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DEGREE OF UNIFORM APPROXIMATION ON DISJOINT INTERVALS

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In this note, the problem of degree of uniform approximation by polynomials on disjoint intervals is considered. It is interesting to note that the error estimates cannot be obtained by extending the given functions to functions on a single interval and applying the one-interval estimates.

1. Introduction. If K is a compact subset of the real line, $C(K)$ will denote, as usual, the Banach space of all real-valued continuous functions on K with the supremum norm $\| \|_K$. For each f in $C(K)$, let $E_n(f; K)$ denote the distance from f to the subspace π_n of all algebraic polynomials with degrees not exceeding n. When there is no possibility of confusion, we will also use $\| \|$ for $\| \|_K$ and $E_n(f)$ for $E_n(f; K)$.

Let $K = [-b, -a] \cup [a, b]$ where $0 < a < b$ and $f \in C(K)$. In this paper, we will show that $E_n(f; K) = O(n^{-r})$ where $r > 0$, if both $E_n(f; [-b, -a])$ and $E_n(f; [a, b])$ are of order $O(n^{-r})$. We will also show that this is a disjoint interval result, in the sense that it cannot be obtained by extending f to a function in $C[-b, b]$. If f has the property that $f|_{[-b,-a]}$ and $f|_{[a,b]}$ are restrictions of functions analytic in the left and right half planes respectively, we will show that $E_n(f; K)$ decreases not slower than a geometric progression. Our proofs of the above results are very elementary, using only the well known classical results of Bernstein (cf. [**4**]).

When entire functions are considered, Fuchs [**2**] obtained very sharp estimates for the case where K is the union of finitely many mutually exterior Jordan curves satisfying certain smoothness conditions. The main tools in [**2**] are function-theoretic techniques and the results in Widom [**6**]. In a private communication, Professor Fuchs also pointed out that his results in [**2**] are also valid for disjoint intervals.

Our proof of the analytic functions result will rely on the following theorem of Bernstein (cf. [**4**], p. 76).

THEOREM A. *Let $f \in C[-1, 1]$. Then*

$$\overline{\lim_{n\to\infty}} \, (E_n(f))^{1/n} \le \frac{1}{\rho}, \qquad \rho > 1,$$

if and only if f is the restriction of a function analytic in the interior of the ellipse with foci at ± 1 and vertices at $\pm(\rho + (1/\rho))/2$.

Hence, if $f|_{[-b,-a]}$ and $f|_{[a,b]}$ are restrictions of two different entire functions and $\tilde{f} \in C[-b, b]$ is any extension of f from K to $[-b, b]$, then Theorem A implies that

$$\overline{\lim_{n\to\infty}} \, \left(E_n(\tilde{f}; [-b, b])\right)^{1/n} = 1.$$

In other words it is not possible to prove that $E_n(f; K)$ decreases faster than a geometric progression by extending $f \in C(K)$ to an $\tilde{f} \in C[-b, b]$ and using the inequality $E_n(f; K) \le E_n(\tilde{f}; [-b, b])$.

2. The main results. We will need the following lemma which is a direct consequence of Theorem A by using the change of variable

$$u = \frac{2x}{b^2 - a^2} - \frac{a^2 + b^2}{b^2 - a^2}.$$

LEMMA 1. *Let $0 < a < b$ and $f \in C[a^2, b^2]$. Then f is the restriction of a function analytic in the interior of the ellipse with foci at a^2 and b^2 and a vertex at the origin if and only if*

$$\overline{\lim_{n\to\infty}} \, \left(E_n(f; [a^2, b^2])\right)^{1/n} \le \frac{b - a}{b + a}. \tag{1}$$

Our first result in this paper is the following.

THEOREM 1. *Let $0 < a < b$ and $K = [-b, -a] \cup [a, b]$. Suppose that $f \in C(K)$ such that $f|_{[-b,-a]}$ is the restriction of a function f_1 analytic in the left half plane* Re $z < 0$ *and that $f|_{[a,b]}$ is the restriction of a function f_2 analytic in the right half plane* Re $z > 0$. *Then*

$$\overline{\lim_{n\to\infty}} \, (E_n(f))^{1/n} \le \sqrt{\frac{b - a}{b + a}}. \tag{2}$$

Furthermore, equality is attained if f_1 and f_2 are two different entire functions.

We remark that in our proof of inequality (2), we only need to assume that $f_1(\sqrt{-z})$ and $f_2(\sqrt{z})$ are analytic in the interior of the ellipse with foci at a^2 and b^2 and a vertex at 0. Here and throughout, the principal value of the square root is used. If f_1 and f_2 are different entire functions, the results in **[2]**, which are also valid for disjoint intervals, also give equality in (2) with limit supremum replaced by limit.

Our proof of Theorem 1 is very elementary. Let P_n and Q_n be the polynomials of best uniform approximation on the interval $[a^2, b^2]$ from π_n to the functions $f_2(\sqrt{x})$ and $f_2(\sqrt{x})/\sqrt{x}$ respectively. By Lemma 1, we have

$$\overline{\lim_{n\to\infty}} \left\|P_n(x) - f_2\left(\sqrt{x}\right)\right\|_{[a^2,b^2]}^{1/n} \leq \frac{b-a}{b+a}, \tag{3}$$

$$\overline{\lim_{n\to\infty}} \left\|Q_n(x) - f_2\left(\sqrt{x}\right)/\sqrt{x}\right\|_{[a^2,b^2]}^{1/n} \leq \frac{b-a}{b+a}. \tag{4}$$

Let $R_{2n+1}(x) = [P_n(x^2) + xQ_n(x^2)]/2$. Then $R_{2n+1} \in \pi_{2n+1}$, and since $P_n(x^2)$ is even in x and $xQ_n(x^2)$ is odd in x, it follows from (3) and (4) that

$$\overline{\lim_{n\to\infty}} \|R_{2n+1} - f\|_{[a,b]}^{1/n} \leq \frac{b-a}{b+a}, \tag{5}$$

$$\lim_{n\to\infty} \|R_{2n+1}\|_{[-b,-a]}^{1/n} \leq \frac{b-a}{b+a}. \tag{6}$$

Similarly, we can find $S_{2n+1}(x) \in \pi_{2n+1}$ such that

$$\overline{\lim_{n\to\infty}} \|S_{2n+1} - f\|_{[-b,-a]}^{1/n} \leq \frac{b-a}{b+a}, \tag{7}$$

$$\overline{\lim_{n\to\infty}} \|S_{2n+1}\|_{[a,b]}^{1/n} \leq \frac{b-a}{b+a}. \tag{8}$$

The relationships (5)–(8) imply that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \|R_{2n+1} + S_{2n+1} - f\|_K^{1/n} \\
&\quad \leq \overline{\lim_{n\to\infty}} \Big\{\|R_{2n+1} - f\|_{[a,b]} + \|S_{2n+1}\|_{[a,b]} \\
&\qquad\qquad + \|R_{2n+1}\|_{[-b,-a]} + \|S_{2n+1} - f\|_{[-b,-a]}\Big\}^{1/n} \\
&\quad \leq \frac{b-a}{b+a},
\end{aligned}$$

and (2) follows immediately.

If f_1 and f_2 are two different entire functions, we will show that equality in (2) holds. First, let us assume that f is an even function (that is, $f_2(x) = f_1(-x)$ for $x \in [a, b]$). Let P_{2n} be the polynomial of best uniform approximation to f on K from π_{2n}. By uniqueness, P_{2n} is also an even function, so that $P_{2n}(x) = q_n(x^2)$ for some $q_n \in \pi_n$. This gives

$$E_n\left(f_2\left(\sqrt{x}\right); [a^2, b^2]\right) \leq \left\| q_n(x) - f_2\left(\sqrt{x}\right)\right\|_{[a^2,b^2]} = E_{2n}(f).$$

Assume that equality is *not* attained in (2). Then we have

$$\overline{\lim_{n\to\infty}} \left\{E_n\left(f_2\left(\sqrt{x}\right); [a^2, b^2]\right)\right\}^{1/n} < \frac{b-a}{b+a},$$

and by Lemma 1, the function $f_2(\sqrt{x})$ is analytic at $x = 0$. Since f_2 is an entire function, it is clear that f_2 must be an even function. That is, $f_1(x) = f_2(-x) = f_2(x)$ for $x \in [-b, -a]$, so that $f_1 \equiv f_2$, which is a contradiction. Hence, if f is even and f_1, f_2 are different entire functions, then equality in (2) must hold.

Suppose that f is not even and f_1, f_2 are different entire functions such that $f_1(x) + f_2(-x)$ is not identical with $f_2(x) + f_1(-x)$. Let $F_1(x) := [f_1(x) + f_2(-x)]/2$, $F_2(x) := [f_2(x) + f_1(-x)]/2$, and

$$F(x) := \frac{f(x) + f(-x)}{2} = \begin{cases} F_1(x) & \text{for } x \in [-b, -a], \\ F_2(x) & \text{for } x \in [a, b]. \end{cases}$$

Then $F_1 \not\equiv F_2$ and F is even. Hence, from what we have just proved, equality in (2) holds for F. If P_n is the best uniform approximant of f on K from π_n, then we have

$$\begin{aligned} E_n(F) &\leq \left\| \frac{P_n(x) + P_n(-x)}{2} - F(x) \right\|_K \\ &\leq \left\| \frac{P_n - f}{2} \right\|_K + \left\| \frac{P_n(-x) - f(-x)}{2} \right\|_K = E_n(f), \end{aligned}$$

so that equality in (2) also holds for f.

Finally, suppose that $f_1(x) + f_2(-x) = f_2(x) + f_1(-x)$ for all x. We define $H_1(x) := x[f_1(x) - f_2(-x)]/2$, $H_2(x) := x[f_2(x) - f_1(-x)]/2$, and

$$H(x) := \frac{x[f(x) - f(-x)]}{2} = \begin{cases} H_1(x) & \text{for } x \in [-b, -a], \\ H_2(x) & \text{for } x \in [a, b]. \end{cases}$$

Since $f_1 \not\equiv f_2$, we have $H_1 \not\equiv H_2$. Hence, since H is even, equality in (2) holds for H. Again, let P_n be the best uniform approximant of f on K from π_n. Then

$$E_n(H) \leq \left\| \frac{x[P_n(x) - P_n(-x)]}{2} - H(x) \right\|_K$$
$$\leq \left\| x \frac{P_n(x) - f(x)}{2} \right\|_K + \left\| x \frac{P_n(-x) - f(-x)}{2} \right\|_K \leq bE_n(f),$$

so that equality in (2) also holds for f. This completes the proof of Theorem 1.

For functions which are not necessarily analytic, we have the following result.

THEOREM 2. *Let* $0 < a < b$ *and* $K = [-b, -a] \cup [a, b]$. *Suppose that* $f \in C(K)$, $f_1 = f|_{[-b,-a]}$ *and* $f_2 = f|_{[a,b]}$, *such that* $E_n(f_1; [-b, -a]) = O(n^{-r})$ *and* $E_n(f_2; [a, b]) = O(n^{-r})$ *for some* $r > 0$. *Then* $E_n(f; K) = O(n^{-r})$.

Before we proceed with the proof, we remark that while Theorem A already shows that Theorem 1 is a disjoint interval result, the following example shows that Theorem 2 cannot be obtained by extending f to a function in $C[-b, b]$ either.

Let $f(x) = \sqrt{|x| - 1}$ where $|x| \geq 1$. Since $E_n(\sqrt{x}; [0, 1]) = O(1/n)$ (cf. [**5**, p. 131] or [**3**]), and $E_n(f; [-2, -1]) = E_n(f; [1, 2]) = E_n(\sqrt{x}; [0, 1])$, Theorem 2 implies that $E_n(f; [-2, -1] \cup [1, 2]) = O(1/n)$ also. However, by a result of Bernstein (cf. [**5**, p. 129]), $E_n(\tilde{f}; [-2, 2]) \neq O(n^{-\alpha})$ for any $\alpha > 1/2$, where $\tilde{f}$ is any extension of f to $[-2, 2]$.

Our proof of Theorem 2 relies on the following result of Bernstein (cf. [**4**; p. 42]).

THEOREM B. *Let* $P_n \in \pi_n$ *such that* $|P_n(x)| \leq 1$ *for all* $x \in [-1, 1]$. *Then* $|P_n(z)| \leq \rho^n$ *for all* z *in the interior of the ellipse with foci at* ± 1 *and vertices at* $\pm(\rho + 1/\rho)/2$.

Hence, if $P_n \in \pi_n$ such that $\|P_n\|_{[a,b]} \leq d$, then we have

$$\|P_n\|_{[-b,-a]} \leq dc_1^n \tag{9}$$

where $c_1 = (3b + a + 2\sqrt{2b^2 + 2ab})/(b - a)$.

Consider the function

$$g(x) = \begin{cases} 0 & \text{if } -b \leq x \leq -a, \\ 1 & \text{if } a \leq x \leq b \end{cases}$$

and let $Q_n \in \pi_n$ be the best uniform approximant of g on $K = [-b, -a] \cup [a, b]$ from π_n. By Theorem 1, we have

$$\|Q_n - g\|_K \leq d_1 c_2^{-n} \tag{10}$$

for some constant d_1 and all n, where $c_2 > \sqrt{(b+a)/(b-a)}$. Let P_n be the best uniform approximant of the given function f_2 on $[a, b]$ from π_n. Then $\| P_n \|_{[a,b]} \leq d_2$ for some constant d_2 and all n. By (9), we have

$$\|P_n\|_{[-b,-a]} \leq d_2 c_1^n \tag{11}$$

for all n. For a real number x, let $[x]$ denote, as usual, the integer part of x. Then for $0 < \alpha < 1$, we have, by (10),

$$\begin{aligned} &\|P_{[\alpha n]} Q_{[(1-\alpha)n]} - f_2\|_{[a,b]} \\ &\leq \|P_{[\alpha n]}\|_{[a,b]} \|Q_{[(1-\alpha)n]} - 1\|_{[a,b]} + \|P_{[\alpha n]} - f_2\|_{[a,b]} \\ &\leq d_2 d_1 c_2^{-[(1-\alpha)n]} + E_{[\alpha n]}(f_2; [a, b]). \end{aligned} \tag{12}$$

On the other hand, we have, from (10) and (11),

$$\|P_{[\alpha n]} Q_{[(1-\alpha)n]}\|_{[-b,-a]} \leq d_1 d_2 c_1^{[\alpha n]} c_2^{-[(1-\alpha)n]}. \tag{13}$$

Let F_2 be defined by

$$F_2(x) = \begin{cases} 0 & \text{for } -b \leq x \leq -a, \\ f_2(x) & \text{for } a \leq x \leq b. \end{cases}$$

By choosing α, $0 < \alpha < 1$, such that $c_1^\alpha < c_2^{1-\alpha}$, we can conclude from (12), (13) and the hypothesis $E_n(f_2; [a, b]) = O(n^{-r})$ that $E_n(F_2; K) = O(n^{-r})$. Similarly if we let

$$F_1(x) = \begin{cases} f_1(x) & \text{for } -b \leq x \leq -a, \\ 0 & \text{for } a \leq x \leq b, \end{cases}$$

we also have $E_n(F_1; K) = O(n^{-r})$. Hence, it follows that

$$E_n(f; K) \leq E_n(F_1; K) + E_n(F_2; K) = O(n^{-r})$$

completing the proof of the theorem.

3. Final remarks. There are many important unanswered questions in the subject of degree of approximation on disjoint intervals. The purpose of this paper is to introduce some elementary techniques and present two results which cannot be obtained by extending the given functions to one interval and applying the one-interval estimates. Unfortunately, the techniques in this paper cannot be easily generalized to handle the case when the two disjoint intervals are of different length.

There is also a big gap between the rates $O(n^{-r})$ and $[(b-a)/(b+a)]^{n/2}$. For instance, if f is analytic in a (smaller) neighborhood of $K=[-b,-a] \cup [a,b]$, we do not know the (exact) rate of convergence of $E_n(f;K)$. As we pointed out in §1, the sharp estimates in Fuchs [**2**] also hold for disjoint intervals when entire functions are considered. The basic techniques are function theoretic and the results in Widom [**6**]. Many related but different results have been obtained by Akhiezer [**1**], where in particular, transfinite diameters of a union of disjoint intervals have been obtained.

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