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SUPER-PRIMITIVE ELEMENTS

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Given an extension, $R \subseteq T$, of commutative integral domains with identity, we say an element $u \in T$ is super-primitive over R, if u is the root of a polynomial $f \in R[x]$ with $c_R(f)^{-1} = R$, i.e., a super-primitive polynomial. The main purpose of this paper is to provide "super-primitive" analogues to some work of Gilmer-Hoffmann and Dobbs concerning primitive elements. (An element $u \in T$ is called primitive over R, if u is the root of a polynomial $f \in R[x]$ with $c_R(f) = R$.)

1. Introduction. Given an extension, $R \subseteq T$, of commutative integral domains with identity, we say an element $u \in T$ is super-primitive over R, if u is the root of a polynomial $f \in R[x]$ with $c_R(f)^{-1} = R$, i.e., a super-primitive polynomial. By $c_R(f)$, we mean the ideal of R generated by the coefficients of f, and when no confusion may result, we will write $c(f)$. Our primary motivation for investigating super-primitive elements is some work of Gilmer and Hoffmann [**6**], and some extensions of that work by Dobbs [**4**]. In particular, their studies dealt with, in the terminology of [**4**], primitive elements. An element $u \in T$ is said to be primitive over R, if u is a root of a polynomial $f \in R[x]$ with $c(f) = R$. It is shown [**4**, Theorem] (in the more general context of commutative rings with identity) that u is primitive over R if and only if $R \subseteq R[u]$ satisfies INC (incomparability). The main purpose of this paper is to consider a natural super-primitive analogue for this result, and to indicate some interesting related ideas.

Throughout this paper, all rings considered will be domains, i.e., commutative integral domains with identity, and any unexplained terminology is standard as in [**5**] or [**11**]. It should be noted that several of the results in these pages could be stated in the generality of commutative rings with identity, however we feel the main thrust of our work lies within the category of domains.

2. Super-primitive elements and associated primes of principal ideals. Let $\mathcal{P}(R) = \{P \in \operatorname{Spec}(R)\colon P$ is minimal over $(a : b)$ for some $a, b \in R\}$. The elements of $\mathcal{P}(R)$ are referred to as the associated primes of principal ideals [**2**]. A useful result concerning $\mathcal{P}(R)$, which we shall employ frequently, is due to Tang [**15**, Theorem E], and is stated as follows: (a) For a finitely generated ideal I of R, $I \subseteq P$ for some $P \in \mathcal{P}(R)$ if and only if $I^{-1} \neq R$; and, (b) $R = \bigcap_{P \in \mathcal{P}(R)} R_P$. It is immediate from this result that $U = R[x] \setminus \bigcup_{P \in \mathcal{P}(R)} P[x]$, where $U = \{f \in R[x]\colon c(f)^{-1} = R\}$. We shall exploit this relationship between the set $\mathcal{P}(R)$ and the super-primitive polynomials in our study of super-primitive elements.

Given an extension of domains $R \subseteq T$ and $P \in \operatorname{Spec}(R)$, we say that the extension satisfies INC at P if distinct comparable primes of T do not contract to P. If $W \subseteq \operatorname{Spec}(R)$, we say that the extension satisfies INC on W if it satisfies INC at each $P \in W$, and as usual, if $W = \operatorname{Spec}(R)$, it is said that $R \subseteq T$ satisfies INC.

The following useful lemma, which is undoubtedly well-known, could not be found in the literature in its present form. Hence, we shall sketch the part of the proof that we have no direct reference for.

PROPOSITION 2.0. *Let $R \subseteq R[x]/I = T$ be a simple extension and let $P \in \operatorname{Spec}(R)$. The following statements are equivalent*:

1. *$R \subseteq T$ satisfies* INC *at P.*
2. *$c(I) \nsubseteq P$ ($c(I)$ = ideal of R generated by all the coefficients of all the polynomials in I).*
3. *$\operatorname{Spec}(T \otimes_R k(P))$ is a finite set.*

Proof. (2) $\Leftrightarrow$ (3) is exactly [**16**, Theorem 3.1]. (1) $\Rightarrow$ (3): By passing to $R_P \subseteq T_{R \setminus P}$, we may assume $R \subseteq T$ satisfies INC at P, where P is the unique maximal ideal in R. Consider the natural map $f\colon R \to T/PT = T \otimes_R k(P)$. Observe that T/PT is of finite type over R via f, and that $f\colon R \to T/PT$ satisfies INC. Hence, $(T/PT \otimes_R R/P) = T/PT$ is module-finite over R/P. [**14**, Proposition 3, p. 50] Whence, T/PT is an Artinian ring, and so $\operatorname{Spec}(T/PT)$ is a finite set. (3) $\Rightarrow$ (1): Again we may assume R is local with maximal ideal P. We claim that T/PT is of finite type over R via $f\colon R \to T/PT$, and that $f\colon R \to T/PT$ satisfies INC. As T/PT is of finite type over R/P, by Noether Normalization [**1**, Theorem 1, p. 344], we conclude that T/PT is module-finite over a polynomial ring A in a finite number of indeterminates with coefficients in R/P. However, $\operatorname{Spec}(T/PT)$ is a finite set, so $A = R/P$, and hence T/PT is module-finite over R/P. Thus, $R \to T/PT$ satisfies INC [**14**, Proposition 3, p. 40], and therefore $R \subseteq T$ satisfies INC at P.

COROLLARY 2.1. [**4**, Theorem]: *Let $R \subseteq T$ be an extension of domains and $u \in T$. Then, u is primitive over R if and only if $R \subseteq R[u]$ satisfies* INC.

Proof. Apply Proposition 2.0, (1) $\Leftrightarrow$ (2).

COROLLARY 2.2. *Let $R \subseteq T$ be an extension of domains and $u \in T$. If u is super-primitive over R, then $R \subseteq R[u]$ satisfies* INC *on $\mathcal{P}(R)$.*

Proof. Let $I = \ker(R[x] \to R[u])$, where the homomorphism is the evaluation map. Since u is super-primitive over R, there exists an $f \in I$ such that $c(f)^{-1} = R$. Hence $c(I) \nsubseteq P$ for each $P \in \mathcal{P}(R)$, [**15**, Theorem E], and so $R \subseteq R[u]$ satisfies INC on $\mathcal{P}(R)$.

It is natural, in view of Corollary 2.1, to ask whether the converse of Corollary 2.2 is true in general. We will show (Example 2.7) that it is not generally true, but we shall indicate some interesting settings where it is true. Before we can accomplish this however, we need to introduce some terminology.

If I is a fractional ideal of R, let $(I^{-1})^{-1} = I_v$. We say I is a v-ideal if $I = I_v$, and a v-ideal is said to be of finite type if there is a finitely generated fractional ideal J of R such that $I = J_v$. A domain R is called a Prüfer v-multiplication domain (PVMD), if the set of v-ideals of R of finite type form a group under the v-multiplication $I \circ J = (IJ)_v$ [**18**], [**12**]. Examples of PVMD's are Prüfer domains, Krull domains, GCD domains, integrally closed coherent domains, etc.

For our immediate purpose, we shall need a somewhat different characterization of PVMD. A domain R is called a P-domain [**12**], if R_P is a valuation domain for each $P \in \mathcal{P}(R)$. Since $R = \bigcap_{P \in \mathcal{P}(R)} R_P$ [**15**, Theorem E], any P-domain is integrally closed. Huckaba and this author proved [**10**, Theorem 3.6], among other things, that R is a PVMD if and only if R is a P-domain and each prime ideal of $R[x]_U$ is extended from a prime ideal of R. (See [**12**] for other interesting characterizations of PVMD's in terms of P-domains.)

It is appropriate now to mention that there exists a P-domain R that is not a PVMD [**12**, Example 2.1]. In fact it is precisely with this ring that we will show that the converse of Corollary 2.2 fails to be true in general. We are now prepared to proceed with this goal in mind.

COROLLARY 2.3. *The following statements are equivalent for a domain R with quotient field K:*

1. *R is a P-domain.*
2. *R is integrally closed and $R \subseteq R[u]$ satisfies* INC *on $\mathcal{P}(R)$ for each $u \in \overline{K}$, where $\overline{K}$ = algebraic closure of K.*
3. *R is integrally closed and $R \subseteq R[u]$ satisfies* INC *on $\mathcal{P}(R)$ for each $u \in K$.*

Proof. It suffices to prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). Assume (1). Then, as mentioned before, R is integrally closed. Let $u \in \overline{K}$. It is enough to show that $R_P \subseteq R_P[u]$ satisfies INC at PR_P for each $P \in \mathcal{P}(R)$. Let $P \in \mathcal{P}(R)$, and set $I = \ker(R_P[x] \to R_P[u])$. Observe that $c_{R_P}(I) = R_P$, since R_P is a valuation domain [**5**, Remark 17.2], and hence an application of Lemma 2.0 completes this direction.

Assume (3) and let $u \in K$. Set $J = \ker(R[x] \to R[u])$, and note by Proposition 2.0 that $J \not\subseteq P[x]$ for each $P \in \mathcal{P}(R)$. Hence, u or u^{-1} is in R_P for each $P \in \mathcal{P}(R)$ [**5**, Lemma 19.14], and so R is a P-domain.

REMARK 2.4. It is straightforward to see that in Corollary 2.3 one can draw analogous conclusions for all the overrings of R, and for all the domains between R and $\overline{K}$.

PROPOSITION 2.5. *Let $\overline{K}$ denote the algebraic closure of the quotient field K of R. Then, R is a* PVMD *if and only if R is integrally closed and each $u \in \overline{K}$ is super-primitive over R.*

Proof. ($\Rightarrow$) Suppose there exists a $u \in \overline{K}$ such that u is not super-primitive over R. Let $I = \ker(R[x] \to R[u])$. Notice that $I \cap U = \varnothing$ and hence $IR[x]_U$ is extended from a prime ideal of R [**10**, Theorem 3.6]. However, $I \cap R = (0)$, and this contradicts the fact that $I \neq (0)$. ($\Leftarrow$) By Corollaries 2.2 and 2.3, it suffices to show that each prime ideal of $R[x]_U$ is extended from a prime ideal of R. Let $(0) \neq Q \in \operatorname{Spec}(R[x])$ such that $Q \cap U = \varnothing$. Set $P = Q \cap R$, and we claim that $P \neq (0)$. For if $P = (0)$, then $R \subseteq R[x]/Q = R[u]$, where $u \in \overline{K}$. Thus, by assumption, there exists an $f \in Q \cap U$, which is a contradiction. We now wish to show that $P[x] = Q$. Suppose not, and let $f \in Q \setminus P[x]$. Observe that $Q = \bigcup_{Q' \subseteq Q, Q' \in \mathcal{P}(R[x])} Q'$ [**9**, Comment following Proposition 2.5] and by the argument above, $Q' \cap R = P' \neq (0)$ provided $Q' \neq (0)$. Hence, $Q = \bigcup_{P' \subseteq P, P' \in \mathcal{P}(R)} P'[x]$ [**2**, Corollary 8], and so $f \in P'[x]$ for some $P' \in \mathcal{P}(R)$. Therefore $f \in P[x]$, which is a contradiction.

REMARK 2.6. It is worthwhile to point out that Proposition 2.5 can be viewed as a restatement of [**12**, Theorem 3.4], and while their proofs are substantially different in content and spirit, they do share some crucial components (e.g. [**2**, Corollary 8]).

We are now prepared to show that the converse of Corollary 2.2 is not generally true.

EXAMPLE 2.7. Let R be a P-domain that is not a PVMD [**12**, Example 2.1]. By Corollary 2.3 we have that $R \subseteq R[u]$ satisfies INC on $\mathcal{P}(R)$ for each $u \in \overline{K}$. However, Proposition 2.5 guarantees the existence of an element $w \in \overline{K}$ such that w is not super-primitive over R. Therefore, the converse of Corollary 2.2 is not true in general.

3. Compactness of $\mathcal{P}(R)$. In this section we shall indicate a large class of domains for which the converse of Corollary 2.2 is true.

PROPOSITION 3.0. *Let R be a domain, and assume $\mathcal{P}(R)$ is a compact subspace of* Spec(R) *in the Zariski topology. Let $u \in \overline{K}$. If $R \subseteq R[u]$ satisfies* INC *on $\mathcal{P}(R)$, then u is super-primitive over R.*

Proof. Let $I = \ker(R[x] \to R[u])$. By Proposition 2.0, $c(I) \nsubseteq P$ for each $P \in \mathcal{P}(R)$. Choose $a_p \in c(I) \setminus P$ for each $P \in \mathcal{P}(R)$, and note that

$$\mathcal{P}(R) = \bigcup_{P \in \mathcal{P}(R)} \left(X_{a_P} \cap \mathcal{P}(R)\right), \quad \text{where } X_{a_P} = \{Q \in \operatorname{Spec}(R) \colon a_p \notin Q\}.$$

Since this is an open cover of $\mathcal{P}(R)$, there exists by compactness a finite subcover. Namely,

$$\mathcal{P}(R) = \bigcup_{i=1}^{n} \left(X_{a_i} \cap \mathcal{P}(R)\right).$$

Let $J = (a_1, \dots, a_n)$. Observe that $J \subseteq c(I)$, and that $J \nsubseteq P$ for each $P \in \mathcal{P}(R)$. Hence, $J^{-1} = R$ [**15**, Theorem E]. We now claim that there exists an $f \in I$ such that $c(f)^{-1} = R$, i.e., that u is super-primitive over R. Since $a_1 \in c(I)$, $a_1 = \sum_{i=1}^{w} r_i \alpha_i$ where $r_i \in R$, and α_i is a coefficient of some $f_i \in I$. Write

$$f_i = \beta_0 + \beta_1 x + \cdots + \alpha_i x^{t_i} + \cdots + \beta_{n_i} x^{n_i}, \quad \text{and let } t = \max\{t_i\}.$$

Then,

$$\sum_{i=1}^{m} x^{t - t_i}(r_i f_i) = g_1 \in I \quad \text{and} \quad a_1 \in c(g_1).$$

Thus, each a_j is the coefficient of some $g_j \in I$. Write

$$g_j = C_{j0} + C_{j1} x + \cdots + a_j x^{v_j} + \cdots + C_{jm_j} x^{m_j}.$$

Let $m > \max\{m_j\}$, and set

$$f = x^m g_1 + x^{2m} g_2 + \cdots + x^{nm} g_n.$$

Then $J \subseteq c(f)$, and so $c(f)^{-1} = R$.

It should be noted that the compactness of $\mathcal{P}(R)$ is sufficient to guarantee the implication in Proposition 3.0, but it is not necessary (see comments following Corollary 3.5). However, it is interesting to investigate when $\mathcal{P}(R)$ is compact (in the Zariski topology).

Clearly if Spec(R) is a Noetherian space, then $\mathcal{P}(R)$ is compact. The next Lemma helps us determine other contexts for which $\mathcal{P}(R)$ is compact.

LEMMA 3.1. *For a domain R, $\mathcal{P}(R)$ is compact if and only if given any ideal I of R with $I \not\subseteq P$ for each $P \in \mathcal{P}(R)$, then there exists a finitely generated ideal $J \subseteq I$ such that $J \not\subseteq P$ for each $P \in \mathcal{P}(R)$.*

Proof. ($\Rightarrow$) Suppose I is an ideal of R such that $I \not\subseteq P$ for each $P \in \mathcal{P}(R)$. We may choose $a_P \in I \setminus P$ for each $P \in \mathcal{P}(R)$ and argue as in Proposition 3.0 to obtain the desired conclusion. ($\Leftarrow$) It clearly suffices to show that any open cover of $\mathcal{P}(R)$ consisting of basic open sets has a finite subcover. Suppose $\mathcal{P}(R) = \bigcup_{\alpha \in \Lambda}(X_{a_\alpha} \cap \mathcal{P}(R))$, and let I be the ideal of R generated by all the a_α. Thus $I \not\subseteq P$ for each $P \in \mathcal{P}(R)$, and so there exists a finitely generated ideal J with $J \subseteq I$ such that $J \not\subseteq P$ for each $P \in \mathcal{P}(R)$. Write $J = (b_1, \ldots, b_m)$ and note that for $1 \leq i \leq m$, $b_i = \sum_{j=1}^{m_i} r_{ij} a_{\alpha_{ij}}$, where $r_{ij}, a_{\alpha_{ij}} \in R$, and $\alpha_{ij} \in \Lambda$. Hence,

$$\mathcal{P}(R) = \bigcup_{i,j} \left(X_{a_{\alpha_{ij}}} \cap \mathcal{P}(R)\right)$$

and the proof is complete.

PROPOSITION 3.2. *Let R be a domain and x an indeterminate over R. Then, $\mathcal{P}(R)$ is compact if and only if $\mathcal{P}(R[x])$ is compact.*

Proof. ($\Leftarrow$) Let F: Spec($R[x]$) $\to$ Spec(R) denote the contraction map. Observe that $F(\mathcal{P}(R[x])) = \mathcal{P}(R)$ [**2**, Corollary 8]. Hence $\mathcal{P}(R)$ is compact. ($\Rightarrow$) Let N be an ideal of $R[x]$ such that $N \not\subseteq Q$ for each $Q \in \mathcal{P}(R[x])$. Thus $c(N) \not\subseteq P$ for each $P \in \mathcal{P}(R)$ [**2**, Corollary 8]. Hence there exists a finitely generated ideal J of R with $J \subseteq c(N)$ such that $J \not\subseteq P$ for $P \in \mathcal{P}(R)$. Write $J = (a_0, \ldots, a_n)$, and let $J' = (f_0, \ldots, f_n)$, where $f_i \in N$ and a_i and $c(f_i)$. Choose $0 \neq b \in N \cap R$ and set $I = (J', b)$. Thus, I is a finitely generated ideal of $R[x]$, $I \subseteq N$, and $I \not\subseteq Q$ for each $Q \in \mathcal{P}(R[x])$, since $Q \cap R = (0)$ or $Q = (Q \cap R)[x]$, where $Q \cap R \in \mathcal{P}(R)$ [**2**, Corollary 8].

Recall that a domain R is said to be treed in case Spec(R), considered as a poset under inclusion, is a tree. Also, we shall denote the maximal ideal space of R by Max(R).

PROPOSITION 3.3. *Let R be a treed domain. Then, $\mathcal{P}(R)$ is compact if and only if* Max(R) $\subseteq \mathcal{P}(R)$.

Proof. ($\Rightarrow$) Let $M \in$ Max(R), and assume $M \notin \mathcal{P}(R)$. Thus, $M \not\subseteq P$ for each $P \in \mathcal{P}(R)$, and so by Lemma 3.1 there exists a finitely generated ideal J, with $J \subseteq M$ and $J \not\subseteq P$ for each $P \in \mathcal{P}(R)$. Hence $M = \bigcup_{P \subsetneq M, P \in \mathcal{P}(R)} P$ [**9**, Comment following Proposition 2.5]. However, since

R is treed, we see that $J \subseteq P$ for some $P \in \mathcal{P}(R)$, which is a contradiction. Therefore $M \in \mathcal{P}(R)$, and this direction is complete. ($\Leftarrow$) Assume $\mathrm{Max}(R) \subseteq \mathcal{P}(R)$, and suppose I is an ideal of R such that $I \not\subseteq P$ for each $P \in \mathcal{P}(R)$. Then, $I = R$, and $\mathcal{P}(R)$ is compact by Lemma 3.1.

COROLLARY 3.4. *Let R be treed. Then $\mathcal{P}(R)$ is compact if and only if $\mathcal{P}(R_M)$ is compact for each $M \in \mathrm{Max}(R)$.*

Proof. ($\Rightarrow$) Since $\mathcal{P}(R_M)$ is compact if and only if $MR_M \in \mathcal{P}(R_M)$, this part of the proof is complete by observing that $M \in \mathcal{P}(R)$ implies $MR_M \in \mathcal{P}(R_M)$. The other direction follows in a similar manner.

COROLLARY 3.5. *Let R be treed. Assume $\mathcal{P}(R)$ is compact and $u \in \overline{K}$. If $R \subseteq R[u]$ satisfies* INC *on $\mathcal{P}(R)$, then $R \subseteq R[u]$ satisfies* INC *on* $\mathrm{Spec}(R)$.

Proof. Let $I = \ker(R[x] \to R[u])$, and recall from Proposition 2.0 that $c(I) \not\subseteq P$ for each $P \in \mathcal{P}(R)$. Thus, by Proposition 3.3, $c(I) = R$. Another appeal to Proposition 2.0 produces the desired conclusion.

It should be noted that the "if" direction of Proposition 3.3 follows without the treed assumption, whereas the "only if" part requires the treed property. To see this, let R be a regular local ring of dimension 2 with maximal ideal M. Then $M \notin \mathcal{P}(R)$ [**11**, Exercise 1, p. 102], yet $\mathcal{P}(R)$ is compact since $\mathrm{Spec}(R)$ is a Noetherian space.

Also note that not all treed domains R have $\mathcal{P}(R)$ compact. For instance, if V is a valuation ring with maximal ideal N such that $N = \bigcup_{P \in \mathrm{Spec}(R), P \subsetneq N} P$, then $N \notin \mathcal{P}(V)$. However, given any valuation ring W, it is possible to find a domain R with $\mathrm{Spec}(R) = \mathrm{Spec}(W)$, and with $\mathcal{P}(R)$ compact, independent of whether $\mathcal{P}(W)$ is compact or not. To produce such an example let $k \subsetneq K$ be fields and let W be a valuation ring of the form $K + M$. Set $R = k + M$, and we claim $\mathcal{P}(R)$ is compact. To show $M \in \mathcal{P}(R)$ it suffices to prove that $M^{-1} \neq R$, for if $a/b \in M^{-1} \setminus R$, then $M = (b : a)$. The proof will be complete when we verify that $M^{-1} = W$. (Throughout this example M^{-1} means: $M^{-1} = \{x \in$ quotient field of R: $xM \subseteq R\}$.) Clearly $W \subseteq M^{-1}$, and so let $u \in M^{-1} \setminus W$. Thus $u^{-1} \in M$, and so either $uM = M$ or $uM = R$. In the first case $1 \in M$. As for the second case $M = Ru^{-1}$, and by the argument used in [**3**, Lemma 1], we get that $k = K$. These contradictions establish the proof.

EXAMPLE 3.6. The following is an example of a local treed domain R that is not a valuation domain, and $\mathcal{P}(R)$ is not compact. Let V be a nontrivial valuation ring with maximal ideal N and quotient field k, such that $\mathcal{P}(V)$ is not compact, and let K be a proper field extension of k. Assume W is a nontrivial valuation ring of the form $K + M$, and set $R = V + M$. We claim that R is the desired example, i.e., R is a local treed domain that is not a valuation ring [5, Exercise 13, p. 203], and $\mathcal{P}(R)$ is not compact. It suffices to show that $N + M \notin \mathcal{P}(R)$. Suppose $N + M \in \mathcal{P}(R)$. Then there exist elements $x, y \in R$ such that $N + M$ is minimal over $(x \underset{R}{:} y)$. Observe that $(x \underset{R}{:} y) \neq (0)$, and so $x \neq 0$. Hence, given any nonzero element $n \in N$ there is a $z \in N + M$ (which depends on n) so that $N + M$ is minimal over $(n \underset{R}{:} z)$ [2, Theorem 3]. Write $z = v + m$, where $v \in V$ and $m \in M$. To complete the proof we will show that N is minimal over $(n \underset{V}{:} v)$, and hence $N \in \mathcal{P}(V)$, a contradiction. First we establish that $(n \underset{V}{:} v) \subseteq N$. Let $u \in (n \underset{V}{:} v)$, and note that $u \in (n \underset{R}{:} z)$. Indeed, $uz = uv + um \in nV + M$. But $M = nM$, and so $nR = nV + M$. Hence, $uz \in nR$. Thus $u \in N + M$, and whence $u \in N$ since $u \in V$. Now we shall verify that N is minimal over $(n \underset{V}{:} v)$. Suppose $(n \underset{V}{:} v) \subseteq P \subseteq N$, where $P \in \operatorname{Spec}(V)$. Observe that $(n \underset{R}{:} z) \subseteq P + M \subseteq N + M$. To see this let $w \in (n \underset{R}{:} z)$, and write $w = a + b$, $a \in V$, $b \in M$. It is enough to show that $a \in P$. Consider, $wz = (a + b)(v + m) \in nR = nV + M$, and so $a \in (n \underset{V}{:} v) \subseteq P$.

REMARK 3.7. For an interesting related study on a class of domains R having $\mathcal{P}(R)$ compact, see [8].

4. Finitely generated uppers of 0. Let R be a domain with quotient field K, and denote the algebraic closure of K by $\overline{K}$. Let $u \in \overline{K}$, and set $I = \ker(R[x] \to R[u])$. Note that if R is a Prüfer domain or more generally an integrally closed coherent domain [13, 11.13] or [7, Corollary 2.4], or a GCD domain [15, Theorem I], then I is finitely generated. Since each of these domains are PVMD's, it is natural to ask whether I is finitely generated for an arbitrary PVMD R. The answer is no, and we shall give a Krull domain counterexample (Example 4.1). Even though I need not always be finitely generated for an arbitrary PVMD or more generally when u is a super-primitive element, it would be interesting to determine what additional conditions force I to be finitely generated. For example, if R is coherent and $u \in \overline{K}$ is super-primitive, then I is finitely generated [7, Theorem 2.5].

Conversely, it is appropriate to mention that if I is finitely generated, and $R \subseteq R[u]$ satisfies INC on $\mathcal{P}(R)$, then u is super-primitive. Indeed, $c(I)$ is a finitely generated ideal of R, and $c(I) \nsubseteq P$ for each $P \in \mathcal{P}(R)$ (Proposition 2.0). Thus, $c(I)^{-1} = R$, and by arguing as in Proposition 3.0 we obtain a super-primitive polynomial in I.

The following result will be used to produce the promised counterexample.

PROPOSITION 4.0. *The following are equivalent for an integrally closed domain R with elements a and b where $b \neq 0$:*
1. *$R \subseteq R[a/b]$ is a finitely presented R-algebra extension.*
2. *$(a) \cap (b)$ is a finitely generated ideal of R.*
3. *$(b : a)$ is a finitely generated ideal of R.*

Proof. We assume $a \neq 0$, and thus (2) $\Leftrightarrow$ (3), since $(b : a) = ((a) \cap (b))a^{-1}$. (1) $\Rightarrow$ (2): Let K denote the quotient field of R, and set $I = \ker(R[x] \to R[a/b])$. Notice that $IK[x] = (bx - a)K[x]$, and hence by [5, Corollary 34.9],

$$I = (bx - a)K[x] \cap R[x] = (bx - a)(a, b)^{-1}R[X].$$

We claim that $(a, b)^{-1}$ is a finitely generated R-module, since I is a finitely generated ideal of $R[x]$. For if $I = (f_1, \ldots, f_n)$, $f_i \in R[x]$, then $f_i = \Sigma_j g u_{ij} h_{ij}$, where $g = bx - a$, $h_{ij} \in R[x]$ and $u_{ij} \in (a, b)^{-1}$. It is straightforward to prove that $(a, b)^{-1} = \Sigma_{i,j} R u_{ij}$. Therefore, $(a) \cap (b)$ is a finitely generated ideal of R, since $(a, b)^{-1} = ((a) \cap (b))/ab$. (2) $\Rightarrow$ (1): As in the previous part, $I = (bx - a)(a, b)^{-1}R[x]$. Hence, since $(a, b)^{-1} = ((a) \cap (b))/ab$ is a finitely generated R-module, we see that I is a finitely generated ideal of $R[x]$.

EXAMPLE 4.1. There exists a PVMD (in particular a Krull domain) R, and an element a/b in the quotient field K of R such that $I = \ker(R[x] \to R[u])$ is not a finitely generated ideal of $R[x]$. Let R be a Krull domain with at least one height one prime ideal not finitely generated [5, Exercise 4, p. 537]. We claim R is the desired example. Indeed, if for each $a/b \in K$, $R \subseteq R[a/b]$ is a finitely presented R-algebra extension, then by applying Proposition 4.0 and the proof of [17, Corollary 6.11], it follows that each height one prime ideal of R is finitely generated. This contradiction establishes our claim.

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