $\cap \{K \mid (H/K, T) \in \Delta\} = \{0\}$, each l-group G_i is isomorphic to G_i', and also H and H' are isomorphic as groups. To show that $H(\Delta) \cong G$ we use Proposition 2.1. To this end let L' be an abelian l-group, and φ: $H' \to L'$ an admissible group homomorphism. Without loss of generality we may assume that L' is totally ordered. Set $K' = \operatorname{Ker} \varphi$ and let T' be the positive cone of the total order on H'/K' inherited from L'. As the restriction of φ on each G_i' is an l-homomorphism, each map $\langle g_i \rangle \mapsto \langle g_i \rangle + K'$: $G_i' \to (H'/K', T')$ is an l-homomorphism, and thus

$(H'/K', T')$ is isomorphic to a member of Λ. Now suppose $\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \varphi(\langle h_{jk}\rangle) \neq 0$ in L'. If $\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \varphi(\langle h_{jk}\rangle) > 0$, then there exists $j \in \mathcal{J}$ such that $\bigwedge_{k\in\mathcal{K}} \varphi(\langle h_{jk}\rangle) > 0$. Hence for this j, $\bigwedge_{k\in\mathcal{K}}(\langle h_{jk}\rangle + K') > 0$. As Δ separates points, we can find $(H/K, T) \in \Delta$ such that $\bigwedge_{k\in\mathcal{K}}(h_{jk} + K) > 0$. Thus, $\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \langle h_{jk}\rangle \neq 0$ in $H(\Delta)$. The case $\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \varphi(\langle h_{jk}\rangle) < 0$ is treated similarly. By Lemma 2.3, φ can be extended to an l-homomorphism φ': $H(\Delta) \to L'$. In view of Proposition 2.1, $H(\Delta) \cong G$.

Conversely, suppose that $H(\Delta) \cong G$ under Ψ. To begin with, it is clear that $\cap\{K \mid (H/K, T) \in \Delta\} = \{0\}$. Next, let L be a totally ordered abelian group, φ: $H \to L$ an admissible group homomorphism, and $(h_j \mid \in \mathcal{J}) \Subset H$ such that $\bigwedge_{j\in\mathcal{J}} \varphi(h_j) > 0$. By Proposition 2.1 there exists an l-homomorphism φ': $H(\Delta) \to L$ such that $\varphi'(\langle h\rangle) = \varphi(h)$ for all $h \in H$. Hence, $\bigwedge_{j\in\mathcal{J}} \varphi'(\langle h_j\rangle) > 0$ and so $\bigwedge_{j\in\mathcal{J}} \langle h_j\rangle \neq 0$. But this simply means that there is $(H/K, T) \in \Delta$ such that $\bigwedge_{j\in\mathcal{J}}(h_j + K) > 0$ in $(H/K, T)$. The proof of the theorem is now complete. □

The natural question to ask now is: What subsets Δ of Λ separate points? It should be clear that Λ itself separates points so that for any collection of abelian l-groups a representation of their free product can be found in Theorem 2.4.

Let us consider the case where each l-group G_i is totally ordered. Let

$$\Lambda' = \{(H, T) \mid T \text{ is the positive cone of a total order on } H \text{ extending the cardinal order}\}.$$

Then $H(\Lambda') = F(\boxplus_{i\in\mathcal{I}} G_i)$ (See Weinberg [31].) Hence Proposition 2.2 and Theorem 2.4 yield.

PROPOSITION 2.5. *If $(G_i \mid i \in \mathcal{I})$ is a family of totally ordered abelian groups, then the set Λ' separates points.*

We will now show that for an arbitrary family $(G_i \mid i \in \mathcal{I})$ of abelian l-groups, there is a nice subset of Λ which separates points. More specifically, we consider the set

$$\Lambda_0 = \left\{\left(\bigoplus_{i\in\mathcal{I}} (G_i/P_i), T\right) \mid P_i \text{ is a prime subgroup of } G_i \text{ for each } i, \text{ and } T \text{ is the positive cone of a total order on } \bigoplus_{i\in\mathcal{I}} (G_i/P_i) \text{ extending the cardinal order}\right\}.$$

Recall that a subgroup P of an abelian l-group L is prime if it is a convex l-subgroup of L and if L/P is totally ordered. See Bigard, et al. [**6**, p. 44] for several characterizations of prime subgroups. Let $(\oplus_{i\in\mathcal{I}}(G_i/P_i), T) \in \Lambda_0$.

As there is a natural group isomorphism between $H/(\oplus_{i\in\mathcal{I}}P_i)$ and $\oplus_{i\in\mathcal{I}}(G_i/P_i)$, T induces a total order on $H/(\oplus_{i\in\mathcal{I}}P_i)$. Under this total order, $H/(\oplus_{i\in\mathcal{I}}P_i)$ is clearly a member of Λ. Thus we may and we will consider Λ_0 as a subset of Λ. We will now show that Λ_0 separates points for H. To this end suppose L is a totally ordered abelian group, φ: $H \to L$ an admissible group homomorphism, and $(h_j \mid j \in \mathcal{J}) \Subset H$ such that $\bigwedge_{j\in\mathcal{J}} \varphi(h_j) > 0$. Since for each $i \in \mathcal{I}$, $\varphi(G_i)$ is totally ordered, there exist prime subgroups $Q_i \subseteq G_i$ with $G_i/Q_i \cong \varphi(G_i)$. Consider now $F(\boxplus_{i\in\mathcal{I}}(G_i/Q_i)) \cong \bigsqcup_{i\in\mathcal{I}} G_i/Q_i \cong \bigsqcup_{i\in\mathcal{I}}\varphi(G_i)$. From Proposition 2.5 we see that

$$\Lambda_1 = \left\{\left(\bigoplus_{i\in\mathcal{I}}(G_i/Q_i), T\right) \mid T \text{ is the positive cone of a total order on } \bigoplus_{i\in\mathcal{I}}(G_i/Q_i) \text{ extending the cardinal order}\right\}$$

separates points for $\oplus_{i\in\mathcal{I}}(G_i/Q_i)$. Thus, considering $(h_j + \oplus_{i\in\mathcal{I}}Q_i \mid j \in \mathcal{J})$ as a family of elements of $\oplus_{i\in\mathcal{I}}(G_i/Q_i)$, we get that the infimum of these elements is positive on some member of Λ_1. But $\Lambda_1 \subseteq \Lambda_0$ and thus Λ_0 must separate points for H. This establishes the following representation theorem.

THEOREM 2.6 *The free product of a family $(G_i \mid i \in \mathcal{I})$ of abelian l-groups is the l-subgroup $H(\Lambda_0)$ of $\prod_{\Lambda_0}(\oplus_{i\in\mathcal{I}}(G_i/P_i), T)$.*

There are, of course, other subsets of Λ that separate points for H. However, the representation given in Theorem 2.6 appears to be the most useful. A variation of the representation is the following. Let

$$\Lambda_0' = \left\{\left(\bigoplus_{i\in\mathcal{I}}(G_i/P_i), T\right) \in \Lambda_0 \mid P_i \text{ is a minimal prime subgroup of } G_i, \text{ for each } i\right\}.$$

We ask the reader to verify the following theorem.

THEOREM 2.7. *The free product of a family $(G_i \mid i \in \mathcal{I})$ of abelian l-groups in the l-subgroup $H(\Lambda_0')$ of $\prod_{\Lambda_0'}(\oplus_{i\in\mathcal{I}}(G_i/P_i), T)$.*

It is worth mentioning at this point that a slight variation of Weinberg's representation for free abelian l-groups (see [31]) is a direct consequence of Theorem 2.7 and of the fact that the free abelian l-group on a set $X \neq \varnothing$ is the abelian l-group free product of $|X|$-copies of $\mathbf{Z} \boxplus \mathbf{Z}$.

We close this section by exhibiting a further relationship between $\bigsqcup_{i \in \mathcal{I}} G_i$ and certain abelian l-groups which are free over partially ordered groups. First of all, set

$$\Gamma_i = \{G_i/P_i \mid P_i \text{ is a prime subgroup of some } G_i\}.$$

Let $\Gamma = \bigcup_{i \in \mathcal{I}} \Gamma_i$ and let D consist of all subsets Δ of Γ such that $\Delta \cap \Gamma_i$ is a singleton for each $i \in \mathcal{I}$. For $\Delta \in D$ consider $F(\boxplus_{\Delta}(G_i/P_i))$, and let δ_Δ: $\boxplus_{\Delta}(G_i/P_i) \to F(\boxplus_{\Delta}(G_i/P_i))$ be the natural o-monomorphism. For each $i \in \mathcal{I}$ define the map Ψ_i: $G_i \to \Pi_{\Delta \in D} F(\boxplus_{\Delta}(G_i/P_i))$ by

$$\Psi_i(g_i) = (\ldots, \delta_\Delta(g_i + P_i), \ldots).$$

It is easy to see that each Ψ_i is an l-homomorphism. Indeed, if $x \wedge y = 0$ in G_i and if P_i is a prime subgroup of G_i, then either $x \in P_i$ or $y \in P_i$. Hence, $\delta_\Delta(x + P_i) \wedge \delta_\Delta(y + P_i) = 0$.

Now, let $H = \oplus_{i \in \mathcal{I}} G_i$ and define Ψ: $H \to \Pi_{\Delta \in D} F(\boxplus_{\Delta}(G_i/P_i))$ by $\Psi(\Sigma_{i \in \mathcal{I}} g_i) = \Sigma_{i \in \mathcal{I}} \Psi_i(g_i)$. Using this notation we have the following representation theorem for $\bigsqcup_{i \in \mathcal{I}} G_i$.

THEOREM 2.8. *The free product $\bigsqcup_{i \in \mathcal{I}} G_i$ of a family $(G_i \mid i \in \mathcal{I})$ of abelian l-groups is the sublattice G' of $\Pi_{\Delta \in D} F(\boxplus_{\Delta}(G_i/P_i))$ generated by $\Psi(H)$.*

Proof. To begin with note that Ψ is an injective group homomorphism. Suppose φ: $H \to L$ is admissible and let $\varphi_i = \varphi|_{G_i}$ for each $i \in \mathcal{I}$. If $\bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \varphi(h_{jk}) \neq 0$ for some $(h_{jk} \mid j \in \mathcal{J}, k \in \mathcal{K}) \Subset H$, then by Proposition 2.1 and Lemma 2.3 we need only show that $\bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \Psi(h_{jk}) \neq 0$ in G'. We assume that L is totally ordered and consider the case where $\bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \varphi(h_{jk}) > 0$. The case where $\bigvee_{j \in \mathcal{J}} \bigwedge_{k \in \mathcal{K}} \varphi(h_{jk}) < 0$ is handled similarly. Hence, there exists $j \in \mathcal{J}$ such that $\bigwedge_{k \in \mathcal{K}} \varphi(h_{jk}) > 0$. Let $\Delta_0 = \{G_i/P_i \mid p_i \text{ is the kernel of } \varphi_i\}$. Then $\Delta_0 \in D$. Let $\bar{\varphi}$: $\oplus_{\Delta_0}(G_i/P_i) \to L$ be defined by $\bar{\varphi}(\Sigma_{i \in \mathcal{I}}(g_i + P_i)) = \Sigma_{i \in \mathcal{I}}(\varphi_i(g_i))$, and write each $h_{jk} \in H$ as $h_{jk} = \Sigma_{i \in \mathcal{I}} g_{jki}$, where each $g_{jki} \in G_i$. Now,

$$\bigwedge_{k \in \mathcal{K}} \bar{\varphi}\left(\sum_{i \in \mathcal{I}} (g_{jki} + P_i)\right) = \bigwedge_{k \in \mathcal{K}} \left(\sum_{i \in \mathcal{I}} \varphi_i(g_{jki})\right) = \bigwedge_{k \in \mathcal{K}} (\varphi(h_{jk})) > 0.$$

But by Proposition 2.5 the set $\{(\oplus_{\Delta_0}(G_i/P_i), T) \mid T$ is the positive cone of a total order on $\oplus_{\Delta_0}(G_i/P_i)$ extending the cardinal order$\}$ separates points for $\oplus_{\Delta_0}(G_i/P_i)$. Since $\bar{\varphi}$ is admissible, $\bigwedge_{k\in\mathcal{K}}(\Sigma_{i\in\mathcal{I}}\, g_{jki} + P_i) > 0$ on some element of this set. Hence, $\bigwedge_{k\in\mathcal{K}}(\delta_{\Delta_0}(\Sigma_{i\in\mathcal{I}}\, g_{jki} + P_i)) > 0$ which implies that $\bigvee_{j\in\mathcal{J}}\bigwedge_{k\in\mathcal{K}} \Psi(h_{jk}) \neq 0$. □

It is of significance to note that all of the representation theory that has been developed in this section can be easily adapted to yield analogous representations for free products in the important class of vector lattices.

3. The subalgebra property. Let $\mathcal{U}$ be a variety of l-groups. Free products in $\mathcal{U}$ are said to have the *subalgebra property* if for any family $(G_i \mid i \in \mathcal{I})$ in $\mathcal{U}$ with l-subgroups $H_i \subseteq G_i$, ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} H_i$ is simply the l-subgroup of ${}^{\mathcal{U}}\bigsqcup_{i\in\mathcal{I}} G_i$ generated by $\bigcup_{i\in\mathcal{I}} H_i$.

As a consequence of a general result due to Jónsson [**19**], any variety of l-groups having the amalgamation property, satisfies the subalgebra property for free products. Since the amalgamation property holds in the variety $\mathcal{A}$ of abelian l-groups (Pierce [**24**]), $\mathcal{A}$-free products have the subalgebra property. As the considerations leading to the proof of the amalgamation property for $\mathcal{A}$ are somewhat involved, we present a short proof of the subalgebra property based on the results of §2.

We begin with the following simple preliminary lemma.

LEMMA 3.1. *Let H be an l-subgroup of an l-group G. If P is a prime subgroup of H, then there is a prime subgroup Q of G such that $P = Q \cap H$.*

Proof. Let $\mathcal{C}(G)$ denote the collection of convex l-subgroups of G. An easy application of Zorn's Lemma to the set $\mathcal{P} = \{K \in \mathcal{C}(G) \mid H \cap K = P\}$ yields the required prime Q in G. □

THEOREM 3.2. *$\mathcal{A}$-free products satisfy the subalgebra property.*

Proof. Consider a family $(G_i \mid i \in \mathcal{I}) \subseteq \mathcal{A}$, and let H_i be an l-subgroup of G_i, for each $i \in \mathcal{I}$. Write $G = \bigsqcup_{i\in\mathcal{I}} G$ and $H = \bigsqcup_{i\in\mathcal{I}} H$. It is to be shown that H is isomorphic to the l-subgroup H^* of G generated by $\bigcup_{i\in\mathcal{I}} H_i$. Consider the unique l-homomorphism $\varphi\colon H \to G$ extending the inclusion maps $H_i \to G_i$. It is clear that $\varphi(H) = H^*$. We proceed to show that φ is injective. Consider a non-zero element h in H. There is $(x_{jkl} \mid j \in \mathcal{J}, k \in \mathcal{K}, l \in \mathcal{L}) \Subset \bigcup_{i\in\mathcal{I}} H_i$, $x_{jkl} \in H_l$, such that $h = \bigvee_{j\in\mathcal{J}}\bigwedge_{k\in\mathcal{K}}\Sigma_{l\in\mathcal{L}}\, x_{jkl}$. Note that $\varphi(h) = \bigvee_{j\in\mathcal{J}}\bigwedge_{k\in\mathcal{K}}\Sigma_{l\in\mathcal{L}}\, x_{jkl}$, where now the operations take place in G. In view of Theorem 2.6, for each $i \in \mathcal{I}$, there is a prime subgroup P_i of H_i, and a total order on $\oplus_{i\in\mathcal{I}}(H_i/P_i)$ with positive cone T, so that

$$\bigvee_{j\in\mathcal{J}}\ \bigwedge_{k\in\mathcal{K}}\ \sum_{l\in\mathcal{L}} \left(x_{jkl} + P_l\right) = \sum_{l\in\mathcal{L}} \left(x_{j_1k_1l} + P_l\right) \neq 0$$

in $(\oplus_{i\in\mathcal{I}}(H_i/P_i), T)$, for some $j_1 \in \mathcal{J}$ and $k_1 \in \mathcal{K}$. By virtue of Lemma 3.1, for each $i \in \mathcal{I}$, there is a prime subgroup Q_i of G_i such that $Q_i \cap G_i = P_i$. Let μ: $\oplus_{i\in\mathcal{I}}(H_i/P_i) \to \oplus_{i\in\mathcal{I}}(G_i/Q_i)$ be defined by $\mu(\Sigma_{i\in\mathcal{I}}(h_i + P_i)) = \Sigma_{i\in\mathcal{I}}(h_i + Q_i)$. It is clear that μ is an injective homomorphism. Let T^* be the positive cone of a total order on $\oplus_{i\in\mathcal{I}}(G_i/Q_i)$ extending the total order on $(\oplus_{i\in\mathcal{I}}(H_i/P_i))$ induced by T (See Fuchs [**9**, p. 39]). Then,

$$\bigvee_{j\in\mathcal{J}} \bigwedge_{k\in\mathcal{K}} \sum_{l\in\mathcal{L}} (x_{jkl} + Q_l) = \sum_{l\in\mathcal{L}} (x_{j_1k_1l} + Q_l) \neq 0$$

in $\oplus_{i\in\mathcal{I}}(G_i/Q_i), T^*) \in \Lambda_0$. By Theorem 2.6, $\varphi(h) \neq 0$. This shows that φ is injective, and the proof of the theorem is complete. □

It is worth mentioning at this point that the subalgebra property (and hence the amalgamation property) fails in many varieties of l-groups. For example, let n be an integer greater than one and consider the variety $\mathcal{L}_n$ of all l-groups satisfying the law $nx + ny = ny + nx$. It is not hard to show that $\mathcal{L}_n$-free products do not have the subalgebra property. Indeed, since $\mathcal{L}_n$ properly contains $\mathcal{A}$ (see Martinez [**22**]), the $\mathcal{L}_n$-free l-group on two generators, or what amounts to the same, the $\mathcal{L}_n$-free product of two copies of $\mathbf{Z} \boxplus \mathbf{Z}$ is not abelian. Let F be the $\mathcal{L}_n$-free l-group on the two element set $\{x_1, x_2\}$, and consider the l-subgroups $G_1 = \langle x_1 \rangle$, $G_2 = \langle x_2 \rangle$, $H_1 = \langle nx_1 \rangle$, $H_2 = \langle nx_2 \rangle$, and $H = \langle nx_1, nx_2 \rangle$. Evidently, $G_1 \cong G_2 \cong H_1 \cong H_2 \cong \mathbf{Z} \boxplus \mathbf{Z}$, H_i is an l-subgroup of G_i $(i = 1, 2)$, $F = G_1^{\mathcal{L}_n} \bigsqcup G_2$, and $H \in \mathcal{A}$. It follows that $H_1^{\mathcal{L}_n} \bigsqcup H_2 \cong H$, and hence $\mathcal{L}_n$-free products do not satisfy the subalgebra property.

Martinez [**23**, Theorem 4.1] showed that if C is an l-ideal of $A \in \mathcal{A}$ and if $B \in \mathcal{A}$, then $C \bigsqcup B$ is an l-subgroup of $A \bigsqcup B$. He asks whether this remains true if C is an arbitrary l-subgroup of A. in view of Theorem 3.2, the answer is clearly affirmative.

4. Free l-groups over group free products. If $(G_i \mid i \in \mathcal{I})$ is a family of totally ordered groups in $\mathcal{A}$, then ${}^{\mathcal{A}}\bigsqcup_{i\in\mathcal{I}} G_i$ is the $\mathcal{A}$-free l-group over the abelian group free product H endowed with the partial order induced by ${}^{\mathcal{A}}\bigsqcup_{i\in\mathcal{I}} G_i$ (i.e., $H = \boxplus_{i\in\mathcal{I}} G_i$; see Proposition 2.2). We will see below that if H is (group) isomorphically embedded in ${}^{\mathcal{A}}\bigsqcup_{i\in\mathcal{I}} G_i$ in some other way, then it becomes a partially ordered abelian group over which the $\mathcal{A}$-free l-group need not be ${}^{\mathcal{A}}\bigsqcup_{i\in\mathcal{I}} G_i$. A similar result arises for other varieties $\mathcal{U}$ containing the variety of representable l-groups.

Example 4.1. Let G be the $\mathcal{A}$-free product of three copies of the integers, and identify these inside G as $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3$. Hence, $\mathbf{Z}_1 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$ is a subgroup of G, and in fact it generates G as a lattice. Consider the elements $w_1 = (1, 1, 1)$, $w_2 = (1, 1, 0)$, and $w_3 = (0, 1, 1)$ of $\mathbf{Z}_1 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$.

Let W_i be the subgroup G generated by w_i for each $i = 1, 2, 3$. Then each W_i is an l-subgroup of G and the abelian group free product $H = W_1 \oplus W_2 \oplus W_3$ is a subgroup of G. If H is given the partial order inherited from G, then $F_{\mathcal{C}}(H) = G$. Indeed, the order on H is the cardinal order on $\mathbf{Z}_1 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$ so any o-homomorphism $H \to L \in \mathcal{C}$ preserves the order on $\mathbf{Z}_1 \boxplus \mathbf{Z}_2 \boxplus \mathbf{Z}_3$. However, $F_{\mathcal{C}}(H)$ cannot be $\bigsqcup_{i=1}^{3} W_i$ since the maps φ_i: $W_i \to \mathbf{Z}$ defined by $\varphi_1(w_1) = 1$, $\varphi_2(w_2) = 2$, and $\varphi_3(w_3) = 0$ cannot be extended to an l-homomorphism on G.

Interestingly enough the preceding example can be adapted to yield the corresponding result for many other l-group varieties. Specifically, we consider any variety containing the class $\mathcal{R}$ of representable l-groups. $\mathcal{R}$ is well-known to be precisely the variety of all l-groups that are subdirect products of totally ordered groups. We first need a result which exhibits a relationship between group free products and l-group free products.

PROPOSITION 4.2. *Let $\mathcal{U}$ be an l-group variety containing the variety $\mathcal{R}$ of representable l-groups. Consider a family $(G_i \mid i \in \mathcal{I})$ of totally ordered groups, and let G be the $\mathcal{U}$-free product of this family. Then the group free product of $(G_i \mid i \in \mathcal{I})$ is (isomorphic to) the subgroup of G generated by $\bigcup_{i \in \mathcal{I}} G_i$.*

Proof. Write H for the group free product $(G_i \mid i \in \mathcal{I})$ and H^* for the subgroup of G generated by $\bigcup_{i \in \mathcal{I}} G_i$. It needs to be shown that H^* is isomorphic to H. By a result due to Vinogradov **[30]** (see also Johnson **[18]**), there is a total order on H extending the total orders of the free factors G_i. Let T be the positive cone of such an order. Evidently $(H, T) \in \mathcal{R} \subseteq \mathcal{U}$. Hence, by the universal property for $\mathcal{U}$-free products, there is an l-homomorphism φ: $G \to (H, T)$ such that $\varphi(g) = g$ for each $g \in G_i$ and each $i \in \mathcal{I}$. Let φ: $H^* \to H$ be the restriction of φ on H^*. It is clear that φ is an onto group homomorphism. Again, by the universal property of the group free product, there is a group homomorphism Ψ: $H \to H^*$ such that $\Psi(g) = g$ for each $g \in G_i$ and $i \in \mathcal{I}$. But then $\Psi \circ \varphi$ is the identity map on H^*, and so φ is injective. It follows that φ is a group isomorphism between H^* and H. □

The above proposition was established by Holland and Scrimger **[16]** for the special case where $\mathcal{U}$ is the variety of all l-groups.

The following proposition for varieties $\mathcal{U} \supseteq \mathcal{R}$ of l-groups is the analogue of Proposition 2.2 which deals with the class $\mathcal{C}$. We note that to establish the equality of ${}^{\mathcal{U}}\bigsqcup_{i \in \mathcal{I}} G_i$ and $F_{\mathcal{U}}(H)$ we do not need to assume the existence of $F_{\mathcal{U}}(H)$. Rather we can use the fact that ${}^{\mathcal{U}}\bigsqcup_{i \in \mathcal{I}} G_i$ exists and show it satisfies the universal property of $F_{\mathcal{U}}(H)$. The details of the proof are left to the reader.

PROPOSITION 4.3 *Let $\mathfrak{U} \supseteq \mathfrak{R}$ be a variety of l-groups, $(G_i \mid i \in \mathfrak{I})$ a family of totally ordered groups, and H the group free product of this family. If H is endowed with the partial order inherited from ${}^{\mathfrak{U}}\bigsqcup_{i \in \mathfrak{I}} G_i$, then*

$$ {}^{\mathfrak{U}}\bigsqcup_{i \in \mathfrak{I}} G_i = F_{\mathfrak{U}}(H). $$

We now are able to extend Example 4.1.

Example 4.4. Let G be the $\mathfrak{U}$-free product of three copies of the integers, say $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$. We have seen (Proposition 4.2) that the group free product H of $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ is simply the subgroup of G generated by $\mathbf{Z}_1 \cup \mathbf{Z}_2 \cup \mathbf{Z}_3$. Let z_i be the unit of $\mathbf{Z}_i$ for each $i = 1, 2, 3$. Clearly, H is the free group of rank 3, and $\{z_1, z_2, z_3\}$ is a set of free generators of H. Now, let

$$ \begin{aligned} w_1 &= z_1 + z_2 + z_3, \\ w_2 &= z_1 + z_2, \text{ and} \\ w_3 &= z_2 + z_3. \end{aligned} $$

Write W_i for the subgroup of G generated by w_i, and note that $\{w_1, w_2, w_3\}$ is also a set of free generators for H. As in Example 4.1 we can readily determine that $F_{\mathfrak{U}}(H) = G$ if H is given the partial order inherited from G. However, also as before, it is easily seen that G cannot be the $\mathfrak{U}$-free product of W_1, W_2, and W_3.

5. Further applications and open problems. There are other possible applications to the representation theory developed in §2. Let us begin by first mentioning a result which is already complete. Let $\mathfrak{D}_e$ be the class of distributive lattices with a distinguished element e. Every abelian l-group can be viewed as a member of $\mathfrak{D}_e$ with $0 = e$. Using Theorem 2.6 we show in a forthcoming paper [**27**] that the $\mathfrak{D}_e$-free product of a family $(G_i \mid i \in \mathfrak{I})$ of abelian l-groups is the sublattice of the abelian l-group free product generated by $\bigcup_{i \in \mathfrak{I}} G_i$. The corresponding result for the class $\mathfrak{L}$ of all l-groups has been established by Franchello in [**10**]. As every member of $\mathfrak{D}_e$ can be embedded (with a $\mathfrak{D}_e$-monomorphism) in an abelian l-group, this establishes an important link between the classes of abelian l-groups and distributive lattices with distinguished element.

Let $\mathfrak{U}$ be a variety of algebraic systems $A = (A, (f_i \mid \in \mathfrak{I}), \wedge, \vee)$, where $|\mathfrak{I}| \leq \aleph_0$, each fundamental operation f_i has finite arity, and $(A, \wedge, \vee)$ is a lattice. An algebra $A \in \mathfrak{U}$ is said to satisfy the countable chain condition if each chain in A has cardinality $\leq \aleph_0$. If $a \in A \in \mathfrak{U}$, call a non-empty subset $S \subseteq A$ a-disjoint provided that $x \wedge y = a$, whenever x and y are distinct elements of S.

THEOREM 5.1 *Let $\mathfrak{U}$ be as in the preceding paragraph, and let F be the $\mathfrak{U}$-free algebra on a non-empty set X. Then*:

(i) *F satisfies the countable chain condition.*

(ii) *For each $a \in F$, any a-disjoint subset of F has cardinality $\leq \aleph_0$.*

Theorem 5.1 (i) was established by Galvin and Jónsson [**12**, Lemma 5] for the class of lattices. However, their proof can be easily modified to yield the aforementioned general theorem by concentrating only on those automorphisms of F induced by permutations on X instead of using all automorphisms of F. A slightly weaker statement than that of Theorem 5.1(ii) is implicit in an interesting but not well-known paper of Amemiya [**3**]. Several special cases of the theorem above have appeared in the literature. In addition to the Galvin-Jónsson paper [loc. cit.], see for example Adams and Kelly [**1**] and [**2**], Balbes [**4**], Bleier [**7**], Horn [**17**], Jónsson [**20**], Sanin [**28**], and Weinberg [**32**]. Balbes [loc. cit.] proved the stronger result that in a free distributive lattice every a-disjoint set is finite.

Specializing Theorem 5.1 in the context of l-groups, we see that free objects in any l-group variety $\mathfrak{U}$ satisfy conditions (i) and (ii). Now, each such object is the $\mathfrak{U}$-free product of copies of $\mathbf{Z} \boxplus \mathbf{Z}$, and, of course, $\mathbf{Z} \boxplus \mathbf{Z}$ satisfies the countable chain condition and contains only finite disjoint subsets. This motivates the next problem.

*Problem*5.2. Let $\mathfrak{U}$ be a variety of l-groups and $(G_i \mid i \in \mathcal{I}) \subseteq \mathfrak{U}$.

(i) Does ${}^{\mathfrak{U}}\bigsqcup_{i \in \mathcal{I}} G_i$ satisfy the countable chain condition if each G_i satisfies this condition?

(ii) Does every disjoint subset of ${}^{\mathfrak{U}}\bigsqcup_{i \in \mathcal{I}} G_i$ have cardinality $\leq \aleph_0$ if each disjoint subset of every factor G_i is finite (or has cardinality $\leq \aleph_0$)? In regards to the preceding problem see Adams and Kelly [**1**] and [**2**], Grätzer and Lakser [**14**], and Lakser [**21**]. The representation theory developed in this paper should prove useful in settling Problem 5.2 in the case where $\mathfrak{U}$ is the variety of abelian l-groups.

Problem 5.3. Are there any nontrivial l-group varieties that have the refinement property for free products?

The representation given by Weinberg [**31**] for free abelian l-groups has been generalized and adapted to other classes as was mentioned in the introduction. This inspires the next problem.

Problem 5.4. Give a reasonable representation for free products in each of the following classes: all l-groups, representable l-groups, archimedean l-groups, f-modules.

References

1. M. E. Adams and D. Kelly, *Chain conditions in free products of lattices*, Algebra Universalis, **7** (1977), 235-243.
2. M. E. Adams and D. Kelly, *Disjointness conditions in free products of lattices*, Algebra Universalis, **7** (1977), 245–258.
3. I. Amemiya, *Countable decomposability of vector lattices*, J. Fac. Hokkaido Univ., **19** (1966), 111–113.
4. R. Balbes, *Projective and injective distributive lattices*, Pacific J. Math., **21** (1967), 405–420.
5. A. Bigard, *Free lattice-ordered modules*, Pacific J. Math., **49** (1973), 1–6.
6. A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et Anneaux Réticulés*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.
7. R. D. Bleier, *Free l-groups and vector lattices*, J. Austral. Math. Soc., **19** (1975), 337–342.
8. P. Conrad, *Lattice ordered groups*, Tulane University, 1970.
9. _____, *Free lattice-ordered groups*, J. Algebra, **16** (1970), 191–203.
10. J. D. Franchello, *Sublattices of free products of lattice ordered groups*, Algebra Universalis, **8** (1978), 101–110.
11. L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.
12. F. Galvin and B. Jónsson, *Distributive sublattices of a free lattice*, Canad. J. Math., **13** (1961), 265–272.
13. G. Grätzer, *Universal Algebra*, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin, 1979.
14. G. Grätzer and H. Lakser, *Chain conditions in distributive free products of lattices*, Trans. Amer. Math. Soc., **144** (1969), 301–312.
15. G. Grätzer and J. Sichler, *Free decompositions of a lattice*, Canad. J. Math., **28** (1975), 276–285.
16. W. C. Holland and E. Scrimger, *Free products of lattice ordered groups*, Algebra Universalis, **2** (1972), 247–254.
17. A. Horn, *A property of free Boolean algebrs*, Proc. Amer. Math. Soc., **19** (1968), 142–143.
18. R. E. Johnson, *Free products in varieties of ordered semigroups*, Proc. Amer. Math. Soc., **19** (1968), 697–700.
19. B. Jónsson, *Sublattices of a free lattice*, Canad. J. Math., **13** (1961), 256–264.
20. _____, *Varieties of lattices: Some open problems*, Colloq. Math. Soc. János Bolyai (to appear).
21. H. Lakser, *Disjointness condition in free products of distributive lattices: an application of Ramsey's theorem*, Proc. Univ. of Houston Lattice Theory Conference, Houston, 1973, 156–168.
22. J. Martinez, *Free products in varieties of lattice ordered groups*, Czech. Math. J., **22** (97) (1972), 535–553.
23. _____, *Free products of abelian l-groups*, Czech. Math. J., **23** (98) (1973), 349–361.
24. K. R. Pierce, *Amalgamations of lattice ordered groups*, Trans. Amer. Math. Soc., **172** (1972), 249–260.
25. R. S. Pierce, *Introduction to the Theory of Abstract Algebras*, Holt, Rinehart, and Winston, New York, 1968.
26. W. B. Powell, *Projectives in a class of lattice ordered modules*, Algebra Universalis, **13** (1981), 24–40.
27. W. B. Powell and C. Tsinakis, *The distributive lattice free product as a sublattice of the abelian l-group free product*, J. Austral. Math. Soc., (to appear).

28. N. Sanin, *O proizvedenii topologiceskih prostranstv*, Trudy Mat. Inst. Steklov. **24** (1948).
29. D. Topping, *Some homological pathology in vector lattices*, Canad. J. Math., **17** (1965), 411–428.
30. A. A. Vinogradov, *On the free product of ordered groups*, (Russian) Mat. Sbornik, **25** (1949), 163–168.
31. E. C. Weinberg, *Free lattice-ordered abelian groups*, Math. Ann., **151** (1963), 187–199.
32. ______, *Free lattice-ordered abelian groups II*, Math. Ann., **159** (1965), 217–222.

Received January 1, 1981.

OKLAHOMA STATE UNIVERSITY
STILLWATER, OK 74078
AND
VANDERBILT UNIVERSITY
NASHVILLE, TN 37235