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CLOSED IDEALS OF $l^1(\omega_n)$ WHEN $\{\omega_n\}$ IS STAR-SHAPED

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Let $A = l^1(\omega_n)$ be a radical Banach algebra of power series where the weight $\{\omega_n\}$ is star-shaped. Let T be the operator of right translation on A. We give sufficient conditions for all closed ideals of A to be standard. These cases are more general than those previously considered, since in all these cases, T is unicellular but not a basis operator. We also construct a large class of such algebras A in which there are elements x such that the closed ideal $(Ax)^-$ is standard, but the algebraic ideal Ax contains no power of z.

1. Introduction. In this paper we study algebras $A = l^1(\omega_n)$ where

$$l^1(\omega_n) = \left\{ \sum_{n=0}^{\infty} \alpha_n z^n : \sum_{n=0}^{\infty} |\alpha_n| \omega_n < \infty \right\}.$$

We shall be concerned entirely with the case when $\{\omega_n\}$ is a star-shaped weight, i.e. essentially that the region below the graph of $\ln \omega_n$ is illuminated by the origin (see Definition 2.1). For these weights A is a radical Banach algebra of power series with unit adjoined, although in the following we shall refer to these algebras simply as radical Banach algebras. The multiplication is, of course, the usual multiplication of formal power series. There are obvious closed ideals in $A = l^1(\omega_n)$, the so called *standard* ideals:

$$K(\infty) \equiv \{0\}$$

and

$$K(n) \equiv \left\{ \sum_{j=n}^{\infty} \alpha_j z^j \in A \right\}, \qquad n = 0, 1, 2, \ldots.$$

Any other closed ideals are referred to as *non-standard* ideals. At present it is not known whether there are *any* weights $\{\omega_n\}$ such that $l^1(\omega_n)$ is a radical Banach algebra and contains a non-standard ideal. So called *Schauder type* ideals, which would have to be non-standard, have been conjectured to exist and an erroneous construction [**5**, p. 205] appears in the literature (see [**7**, 2. Schauder Type Ideals] for a specific discussion of the error). We note that if one removes the restriction that $l^1(\omega_n)$ be an *algebra*, examples can be given where the right shift operator on The Banach space $l^1(\omega_n)$ is quasinilpotent and has non-standard closed invariant subspaces [**6**]. In these examples $l^1(\omega_n)$ is very far from being an algebra. In Section two we show that with some regularity conditions, these ideals cannot exist in $A = l^1(\omega_n)$ if $\{\omega_n\}$ is a star-shaped weight (see Theorem 2.8). Also in Section two we show that if $\{\omega_n\}$ is star-shaped and $(\omega_n)^{1/n^2} \to 0$, a simplification of [7, Theorem 4.9], giving necessary and sufficient conditions for the algebraic ideal Ax to contain a power of z, can be given (see Theorem 2.10). We use this result later in the proof of Proposition 4.2. Hence, the situation of $l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight is somewhat nicer than the general case of an arbitrary weight.

It is well known that a non-trivial closed ideal is standard if and only if it contains a power of z [4, Lemma 4.5]. This essentially follows from the fact that a closed ideal of finite codimension must be standard. It is also clear that all closed ideals are standard if and only if all principal closed ideals are standard. A condition equivalent to all closed ideals being standard is that the operator T of right translation be *unicellular*, i.e. its closed invariant subspaces are totally ordered [5, p. 189]. This will clearly follow if all non-trivial principal algebraic ideals contain a power of z. Certain equivalent conditions and sufficient conditions for this are known, [2], [3, Theorem 3.15], [4, Theorem 4.1]. A stronger condition is to require that $l^1(\omega_{n+p})$ be a Banach algebra for each p. This is equivalent [5, Theorem 1, p. 91] to requiring T to be a *basis operator* [5, Definition, p. 189]. Hence, we have the following three conditions on $A = l^1(\omega_n)$, each one implying the next:

(I) T is a *basis operator* [5, p. 189].

(II) The algebraic ideal Ax contains a power of z for all x non-zero in A.

(III) All closed ideals $(Ax)^-$ contain a power of z and hence are standard, x non-zero in A. Equivalently, T is *unicellular*.

In Section three we give sufficient conditions on a star-shaped weight $\{\omega_n\}$ for (III) to hold (see Theorem 3.7, Theorem 3.9). These cases are more general than those previously considered, since in all these cases, T is *not* a basis operator. Hence (III) does not imply (I) and this answers the conjecture [5, p. 192] whether a unicellular operator need be a basis operator in the negative (see Corollary 3.8).

Badé, Dales, and Laursen [1] have given the first example of an algebra $A = l^1(\omega_n)$ and an element x in A such that either $(Ax)^-$ is nonstandard or $(Ax)^-$ is standard and contains no power of z. The weight used is not star-shaped. In the fourth section we give a large class of examples $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight, containing an element x such that $(Ax)^-$ is standard but Ax contains no power of z. In these cases T is unicellular (see Proposition 4.2). This shows that (III) doesn't even imply (II). Also, in these cases $(\omega_n)^{1/n^2} \to 0$ so we make use of our earlier results in this situation from Section two.

A remaining question is whether (II) implies (I). The author very recently has been informed that K. B. Laursen has constructed a weight such that (II) holds but not (I). This would also give an example of a unicellular operator which is not a basis operator but in the other direction to our examples in Proposition 4.2.

2. Star-shaped weights. We shall be considering the following class of weights.

DEFINITION 2.1. We say that $\{\omega_n\}$ is a *star-shaped* weight provided the following conditions hold:

(i) $\omega_0 = 1$

(ii) $(\omega_n)^{1/n} \to 0$ as $n \to \infty$

(iii) $t > s$ implies $(\omega_t)^s \leq (\omega_s)^t$.

If $\{\omega_n\}$ is a star-shaped weight, it is easily seen that $\omega_{n+m} \leq \omega_n\omega_m$. It is also clear that ω_n and $(\omega_n)^{1/n}$ are non-increasing. Hence $l^1(\omega_n)$ is a radical Banach algebra of power series. We use the term star-shaped since condition (iii) above implies that the set $\{(n, m) \in \mathbf{Z}^2\colon\ m \leq \ln \omega_n\}$ is "illuminated" by the origin. We wish to note that star-shaped weights need not be algebra weights for l^p if $p > 1$. It is easy to construct star-shaped weights since they are determined by prescribed drops of $(\omega_n)^{1/n}$. To simplify notation we let $u_n = -\log \omega_n$ in *all* the following. We have

DEFINITION 2.2. Let $\{n(k)\}$ be an increasing sequence of nonnegative integers with $n(0) = 0$. We say that the *star-shaped* weight $\{\omega_n\}$ is *induced* by $\{\omega_{n(k)}\}$ provided the following hold:

(i) $n(k+1)u_{n(k)} < n(k)u_{n(k+1)}$, all $k \geq 1$.

(ii) For $n(k) \leq j < n(k+1)$ and $k \geq 0$ we have

$$u_j = (j/n(k))u_{n(k)}.$$

In the converse direction we have the following

PROPOSITION 2.3. *Let $\{n(k)\}$ be an increasing sequence of nonnegative integers with $n(0) = 0$. Suppose values $\{\omega_{n(k)}\}$ are assigned satisfying:*

(i) $n(k+1)u_{n(k)} < n(k)u_{n(k+1)}$, *all* $k \geq 1$,

(ii) $(1/n(k))u_{n(k)} \to \infty$ *as* $k \to \infty$,

(iii) $u_{n(0)} = 0$, $u_{n(1)} \geq 0$.

Then there is a unique star-shaped weight $\{\omega_n\}$ induced by $\{\omega_{n(k)}\}$.

Proof. Let $k \geq 0$ and for $n(k) \leq j < n(k+1)$ define

$$u_j = (j/n(k))u_{n(k)}.$$

That $\{\omega_n\}$ is a star-shaped weight is easily verified, and it is clear that $\{\omega_n\}$ is induced by $\{\omega_{n(k)}\}$.

It may happen that a star-shaped weight $\{\omega_n\}$ is only induced by itself. This will happen if $(\omega_n)^{1/n}$ is strictly decreasing, e.g. in the case $\omega_n = e^{-n^2}$. We are more interested in the cases when $\{\omega_n\}$ is induced by $\{\omega_{n(k)}\}$ where the sequence $n(k)$ increases fairly rapidly, for the following reason.

LEMMA 2.4. *Let $\{\omega_n\}$ be a star-shaped weight induced by $\{\omega_{n(k)}\}$. Suppose $n(k+1) \geq 2n(k)$ eventually. Let $\overline{\omega}_n = \omega_{n+1}$, $n = 0, 1, 2, \ldots$. Then*

(i) *$l^1(\overline{\omega}_n)$ is not a Banach algebra.*

(ii) *The right shift operator T on $l^1(\omega_n)$ is not a basis operator.*

Proof. As before let $u_n = -\log \omega_n$ and $\bar{u}_n = -\log \overline{\omega}_n$. For large k,

$$2\bar{u}_{n(k)-1} = 2u_{n(k)} + \bar{u}_{2n(k)-2} - u_{2n(k)-1}.$$

Since $n(k) \leq 2n(k) - 1 < n(k+1)$

$$u_{2n(k)-1} = ((2n(k) - 1)/n(k))u_{n(k)},$$

and hence

$$2\bar{u}_{n(k)-1} - \bar{u}_{2n(k)-2} = (1/n(k))u_{n(k)} \to \infty \quad \text{as } k \to \infty,$$

and it follows

$$\{\bar{u}_n + \bar{u}_m - \bar{u}_{n+m}\} \quad \text{is unbounded above.}$$

Hence $l^1(\overline{\omega}_n)$ cannot be a Banach algebra. By [**5**, Theorem 1, p. 191] it follows that T cannot be a *basis operator*.

Hence star-shaped weights of the above type are a good source for algebras where T, the operator of right translation, is not a basis operator. With some regularity conditions, $l^1(\omega_n)$, where $\{\omega_n\}$ is a star-shaped weight doesn't contain *Schauder type* ideals either. We give the definition of an *admissible Schauder type* ideal below. It follows by [**7**, Proposition 2.1] that certain restrictions are placed on a *Schauder type* ideal. We refer the reader to [**7**, §2] for a technical discussion which motivates the definition below. As noted in the introduction, it is not known if non-standard ideals exist in *any* algebra $l^1(\omega_n)$ and a Schauder type ideal would be non-standard (see [**7**, Definition 1.2 and discussion following]).

DEFINITION 2.5. We say that $x = \sum_{k=1}^{\infty} x_k z^{m(k)}$ generates an *admissible Schauder type* ideal if there are disjoint intervals of non-negative integers I_k and a positive integer c such that:

(i) The set of non-negative integers is the disjoint union $\bigcup_{k=0}^{\infty} I_k$.

(ii) For $k \geq c$, $m(k) \in I_{k-1}$.

(iii) For $m \in I_{k-1}$ and $k \geq c$, then $m(k) + m < m(k+1)$.
(iv) For $m \in I_{k-1}$ let

$$a(m) = \left\| \frac{T^m x}{x_k \omega_{m(k)+m}} - \frac{z^{m(k)+m}}{\omega_{m(k)+m}} \right\|.$$

Then

$$\sum_{k=1}^{\infty} \sum_{m \in I_{k-1}} a(m) < \infty.$$

Condition (ii) essentially requires fairly rapid growth of the $m(k)$'s. For example, $m(k)$ must increase faster than 2^k. It is the use of $m(k) \approx 2^k$ which makes the construction in [5] fail. We now have the following lemma. We *emphasize* that $\{\omega_n\}$ is *not* assumed to be induced by $\{\omega_{m(k)}\}$.

LEMMA 2.6. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight. Suppose $x = \sum_{k=1}^{\infty} x_k z^{m(k)}$ in A generates an admissible Schauder type ideal. Then there is a constant B and an integer k_0 such that*

$$\left(\omega_{m(k+1)}\right)^{1/m(k+1)} \geq \left[\left(\frac{|x_1|}{|x_k| \, \omega_{m(k)} B}\right)\left(\omega_{m(k)}\right)\right]^{1/(m(k)-m(1))},$$

for all $k \geq k_0$.

Proof. Since $\Sigma\, a(m) < \infty$ by condition (iv) in Definition 2.5, there is a constant C such that $a(m) \leq C$, all m. By (ii) and (iii) of Definition 2.5 it follows that the I_k are eventually *consecutive*, i.e. if $I_k = [a, b]$, $I_{k+1} = [c, d]$ then $c = b + 1$. Choose k_0 sufficiently larger than $\max\{c, 2\}$ so this holds for $k \geq k_0$ and for such a k let h be the largest integer in I_{k-1}. Let $m = m(k+1) - m(k) - h$. Then $m(k+1) - m(1) - m \in I_k$ and

$$C \geq \left| \frac{x_1 \omega_{m(k+1)-m}}{x_{k+1} \omega_{m(k+1)+m(k+1)-m(1)-m}} \right|.$$

Let $s = m(k+1) + m(k+1) - m(k) - m$ and $t = m(k+1) + m(k+1) - m(1) - m$. Since $\{\omega_n\}$ is star-shaped and $s \leq t$, it follows that

$$\omega_t \leq (\omega_s)^{t/s} = (\omega_s)(\omega_s)^{(m(k)-m(1))/s}.$$

This, together with the first inequality, implies

$$C \geq \left| \frac{x_k \omega_{m(k+1)-m} x_1}{x_{k+1} \omega_s x_k (\omega_s)^{(m(k)-m(1))/s}} \right|.$$

But $m(k+1)-m-m(k)=h\in I_{k-1}$, so $C\,|\,x_k\omega_{m(k+1)-m}\,|\geq|\,x_{k+1}\omega_s\,|$ and

$$C\geq\frac{1}{C}\left|\frac{x_1}{x_k(\omega_s)^{(m(k)-m(1))/s}}\right|$$

$$\geq\frac{1}{C}\left|\frac{x_1}{x_k(\omega_{m(k+1)})^{(m(k)-m(1))/m(k+1)}}\right|$$

since $s>m(k+1)$

$$\geq\left|\frac{x_1\omega_{m(k)}}{Cx_k\omega_{m(k)}(\omega_{m(k+1)})^{(m(k)-m(1))/m(k+1)}}\right|.$$

If we let $B=C^2$ we then have

$$(\omega_{m(k+1)})^{1/m(k+1)}\geq\left[\left(\frac{|\,x_1\,|}{|\,x_k\,|\,\omega_{m(k)}B}\right)(\omega_{m(k)})\right]^{1/(m(k)-m(1))}$$

and the result follows.

COROLLARY 2.7. *Suppose the hypotheses of Lemma 2.6 hold. Let* $\varepsilon_k=m(1)/(m(k)-m(1))$. *Then there exists* $r\geq\max\{c,2\}$ *such that*:

$$\left(\frac{1}{m(k+1)}\right)u_{m(k+1)}\leq\left(\prod_{s=r}^{k}(1+\varepsilon_s)\right)\left(\frac{1}{m(r)}\right)u_{m(r)}$$

for all $k\geq r$.

Proof. Note that

$$\left(\frac{|\,x_1\,|}{|\,x_k\,|\,\omega_{m(k)}B}\right)\geq 1$$

eventually since $\|x\|=\Sigma\,|\,x_k\,|\,\omega_{m(k)}<\infty$. Also note that

$$\left(\frac{1}{m(k)-m(1)}\right)u_{m(k)}=(1+\varepsilon_k)\left(\frac{1}{m(k)}\right)u_{m(k)}$$

Induction and Lemma 2.6 complete the proof, where r is taken sufficiently larger than the k_0 of Lemma 2.6.

THEOREM 2.8. *Let* $A=l^1(\omega_n)$ *where* $\{\omega_n\}$ *is a star-shaped weight. Then* A *has no admissible Schauder type ideals.*

Proof. Suppose an admissible Schauder type ideal exists, generated by $x = \Sigma\, x_k z^{m(k)}$. By the previous corollary and the fact that

$$(\omega_{m(k+1)})^{1/m(k+1)} \to 0$$

as $k \to \infty$, it follows that

$$\prod_{s=r}^{k} (1 + \varepsilon_s) \to \infty$$

as $k \to \infty$. But then $\Sigma\, \varepsilon_s = \infty$, i.e.

$$\sum_{s=r}^{\infty} \left(\frac{m(1)}{m(s) - m(1)} \right) = \infty.$$

Since $m(k+1) \in I_k$, $k \geq c$, by condition (ii) of Definition 2.5 we have by condition (iii) of that same definition that $m(r+s) > 2^s m(r)$ and

$$\frac{m(1)}{m(r+s) - m(1)} \leq 2^{-s} \left(\frac{m(1)}{m(r) - \dfrac{m(1)}{2^s}} \right)$$

which forces $\Sigma\, \varepsilon_s$ to be convergent, a contradiction, and the result follows.

In the next sections we shall be looking at star-shaped weights where generally

$$(\omega_n)^{1/n^2} \to 0 \quad \text{as } n \to \infty.$$

For such weights we have the following result concerning *equivalent* algebras (see [7, Definition 3.1]).

LEMMA 2.9. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight also satisfying $(\omega_n)^{1/n^2} \to 0$ as $n \to \infty$. Let $A_1 = l^1(\tilde{\omega}_n)$ be another radical Banach algebra of power series such that for some $C > 0$*

$$-Cn \leq u_n - \tilde{u}_n \leq Cn$$

$$\textit{where } u_n = -\log \omega_n \quad \textit{and} \quad \tilde{u}_n = -\log \tilde{\omega}_n.$$

Then if $a \in A_1$, $za \in A$.

Proof. Note:

$$\begin{aligned} u_{n+1} &= u_{n+1} - u_n + u_n \geq ((n+1)/n)u_n - u_n + \tilde{u}_n - Cn \\ &= (1/n)u_n - Cn + \tilde{u}_n = n\big((1/n^2)u_n - C\big) + \tilde{u}_n. \end{aligned}$$

Eventually $u_{n+1} \geq \tilde{u}_n$, which implies the result.

We can then give a simplification, in this case, of [7, Theorem 4.9] which tells when the algebraic ideal Ax contains a power of z. The proof is very similar to that of [7, Theorem 4.9] using [7, Proposition 4.5 and Proposition 4.7] together with the above lemma and we omit it.

THEOREM 2.10. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight also satisfying $(\omega_n)^{1/n^2} \to 0$ as $n \to \infty$. If x is non-zero in A and $(Ax)^-$ is a standard ideal, then the following are equivalent*:

(1) *The algebraic ideal Ax contains a power of z.*

(2) *For some l, $\sup_n \|x_n^*\| \omega_{n+l} < \infty$, where $\{x_n^*\}$ are the biorthogonal functionals* (i.e. $x_n^*(T^m x) = \delta_{n,m}$) *in $((Ax)^-)^*$.*

We shall return to these considerations again in Section four.

3. Unicellular right shift operators. In this section we will consider algebras $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Let $y = \sum_{j=n(c)}^{\infty} \zeta_j z^j$ be an element of A with $\zeta_{n(c)} = 1$. We shall need some results converning the associated sequence $\{c_n\}$ discussed in [7]. However, the following definition and list of results are easily verified, so our treatment here is essentially self-contained.

DEFINITION 3.1. Let y be as above. Let

$$c_0 = \frac{1}{\zeta_{n(c)}} = 1$$

and if $c_0, c_1, \ldots, c_{n-1}$ have been defined, let

$$c_n = -\frac{1}{\zeta_{n(c)}} \sum_{k=0}^{n-1} c_k \zeta_{n(c)+n-k} = -\sum_{k=0}^{n-1} c_k \zeta_{n(c)+n-k}.$$

We shall refer to $\{c_n\}$ as the *associated* sequence for y.

If $A = l^1(\omega_n)$, it is elementary that the dual $A^* = l^\infty(1/\omega_n)$. Let the canonical dual weak-star basis be denoted by $\{e_n^*\}$ (i.e. $e_n^*(z^m) = \delta_{n,m}$). It follows that [7, Lemma 4.2]:

(3.1) (i) $\sum_{n=0}^{N} c_n T^n y = z^{n(c)}$ on $[n(c), n(c) + N]$ where $Tf = zf$ as before, $f \in A = l^1(\omega_n)$.

(3.2) (ii) $\left(\sum_{n=0}^{\infty} c_n z^n\right) y = z^{n(c)}$ as *formal* power series.

(3.3) (iii) If $\chi_m^* \equiv \sum_{n=0}^{m} c_n e_{n(c)+m-n}^* \in A^*$, then $\chi_m^*(T^n y) = \delta_{n,m}$.

DEFINITION 3.2. Let Q_n denote the natural projection of an element in $l^1(\omega_n)$ by restriction to $[n, \infty)$, i.e.

$$Q_n\left(\sum_{i=0}^{\infty} \alpha_i z^i\right) = \sum_{i=n}^{\infty} \alpha_i z^i.$$

We have the following sufficient condition for an element to generate a standard ideal, which is motivated by condition (3.2).

LEMMA 3.3. *Let $A = l^1(\omega_n)$ and $y \in A$, where $y = \sum_{j=n(c)}^{\infty} \zeta_j z^j$, $\zeta_{n(c)} = 1$. Let $\{c_n\}$ be the associated sequence for y. If there is a positive integer m such that for every $\varepsilon > 0$ there exists $k = k(\varepsilon)$, with $n(k+1) > n(c) + m$ satisfying*:

$$\sum_{n=0}^{n(k+1)-n(c)-1} |c_n| \, \|Q_{n(k+1)} T^{n+m} y\| < \varepsilon.$$

Then $(Ay)^-$ is standard.

Proof. We shall show that $z^{n(c)+m} \in (Ay)^-$. This will imply the result since a non-trivial *closed* ideal is standard if and only if it contains a power of z [**4**, Lemma 4.5]. Let $\varepsilon > 0$ and let $k = k(\varepsilon)$ by hypothesis. By (3.1) $\sum_{n=0}^{r} c_n T^n y$ and $z^{n(c)}$ agree on $[n(c), n(k+1) - 1]$ where $r = n(k+1) - n(c) - 1$. Hence $\sum_{n=0}^{r} c_n T^{n+m} y$ and $z^{n(c)+m}$ agree on $[n(c), n(k+1) - 1 + m]$ and, in particular, on $[n(c), n(k+1) - 1]$. Thus

$$\left\| z^{n(c)+m} - \sum_{n=0}^{r} c_n T^{n+m} y \right\| = \left\| Q\left(z^{n(c)+m} - \sum_{n=0}^{r} c_n T^{n+m} y \right) \right\|$$

where $Q = Q_{n(k+1)}$, and the above is

$$= \left\| \sum_{n=0}^{r} c_n Q T^{n+m} y \right\|$$

since $n(c) + m < n(k+1)$. Then the above is

$$\leq \sum_{n=0}^{r} |c_n| \, \|Q T^{n+m} y\| < \varepsilon.$$

Since ε was arbitrary, $z^{n(c)+m} \in (Ay)^-$ and the result follows. We have also shown that a *subsequence* of the sequence of partial sums of $\sum_a c_n T^{n+m} y$ converges to $z^{n(c)+m}$. To put Lemma 3.3 to use we will see that a weight with large "drops" will suffice.

We need two more lemmas, one concerning the size of $\|QT^n y\|$, the other concerning the size of $|c_n|$. We continue to write $u_n = -\log \omega_n$.

LEMMA 3.4. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Let $y = \sum_{j=1}^{\infty} \zeta_j z^j$ be an element of A. Suppose there is $b \geq 1$ such that*

$$\left(\frac{1}{n(k+1)}\right)^2 u_{n(k+1)} \geq \left(\frac{b}{n(k+1)-1}\right) u_{n(k+1)-1}$$

for fixed k. Then

$$\| Q_{n(k+1)} T^n y \| \leq (\omega_{n(k+1)-1})^{bn} \| y \|,$$

for all n.

Proof. We note that since $n(k) \leq n(k+1) - 1 < n(k+1)$

$$\left(\frac{b}{n(k+1)-1}\right) u_{n(k+1)-1} = \left(\frac{b}{n(k)}\right) u_{n(k)}.$$

Now let $Q = Q_{n(k+1)}$, let n be fixed, and let $s = \max\{n(k+1) - n, 0\}$. Then

$$\begin{aligned}
\| QT^n y \| &= \left\| Q \sum_{i=1}^{\infty} \zeta_i z^{i+n} \right\| = \left\| \sum_{i=s}^{\infty} \zeta_i z^{i+n} \right\| \\
&= \sum_{i=s}^{\infty} |\zeta_i| \omega_{i+n} \\
&= \sum_{i=s}^{n(k+1)-1} |\zeta_i| \omega_{i+n} + \sum_{i=n(k+1)}^{\infty} |\zeta_i| \omega_{i+n}.
\end{aligned}$$

Using the fact that $\{\omega_n\}$ is star-shaped, the above is

$$\leq \sum_{i=s}^{n(k+1)-1} |\zeta_i| (\omega_{n(k+1)})^{(i+n)/n(k+1)} + \sum_{i=n(k+1)}^{\infty} |\zeta_i| (\omega_i)^{(i+n)/i}$$

$$\leq (\omega_{n(k+1)})^{n/n(k+1)} \sum_{i=s}^{n(k+1)-1} |\zeta_i| (\omega_{n(k+1)})^{i/n(k+1)} + \sum_{i=n(k+1)}^{\infty} |\zeta_i| \omega_i (\omega_i)^{n/i}$$

$$\leq \left((\omega_{n(k+1)-1})^{bn}\right) \sum_{i=s}^{n(k+1)-1} |\zeta_i| \omega_i + (\omega_{n(k+1)})^{n/n(k+1)} \sum_{i=n(k+1)}^{\infty} |\zeta_i| \omega_i.$$

Note the crucial use in all the preceding of the star-shaped nature of $\{\omega_n\}$ and the sharp decrease of the weight at $n(k+1)$. The above is then

$$\leq (\omega_{n(k+1)-1})^{bn} \sum_{i=s}^{\infty} |\zeta_i| \omega_i \leq (\omega_{n(k+1)-1})^{bn} \| y \|,$$

and the result follows.

Although no special assumptions on y were needed in the previous lemma, we will need them in the next lemma.

LEMMA 3.5. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Let $y = \sum_{j=n(c)}^{\infty} \zeta_j z^j$ be an element of A where $\zeta_{n(c)} = 1$. Let k be fixed, $k > c$, and let $d = n(k) - n(c) \geq 4$. Define*

(i) $N \equiv (\underset{n(c) \leq i < n(k)}{\text{maximum}} |\zeta_i|) \vee 1.$

(ii) $R \equiv (\underset{n(c) \leq i < n(k+1)}{\text{maximum}} |\zeta_i|) \vee 1.$

(iii) $M \equiv (dN)^d$.

Then if $r \leq d^{d-1}$ and $\{c_n\}$ is the associated sequence for y

$$|c_{rd+j}| \leq M^r (dN)^j R^r \leq M^{r+1} R^r$$

for $j = 0, 1, 2, \ldots, d-1$, provided $(rd + j) < n(k+1) - n(c)$.

Proof. It is clear that $c_0 = 1$. Since $c_1 = -c_0 \zeta_{n(c)+1}$ it follows that $|c_1| \leq N \leq dN$. Then since $c_2 = -c_1 \zeta_{n(c)+1} - c_0 \zeta_{n(c)+2}$, it follows that $|c_2| \leq dN^2 + N \leq (dN)^2$. In general it is easily seen that $|c_j| \leq (dN)^j$ for $j = 0, 1, 2, \ldots, d-1$. We suppose the result holds up to and including $|c_{rd+j}|$. We consider $|c_n|$ where $n = rd + j + 1$ and have three cases:

Case I. $2 \leq j + 1 \leq d - 1$. Then using the equation $c_n = -\sum_{k=0}^{n-1} c_k \zeta_{n(c)+n-k}$ and "grouping terms" we have

$$|c_n| \leq \sum_{s=0}^{r-2} \sum_{i=0}^{d-1} |c_{sd+i}| R + \sum_{i=0}^{j+1} |c_{(r-1)d+i}| R + \sum_{i=j+2}^{d-1} |c_{(r-1)d+i}| N$$

$$+ \sum_{i=0}^{j-1} |c_{rd+i}| N + |c_{rd+j}| N$$

$$\leq (r-1)\big(d(M^{r-1}R^{r-1}) + (j+2)(M^r R^r) + (d-j-2)(M^r R^{r-1} N)$$

$$+ (j)\big(M^r (dN)^{j-1} R^r N\big) + \big(M^r (dN)^j R^r N\big)$$

$$\leq 4\big(M^r (dN)^j R^r N\big) \qquad (\text{since } j \geq 1)$$

$$\leq M^r (dN)^{j+1} R^r \leq M^{r+1} R^r.$$

Case II. $j+1=1$, i.e. $n=rd+1$. In a similar manner to case I we may write

$$\begin{aligned}
|c_n| &\le \sum_{s=0}^{r-2}\sum_{i=0}^{d-1}|c_{sd+i}|R+\sum_{i=0}^{1}|c_{(r-1)d+i}|R \\
&\quad+\sum_{i=2}^{d-1}|c_{(r-1)d+i}|N+|c_{rd}|N \\
&\le (r-1)\big(d(M^{r-1}R^{r-1})\big)+\big(2\big(M^{r-1}(dN)^{d-1}R^r\big)\big) \\
&\quad+(d-2)\big(M^{r-1}(dN)^{d-1}R^{r-1}N\big)+M^rR^rN \\
&\le 3(M^rR^rN)\le M^r(dN)R^r\le M^{r+1}R^r.
\end{aligned}$$

Case III. $j+1=d$ and $n=rd+j+1=(r+1)d$. Similar "grouping" as in the previous case shows

$$\begin{aligned}
|c_n| &\le \sum_{s=0}^{r-1}\sum_{i=0}^{d-1}|c_{sd+i}|R+|c_{rd}|R+\sum_{i=1}^{d-2}|c_{rd+i}|N+|c_{rd+d-1}|N \\
&\le \Big(r\big(d\big(M^{r-1}(dN)^{d-1}R^r\big)\big)\Big)+M^rR^{r+1} \\
&\quad+(d-2)\big(M^r(dN)^{d-2}R^rN\big)+M^r(dN)^{d-1}R^rN \\
&\le 4\big(M^r(dN)^{d-1}R^{r+1}N\big)\le M^{r+1}R^{r+1},
\end{aligned}$$

and the result follows.

We now seek conditions on $\{\omega_{n(k)}\}$ which will ensure that $\|QT^{n+1}y\|$ is small enough to cancel $|c_n|$.

LEMMA 3.6. *Let $A=l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Suppose there is an integer $c\ge 4$ such that $2n(k)\le n(k+1)<(\frac{1}{2})n(k)^{((1/2)n(k))}$ for all $k\ge c$. Suppose the following conditions on $\{\omega_{n(k)}\}$ are satisfied*:

(i) $(1/n(k+1))^2u_{n(k+1)}\ge(3/(n(k+1)-1))u_{n(k+1)-1}, \quad k\ge c.$

(ii) $u_{n(k)}\ge 3n(k+1)\log n(k), \qquad k\ge c.$

Let $y=\sum_{j=n(c)}\zeta_jz^j$ be an element of A with $\zeta_{n(c)}=1$. Then $(Ay)^-$ is standard.

Proof. We remark that by the term "induced" we are always assuming $n(k+1)u_{n(k)} < n(k)u_{n(k+1)}$, $k \geq 1$, and $(\omega_n)^{1/n} \to 0$ (although (ii) implies the latter also). Furthermore, we note condition (i) is equivalent to

$$\left(\frac{1}{n(k+1)}\right)^2 u_{n(k+1)} \geq \left(\frac{3}{n(k)}\right) u_{n(k)} \qquad k \geq c.$$

For fixed y it is clear there exists $S \geq 1$ such that $|\zeta_i| \leq S(\omega_i)^{-1}$, all i. Let k be fixed and sufficiently larger than c so that

$$d = n(k) - n(c) > \left(\frac{2}{3}\right) n(k). \tag{3.4}$$

Suppose $0 \leq n < n(k+1) - n(c)$. Then $n = rd + j$ where $0 \leq j \leq d-1$ and $rd + j < n(k+1) - n(c)$. Thus $r < n(k+1)/d$ and since $(1/2)n(k) < d$ we have that $r < d^{d-1}$. Also note that since $c \geq 4$, then $d \geq 4$. Let N, R and $M = (dN)^d$ be as in Lemma 3.5. Then Lemma 3.4, Lemma 3.5 and condition (i) imply that

$$|c_n| \, \|QT^{n+1}y\| \leq M^r (dN)^j R^r (\omega_{n(k+1)-1})^{3n+3} \|y\|.$$

where $Q = Q_{n(k+1)}$, and the above is

$$\leq ABC(\omega_{n(k+1)-1})^2 \|y\|$$

where

$$A = d^{rd+j}(\omega_{n(k+1)-1})^n,$$

$$B = N^{rd+j}(\omega_{n(k+1)-1})^n$$

and

$$C = R^r(\omega_{n(k+1)-1})^n.$$

First, note that

$$\begin{aligned} A &\leq d^{rd+j}(\omega_{n(k)})^n \\ &\leq d^{rd+j}(n(k))^{-3n(k+1)n} \quad \text{by (ii)} \\ &\leq 1, \quad \text{since } d \leq n(k). \end{aligned}$$

Second,

$$\begin{aligned} B &\leq N^{rd+j}(\omega_{n(k)})^n \\ &\leq N^{rd+j}(\omega_{n(k)-1})^{3n(n(k))^2/(n(k)-1)} \quad \text{by (i)} \\ &\leq N^n(\omega_{n(k)-1})^n. \end{aligned}$$

But since $N\omega_{n(k)-1} \le S$, the above is

$$\le S^n.$$

Third, we note first that $r \le n$. We also note that $R\omega_{n(k+1)-1} \le S$, hence

$$\begin{aligned} C &\le R^r(\omega_{n(k+1)-1})^n \\ &\le S^n. \end{aligned}$$

Thus

$$|c_n| \, \|QT^{n+1}y\| \le S^{2n(k+1)}(\omega_{n(k+1)-1})^2 \|y\| \le S^{2n(k+1)}(\omega_{n(k)})(\omega_{n(k)})\|y\|$$

$$\le \left(\frac{S}{n(k)}\right)^{3n(k+1)} (\omega_{n(k)})\|y\|, \quad \text{by (ii)}.$$

Since the above holds for all n such that $0 \le n < n(k+1) - n(c)$, we have that

$$\sum_{n=0}^{n(k+1)-n(c)-1} |c_n| \, \|QT^{n+1}y\| \le n(k+1)(\omega_{n(k)})\left(\frac{S}{n(k)}\right)^{3n(k+1)} \|y\|$$

where $Q = Q_{n(k+1)}$. This holds for all sufficiently large k. But $n(k+1)(\omega_{n(k)}) \to 0$ as $k \to \infty$ by (ii). Also, eventually $S/n(k)$ is less than one. Hence we conclude that given $\varepsilon > 0$ there is $k = k(\varepsilon)$ such that

$$\sum_{n=0}^{n(k+1)-n(c)-1} |c_n| \, \|QT^{n+1}y\| < \varepsilon,$$

where $Q = Q_{n(k+1)}$. By Lemma 3.3, $(Ay)^-$ is then standard, completing the proof of the lemma.

We finally come to our major result. Note (ii) implies there is a sharp decrease in the weight at $n(k+1)$.

THEOREM 3.7. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Suppose the following conditions eventually hold*:

(i) $2n(k) \le n(k+1) < (\frac{1}{2})n(k)^{((1/2)n(k))}$.

(ii) $(1/n(k+1))^2 u_{n(k+1)} \ge (3/(n(k+1)-1))u_{n(k+1)-1}$.

(iii) $u_{n(k)} \ge 3n(k+1)\log n(k)$.

Then the only closed ideals in A are the standard ideals and the right shift operator T is unicellular but not a basis operator.

Proof. It is clear that the hypotheses of Lemma 3.6 are satisfied as long as c is sufficiently large. If $x \in A$ we note that $(Ax)^-$ is standard if and only if $(A(\alpha z^r x))^-$ is standard for $\alpha \neq 0$ and r a positive integer.

Hence, it suffices to show that $(Ay)^-$ is standard where $y = \sum_{j=n(c)}^{\infty} \zeta_j z^j$, $\zeta_{n(c)} = 1$, for c sufficiently large. This is precisely what Lemma 3.6 does; hence all closed ideals of A are standard and T is unicellular. Lemma 2.4 implies that T is not a basis operator, and completes the proof.

Since it is easy to construct star-shaped weights $\{\omega_n\}$ induced by $\{\omega_{n(k)}\}$ where $\omega_{n(k)}$ is chosen inductively to satisfy the above hypotheses and the hypotheses of Proposition 2.3, we have the following:

COROLLARY 3.8. *There exist radical Banach algebras $l^1(\omega_n)$ of power series, where the weight $\{\omega_n\}$ is star-shaped, such that the right shift operator T is unicellular but not a basis operator.*

When the growth of the sequence $\{n(k)\}$ is more restricted, we can simplify Theorem 3.7 as follows. As before $u_n = -\log \omega_n$.

THEOREM 3.9. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Suppose the following conditions eventually hold:*

(i) $2n(k) \leq n(k+1) \leq (k+1)n(k)$

(ii) $(1/n(k+1))^2 u_{n(k+1)} \geq ((3k+3)/n(k))u_{n(k)}$.

Then $(\omega_n)^{1/n^2} \to 0$, the only closed ideals in A are the standard ones and the right shift operator T is unicellular but not a basis operator.

Proof. We first note that since $n(k+1) \leq (k+1)n(k)$ eventually,

$$\left(\frac{1}{n(k+1)}\right)^2 u_{n(k+1)} \geq \left(\frac{3n(k+1)}{n(k)^2}\right) u_{n(k)} \geq \left(\frac{3}{n(k)}\right) u_{n(k)}$$

eventually by (ii). Since $\{\omega_n\}$ is a star-shaped weight, $(\omega_n)^{1/n} \to 0$ and hence eventually $\omega_n \leq 2^{-n}$. Then $n^4\omega_n \to 0$ also and eventually $\omega_n \leq n^{-4}$. Thus, eventually

$$\begin{aligned} u_{n(k)} &\geq \left(\frac{3kn(k)^2}{n(k-1)}\right) u_{n(k-1)} \geq (3kn(k))u_{n(k-1)} \\ &\geq (3kn(k))\log\left(n(k-1)^4\right) \\ &\geq (6kn(k))\log\left(n(k-1)^2\right) \geq (6kn(k))\log n(k). \end{aligned}$$

Again since $n(k+1) \leq (k+1)n(k)$ eventually

$$u_{n(k)} \geq 3n(k+1)\log n(k).$$

The hypotheses of Theorem 3.7 are then satisfied. It only remains to show $(\omega_n)^{1/n^2} \to 0$, but if $n(k) \leq j < n(k+1)$,

$$\left(\frac{1}{j}\right)^2 u_j = \left(\frac{1}{jn(k)}\right) u_{n(k)} \geq \left(\frac{1}{(k+1)n(k)^2}\right) u_{n(k)},$$

since $j < n(k+1) \leq (k+1)n(k)$. The above is then

$$\geq \left(\frac{3k}{n(k-1)(k+1)}\right) u_{n(k-1)} \quad \text{by (ii)}$$

$$\geq \left(\frac{1}{n(k-1)}\right) u_{n(k-1)}$$

since $3k \geq k+1$, and the latter term goes to ∞. Hence $(\omega_n)^{1/n^2} \to 0$ and the result follows.

We shall use Theorem 3.9 later on in the next section.

4. Closed and algebraic principal ideals. Let $A = l^1(\omega_n)$. As before we shall suppose $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. We consider the special case of an element $x = \sum_{k=c}^{\infty} x_k z^{n(k)}$ in A where $x_c = 1$. We shall be interested in both the closed principal ideal $(Ax)^-$ and the algebraic principal ideal Ax. We first need a result in the spirit of Lemma 3.5 for this special case.

LEMMA 4.1. *Let $A = l^1(\omega_n)$ where $\{\omega_n\}$ is a star-shaped weight induced by $\{\omega_{n(k)}\}$. Let $x = \sum_{k=c}^{\infty} x_k z^{n(k)}$ be an element of A where $x_c = 1$. Let k be fixed, $k > c$. Let $d = n(k) - n(c) \geq 4$, $d^{d-1} \geq k$ and $M = (d|x_{k-1}|)^d$. Suppose*

(i) $(\underset{c \leq j < k-1}{\text{maximum}} |x_j|) < |x_{k-1}| < |x_k|$.

(ii) $M^k/|x_k| < \delta < 1/2k$, *some* $\delta > 0$.

Then if $\{c_n\}$ is the associated sequence for y

(1) $|c_{rd+j}| \leq M^r(d|x_{k-1}|)^j |x_k|^r \leq M^{r+1}|x_k|^r$ *for* $j = 0, 1, 2, \ldots, d-1$; $r = 0, 1, 2, \ldots, k$ *and* $rd + j < n(k+1) - n(c)$.

(2) $(1 - r\delta)|x_k|^r \leq |c_{rd}| \leq (1 + r\delta)|x_k|^r$ *for* $r = 1, 2, \ldots, k$.

Proof. Using the notation of Lemma 3.5 note that $N = |x_{k-1}|$ and $R = |x_k|$ by (i). Hence M is the same in both cases, and (1) follows by Lemma 3.5. To show (2), we show the *additional* arguments which must be added to the proof of Lemma 3.5, under our special circumstances. If $r = 1$

$$|c_d| \leq \sum_{i=1}^{d-1} |c_i||x_{k-1}| + |c_0||x_k|$$

and hence

$$|x_k| - M \leq |c_d| \leq |x_k| + M$$

i.e.

$$(1-\delta)|x_k| \leq |c_d| \leq (1+\delta)|x_k| \quad \text{by (ii)}.$$

Suppose (2) holds for values up through r, $r+1 \leq k$. Let $n = (r+1)d$ and referring to this same stage (Case III) in the proof of Lemma 3.5 note that

$$\begin{aligned} |c_n| &\leq |c_{rd}||x_k| + 3M^r(d|x_{k-1}|)^{d-1}|x_k|^r|x_{k-1}| \\ &\leq |c_{rd}||x_k| + M^{r+1}|x_k|^r. \end{aligned}$$

Hence

$$\begin{aligned} |c_n| &\leq (1+r\delta)|x_k|^{r+1} + \frac{M^k}{|x_k|}|x_k|^{r+1} \\ &\leq (1+(r+1)\delta)|x_k|^{r+1} \quad \text{by (ii)}. \end{aligned}$$

Also

$$\begin{aligned} |c_n| &\geq (1-r\delta)|x_k|^{r+1} - \frac{M^k}{|x_k|}|x_k|^{r+1} \\ &\geq (1-(r+1)\delta)|x_k|^{r+1} \quad \text{by (ii)}. \end{aligned}$$

Induction completes the proof.

PROPOSITION 4.2. *Let $\{n(k)\}$ be an increasing sequence satisfying $n(0) = 0$ and*

$$kn(k) < n(k+1) \leq (k+1)n(k), \qquad k = 1, 2, \ldots.$$

Define $x_1 = 1$ and $x_2, x_3, \ldots, x_k, \ldots$ inductively so that the following four conditions hold.

(i) $|x_k| > |x_{k-1}|$.

(ii) $(M(k))^k/|x_k| < 1/4k$

where $d(k) = n(k) - n(1)$ and $M(k) = (d(k)|x_{k-1}|)^{d(k)}$.

(iii) $[(\frac{3}{4})(\frac{1}{k})^{kn(k)}|x_k|^{n(1)/n(k)}k^{n(1)}] \geq k$.

(iv) $|x_k| \geq ((k-1)^{3k}|x_{k-1}|^{3k/n(k-1)})^{n(k)^2}$.

Then letting $\omega_{n(k)} = k^{-n(k)}|x_k|^{-1}$, $k \geq 1$, it follows that there is a unique star-shaped weight $\{\omega_n\}$ induced by $\{\omega_{n(k)}\}$, with $(\omega_n)^{1/n^2} \to 0$. Letting $x = \sum_{k=1}^{\infty} x_k z^{n(k)} \in l^1(\omega_n)$ it also follows that:

(1) *The right shift operator T on $A = l^1(\omega_n)$ is unicellular but not a basis operator.*

(2) *The closed ideal $(Ax)^-$ is standard.*

(3) *The algebraic ideal Ax contains no power of z.*

Proof. We first note that $\omega_{n(1)} = 1$ and $\omega_{n(2)} \le 1/2$. If $k \ge 3$ condition (iv) implies

$$|x_k|^{(1/n(k))^2} \ge (k-1)^{3k} |x_{k-1}|^{3k/n(k-1)}$$

hence

$$\left(\frac{|x_k|}{k^{-n(k)}}\right)^{(1/n(k))^2} \ge \frac{|x_{k-1}|^{3k/n(k-1)}}{(k-1)^{-3k}}$$

$$(4.1) \qquad (\omega_{n(k)})^{(1/n(k))^2} \le (\omega_{n(k-1)})^{3k/n(k-1)}$$

$$\text{i.e. } \left(\frac{1}{n(k)}\right)^2 u_{n(k)} \ge \left(\frac{3k}{n(k-1)}\right) u_{n(k-1)}.$$

Also

$$(4.2) \qquad \left(\frac{1}{n(k)}\right) u_{n(k)} \ge 3kn(k)\left(\frac{1}{n(k-1)}\right) u_{n(k-1)},$$

and

$$(4.3) \qquad \left(\frac{1}{n(k)}\right) u_{n(k)} > \left(\frac{1}{n(k-1)}\right) u_{n(k-1)}, \qquad k \ge 3.$$

Equations (4.2), (4.3) and the fact that $\omega_{n(2)} < \omega_{n(1)} = 1$ show that the hypotheses of Proposition 2.3 are satisfied. Hence there is a unique star-shaped weight $\{\omega_n\}$ induced by $\{\omega_{n(k)}\}$. Condition (4.1) and Theorem 3.9 show that T is unicellular, but not a basis operator and that $(\omega_n)^{1/n^2} \to 0$. Hence all closed ideals are standard and, in particular, $(Ax)^-$ is standard, once it is easily checked that $x \in l^1(\omega_n)$. To prove (3) suppose that Ax contains a power of z. Then by Theorem 2.10, there is an l such that

$$(4.4) \qquad \sup_n \|x_n^*\| \omega_{n+l} < \infty.$$

But equation (3.3) and the fact that $(Ax)^-$ is standard implies that $\chi_n^* = x_n^*$ and $\|x_n^*\| \ge |c_n|/\omega_{n(1)} = |c_n|$. Hence there is B such that

$$(4.5) \qquad \sup_n |c_n| \omega_{n+l} \le B < \infty.$$

Pick any $k > \max\{B, 4\}$ such that $(k-1)n(1) \ge l$. This is possible since $n(1) > 0$. Let $d = n(k) - n(1)$ as before and $n = dk$. It is clear that $d \ge 4$ and $d^{d-1} \ge k$. Equation (4.5) implies that

$$(4.6) \qquad |c_n| \omega_{n+(k-1)n(1)} \le B.$$

But

$$
\begin{aligned}
|c_n|\,\omega_{n+(k-1)n(1)} &\geq |c_{dk}|\,\omega_{kn(k)-n(1)} \\
&\geq \left(1 - k\left(\frac{1}{4k}\right)\right) |x_k|^k \left(\omega_{n(k)}\right)^{(kn(k)-n(1))/n(k)}
\end{aligned}
$$

by (i), (ii) and Lemma 4.1, and the above is

$$
\begin{aligned}
&\geq \left(\frac{3}{4}\right)\left(|x_k|\,\omega_{n(k)}\right)^k \left(\omega_{n(k)}\right)^{-n(1)/n(k)} \\
&\geq \left(\frac{3}{4}\right)\left(\frac{1}{k}\right)^{kn(k)}\left(\frac{1}{|x_k|\,k^{n(k)}}\right)^{-n(1)/n(k)} \\
&\geq \left[\left(\frac{3}{4}\right)\left(\frac{1}{k}\right)^{kn(k)} |x_k|^{n(1)/n(k)} k^{n(1)}\right] \\
&\geq k \qquad \text{by (iii)} \\
&> B,
\end{aligned}
$$

contradicting equation (4.6). Hence Ax contains no power of z and this completes the proof.

Proposition 4.2 gives numerous examples of radical Banach algebras of power series $A = l^1(\omega_n)$, where the weight $\{\omega_n\}$ is star-shaped, which have only the obvious closed ideals, yet not all algebraic principal ideals contain a power of z.

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