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REGULARITY OF THE BERGMAN PROJECTION IN CERTAIN NON-PSEUDOCONVEX DOMAINS

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Suppose D is a smooth bounded domain contained in $\mathbf{C}^n$ ($n \geq 2$) whose Bergman projection satisfies global regularity estimates, and suppose K is a compact subset of D such that $D - K$ is connected. The purpose of this note is to prove that, under these circumstances, the Bergman projection associated to the domain $D - K$ satisfies global regularity estimates.

This result is presently known only in very special cases when both D and K have a particularly simple form. For example, the fundamental paper of Kohn [**5**] reveals that if Ω_1 and Ω_2 are two smooth bounded strictly pseudoconvex domains in $\mathbf{C}^n$ ($n > 2$) such that $\Omega_2 \subset\subset \Omega_1$, then the $\bar{\partial}$-Neumann problem for the domain $\Omega_1 - \bar{\Omega}_2$ is subelliptic. Kohn's formula, $P = I - \bar{\partial}^* N \bar{\partial}$, which relates the Bergman projection P to the $\bar{\partial}$-Neumann operator N, shows that the Bergman projection associated to $\Omega_1 - \bar{\Omega}_2$ satisfies global regularity estimates. Recently, Derridj and Fornaess [**3**] have shown that if Ω_1 and Ω_2 are two pseudoconvex domains with real analytic boundaries in $\mathbf{C}^n$ with $n \geq 3$ and $\Omega_2 \subset\subset \Omega_1$, then the $\bar{\partial}$-Neumann operator for $\Omega_1 - \bar{\Omega}_2$ satisfies subelliptic estimates. Hence, the Bergman projection associated to $\Omega_1 - \bar{\Omega}_2$ satisfies global estimates in this case, also.

In Bell and Boas [**2**], it is proved that the Bergman projection associated to a smooth bounded complete Reinhardt domain satisfies global regularity estimates. Thus, there are more subtle examples of non-pseudoconvex domains for which regularity of the Bergman projection holds than those addressed by the theorem of the present work. Recently, the techniques used in [**2**] have been refined by David E. Barrett [**1**] to prove that the Bergman projection associated to a smooth bounded domain with a Lie group of transverse symmetries satisfies global regularity estimates.

The question as to whether or not the Bergman projection associated to a domain satisfies global regularity estimates is very important in problems relating to boundary behavior of holomorphic mappings (see [**2**]).

The Bergman projection P associated to a bounded domain D contained in $\mathbf{C}^n$ is the orthogonal projection of $L^2(D)$ onto $H(D)$, the closed subspace of $L^2(D)$ consisting of L^2 holomorphic functions. The space $C^\infty(\bar{D})$ is defined to be the set of functions in $C^\infty(D)$, all of whose derivatives are bounded functions on D. The family of derivative sup-norms exhibits the Frechet space topology on $C^\infty(\bar{D})$.

We shall say that a bounded domain D satisfies *condition R* whenever P is a continuous operator from $C^\infty(\bar{D})$ to $C^\infty(\bar{D})$. We can now state the main result of this paper.

THEOREM. *If $D \subset \mathbf{C}^n$ $(n \geq 2)$ is a smooth bounded domain which satisfies condition R and K is a compact subset of D such that $D - K$ is connected, then $D - K$ satisfies condition R.*

Examples of domains for which condition R is known to hold include smooth bounded strictly pseudoconvex domains (Kohn [**5**]), smooth bounded pseudoconvex domains with real analytic boundaries (Kohn [**6**], Diederich and Fornaess [**4**]), and smooth bounded complete Reinhardt domains (Bell and Boas [**2**]).

Before we prove the theorem, we must define some Sobolev norms and spaces. If D is a smooth bounded domain and s is a positive integer, the space $W^s(D)$ is the usual Sobolev space of complex valued functions on D whose distributional derivatives up to order s are contained in $L^2(D)$. The Sobolev s-norm of a function u is defined via

$$\|u\|_s^2 = \sum_{|\alpha| \leq s} \|\partial^\alpha u\|^2_{L^2(D)}$$

where the symbol ∂^α is the standard differential operator of order α. If $v \in L^2(D)$, we define the *negative* Sobolev s-norm of v via

$$\|v\|_{-s} = \operatorname{Sup}\left\{\left|\int_D v\phi\right| : \phi \in C_0^\infty(D); \ \|\phi\|_s = 1\right\}.$$

If $g \in H(D)$ we define the *special* Sobolev s-norm of g to be

$$|||g|||_s = \operatorname{Sup}\left\{\left|\int_D g\bar{h}\right| : h \in H(D); \ \|h\|_{-s} = 1\right\}.$$

REMARK. It is always true that if D is a smooth bounded domain, then there is a constant C such that

$$|||g|||_s \leq C\|g\|_s$$

for all $g \in H(D)$. This can be proved using techniques similar to those used in [**2**]. The reverse inequality $\|g\|_s \leq C|||g|||_s$ only holds if the Bergman projection associated to D satisfies an estimate of the form $\|P\phi\|_s \leq C\|\phi\|_s$. The norm $|||\ |||_s$ has found fruitful application in the theory of boundary behavior of holomorphic mappings.

Our main theorem is a relatively simple consequence of the following lemma.

LEMMA. *Suppose that D is a smooth bounded domain contained in $\mathbf{C}^n$ which satisfies condition R and that s is a positive integer. There exists a positive integer $M = M(s)$ and a constant $C = C(s)$ such that*

$$\| g \|_s \leq C ||| g |||_{s+M}$$

for all g in $H(D)$.

We shall now prove the theorem, assuming the lemma.

Proof of the Theorem. Let P denote the Bergman projection associated to $D - K$. Let u be a function in $C^\infty(\overline{D - K})$. The function Pu extends to be holomorphic on all of D by Hartog's theorem. We will prove the theorem by showing that for each positive integer s, there are constants $c = c(s)$ and $N = N(s)$ which are independent of u such that

$$\|Pu\|_{W^s(D)} \leq c \operatorname{Sup}\{|\partial^\alpha u(x)| : z \in D - K; |\alpha| \leq N\}.$$

Let s be a fixed positive integer, and let $M = M(s)$ be the constant of the lemma associated to D and s. According to the lemma, $\| Pu \|_s \leq C ||| Pu |||_{s+M}$. Let g be a test function in $H(D)$. To complete the proof of the theorem, we must bound $| \int_D Pu\bar{g} |$ by a constant times

$$\|g\|_{-s-M} \operatorname{Sup}\{|\partial^\alpha u(z)| : z \in D - K; |\alpha| \leq N\}$$

for some integer N, where the constant is independent of g and u. Let Ω be a smooth bounded domain such that $K \subset\subset \Omega \subset\subset D$. Now

$$\int_D Pu\bar{g} = \int_{D-K} Pu\bar{g} + \int_K Pu\bar{g}.$$

The second integral in this sum can be ignored for our purposes because $\| g \|_{L^2(D-K)} \leq$ (constant) $\| g \|_{-s-M}$ and $\| Pu \|_{L^2(K)} \leq$ (constant) $\| u \|_{L^2(D-K)}$. The first integral can be further decomposed:

$$\int_{D-K} Pu\bar{g} = \int_{D-K} u\bar{g} = \int_{D-\Omega} u\bar{g} + \int_{\Omega-K} u\bar{g}.$$

Once again, the second integral in the sum can be ignored because $\| g \|_{L^2(\Omega)} \leq$ (constant) $\| g \|_{-s-M}$. Thus, it remains only for us to estimate the integral $\int_{D-\Omega} u\bar{g}$.

Let $\partial/\partial n$ denote the normal derivative operator on $b(D - \overline{\Omega})$. If ψ is a function such that $\psi = 0 = \partial\psi/\partial n$ on $b(D - \overline{\Omega})$, then $\Delta\psi$ is orthogonal to holomorphic functions on $D - \overline{\Omega}$. This can be seen by performing an integration by parts. We now solve the following elliptic boundary value problem on $D - \overline{\Omega}$:

$$\Delta^m \phi = 0 \quad \text{on } D - \overline{\Omega}$$

where $m = s + M + 2$, and ϕ satisfies the boundary conditions:

$$\begin{cases} \phi = \dfrac{\partial \phi}{\partial n} = 0, \\ \Delta\phi = u, \\ \left(\dfrac{\partial}{\partial n}\right)^t \Delta\phi = \left(\dfrac{\partial}{\partial n}\right)^t u \quad \text{for } t = 1, 2, \ldots, m-3, \end{cases}$$

on bD and $b\Omega$.

The solution ϕ to this problem is such that $u - \Delta\phi$ belongs to the $W^{s+M}(D - \overline{\Omega})$ closure of $C_0^\infty(D - \overline{\Omega})$. To complete the proof of the theorem, observe that

$$\int_{D-\Omega} u\bar{g} = \int_{D-\Omega} (u - \Delta\phi)\bar{g}.$$

The absolute value of this last integral is less than or equal to

$$\|u - \Delta\phi\|_{W^{s+M}(D-\Omega)} \|g\|_{-s-M}.$$

Finally, we must estimate $\|u - \Delta\phi\|_{W^{s+M}(D-\Omega)}$. Now, for each positive integer t, there is a constant C_t which does not depend on u such that $\|\phi\|_t \le C_t \|u\|_{t+Q}$ where Q can be taken to be equal to $(m-3)(m+2)/2$ (see [7]). Hence,

$$\begin{aligned} \|u - \Delta\phi\|_{s+M} &\le C\big(\|u\|_{s+M} + \|\phi\|_{s+M+2}\big) \\ &\le C \operatorname{Sup}\{|\partial^\alpha u(z)| : z \in D - K;\ |\alpha| \le N\} \end{aligned}$$

where $N = s + M + 2 + Q$. This completes the proof of the theorem. □

The proof of the theorem will be legitimate, once we establish the truth of the lemma.

Proof of the Lemma. Since P maps $C^\infty(\overline{D})$ to $C^\infty(\overline{D})$ continuously, there is a positive integer $M = M(s)$ such that $\|P\phi\|_s \le$ (constant) $\|\phi\|_{s+M}$ for all ϕ in $W^{s+M}(D)$.

Let Ω be a relatively compact subset of D, and let g be a function in $H(D)$. The linear functional L on $H(D)$ defined by

$$Lh = \sum_{|\alpha| \le s} \int_\Omega \partial^\alpha h \overline{\partial^\alpha g}$$

is continuous. Hence, There is a function G in $H(D)$ such that $Lh = \langle h, G\rangle_{L^2(D)}$ for all h in $H(D)$. Now

$$\|g\|^2_{W^s(\Omega)} = Lg = \langle g, G\rangle_{L^2(D)} \leq |||g|||_{s+M} \|G\|_{-s-M}.$$

The proof of the lemma will be finished when we prove that $\|G\|_{-s-M} \leq$ (constant) $\|g\|_{W^s(\Omega)}$ where the constant is independent of g and Ω. Indeed, if $\phi \in C_0^\infty(D)$, then

$$\left|\int_D G\bar{\phi}\right| = \left|\int_D G\overline{P\phi}\right| = \left|\sum_{|\alpha|\leq s} \int_\Omega \partial^\alpha g \overline{\partial^\alpha P\phi}\right| \leq \|g\|_{W^s(\Omega)} \|P\phi\|_s$$
$$\leq C\|g\|_{W^s(\Omega)} \|\phi\|_{s+M}.$$

Hence, $\|g\|_{W^s(\Omega)} \leq C|||g|||_{s+M}$. Since the constant C is independent of g and Ω, we obtain that $\|g\|_s \leq C|||g|||_{s+M}$.

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