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H^∞ INTERPOLATION FROM A SUBSET OF THE BOUNDARY

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We obtain necessary and sufficient conditions for a bounded function on an open subset of the boundary of a smooth, bounded domain D in $\mathbf{C}^n$ to be the restriction of a holomorphic function from D into the unit disc. Our condition is a quadratic inequality involving the Szegö kernel of D which is the boundary analogue of the classical Pick-Nevanlinna condition for interpolation in the unit disc.

The classical interpolation theorem of Pick [7] and Nevanlinna [5] provides a necessary and sufficient condition for a function f defined on a subset $\{a_i\}$ of the open unit disc Δ to be the restriction of a holomorphic function from Δ into itself. This condition is a quadratic inequality involving the Szegö kernel of Δ. Specifically, an interpolating function of the required type exists if and only if for every finitely non-zero collection $\{\alpha_i\}$ of complex numbers we have

$$\sum S(a_i, a_j)\left(1 - f(a_i)\overline{f(a_j)}\right)\alpha_i\bar{\alpha}_j \geq 0, \tag{1}$$

where

$$S(z, \zeta) = \left[2\pi(1 - z\bar{\zeta}\,)\right]^{-1}.$$

This result has been generalized by FitzGerald and Horn [4] (under the additional hypothesis that $\{a_i\}$ is a set of uniqueness for holomorphic functions) to more general sesquiholomorphic positive definite kernels defined in arbitrary domains in $\mathbf{C}^n$. In this note we address the analogous problem when the function f is specified on a subset of the boundary. In the case of the disc or the upper half plane, this problem has been considered by FitzGerald [3] and by Rosenblum and Rovnyak [8].

The method presented here applies to a rather general class of domains in $\mathbf{C}^n$. Our condition is similar to (1), but the summation must be replaced by integration. Thus, a bounded function f defined on an open subset E of the boundary (or distinguished boundary) of a domain D gives the boundary values of a holomorphic function from D into Δ if and only if for every $\rho \in L^2(E)$

$$\int_E\int_E S(z, \zeta)\left(1 - f(z)\overline{f(\zeta)}\right)\rho(z)\overline{\rho(\zeta)}\,d\sigma(z)\,d\sigma(\zeta) \geq 0. \tag{2}$$

Here $S(z, \zeta)$ is the Szegö kernel for D and σ is the induced measure on ∂D (or on the distinguished boundary in the case of a product domain).

Because the Szegö kernel is singular on ∂D, some care must be taken to interpret (2) properly. A precise formulation is given in §1.

1. Preliminaries. Let D be bounded a domain in $\mathbf{C}^n$ with C^2 boundary. A *characterizing function* for D is a real valued C^1 function ρ on $\mathbf{C}^n$ such that $D = \{\rho < 0\}$ and $d\rho \neq 0$ on ∂D. If ρ is any characterizing function, then there is an $\varepsilon_0 > 0$ such that the domains $D_\varepsilon = \{\rho < -\varepsilon\}$ are subdomains of D with C^1 boundary for $0 < \varepsilon < \varepsilon_0$. Let σ and σ_ε denote the surface measures on ∂D and ∂D_ε respectively.

For $0 < \rho < \infty$, the *Hardy class* $H^p(D)$ is the class of all holomorphic functions f on D satisfying

$$|||f|||_p = \sup_{0<\varepsilon<\varepsilon_0} \left(\int_{\partial D_\varepsilon} |f|^p \, d\sigma_\varepsilon \right)^{1/p} < \infty.$$

If we replace $|f|^p$ by $\log^+|f| = \max\{\log|f|, 0\}$, we obtain the *Nevenlinna Class* $\mathfrak{N}(D)$. It follows from subharmonicity of $\log|f|$ and $|f|^p$ that these classes are independent of the characterizing function used in the definition. For $1 \leq p < \infty$, $|||\cdot|||_p$ is a norm which makes $H^p(D)$ into a Banach space. Clearly the Nevanlinna class contains all of the Hardy classes.

Any function $f \in \mathfrak{N}(D)$ has non-tangential limits at almost every point of ∂D. (In fact, even more is true. See **[9]**, Chapter 2.) We will denote the boundary value function by f^*. (We will occasionally omit the * and write simply $f(\zeta)$ for $f^*(\zeta)$ when $\zeta \in \partial D$.) If $f \in H^p(D)$ then $f^* \in L^p(\partial D)$ and moreover "dilations" of f converge to f^* in $L^p(\partial D)$. To make this statement precise, let π be the normal projection of a tubular neighborhood of ∂D onto ∂D. Then for ε sufficiently small, $\pi_\varepsilon = \pi|_{\partial D_\varepsilon}$ is a C^1 diffeomorphism of ∂D_ε onto ∂D. For any function u on D we define $u_\varepsilon = u \circ \pi_\varepsilon^{-1}$. Our assertion is that for $f \in H^p(D)$, $0 < p < \infty$, f_ε converges to f^* in $L^p(\partial D)$. The mapping $f \to f^*$ is a linear isomorphism from $H^p(D)$ onto a closed subspace of $L^p(\partial D)$. We will denote the range of this mapping by $H^p(\partial D)$. For $f \in H^p(D)$ we define $\|f\|_p$ to be $\|f^*\|_{L^p(\partial D)}$. For $p \geq 1$, the norms $\|\cdot\|_p$ and $|||\cdot|||_p$ are equivalent. The reader is referred to Stein **[9]** for a thorough investigation of boundary behavior of holomorphic functions.

The space $H^2(D)$ with the norm $\| \quad \|_2$ is a Hilbert space. Moreover, point evaluation at any point of D is a continuous linear functional. It follows that $H^2(D)$ has a reproducing kernel $S(z, \zeta)$ defined on $D \times D$. (See Aronszajn **[1]**). The kernel $S(z, \zeta)$ is the *Szegö kernel* for D, and has the following properties:

1. $S(\cdot, \zeta) \in H^2(D)$ for any fixed $\zeta \in D$. In particular, $S(\cdot, \zeta)$ has non-tangential limits at almost every point of ∂D.

2. For any fixed $z_1, \ldots, z_N \in D$ the matrix $[S(z_i, z_j)]$ is positive definite. In particular, $S(\zeta, z) = \overline{S(z, \zeta)}$ and $S(z, z) > 0$.

3. Let S: $L^2(\partial D) \to H^2(\partial D)$ be the orthogonal projection. Then $Sf = (\tilde{S}f)^*$ where $\tilde{S}$: $L^2(\partial D) \to H^2(D)$ is defined by

$$\tilde{S}f(z) = \langle f, S(\cdot, z)\rangle_{L^2(\partial D)} = \int_{\partial D} f(\zeta) S(z, \zeta)\, d\sigma(\zeta).$$

We will henceforth omit the tilde and use the symbol S to denote either operator.

We can now give a precise formulation of our main result.

(1.1) THEOREM. *Let D be a bounded domain in $\mathbf{C}^n$ with a C^2 boundary, E an open subset of ∂D, and f a bounded, measurable function on E. Then the following conditions are equivalent.*

(i) *There is a holomorphic function $\tilde{f}$ from D into Δ (the open unit disc in $\mathbf{C}$) such that $\tilde{f}^*|_E = f$.*

(ii) *For every $\rho \in L^2(E)$ we have $\|S\chi_E\bar{\rho}\bar{f}\|_2 \le \|S\chi_E\bar{\rho}\|_2$.*

Here χ_E denotes the characteristic function of E, and we have adopted the convention that, if ρ is a function on E, then $\chi_E\rho$ denotes the function on ∂D which agrees with ρ on E and which vanishes on $\partial D \setminus E$. Theorem (1.1) will be proved in §2.

Condition (ii) in the above theorem can be reconciled with the integral condition (2) in the introduction as follows. For any $\rho \in L^2(E)$ we have

$$\begin{aligned}\|S\chi_E\bar{\rho}\|^2 &= \langle S\chi_E\bar{\rho}, S\chi_E\bar{\rho}\rangle_{H^2(\partial D)} = \langle S\chi_E\bar{\rho}, \chi_E\bar{\rho}\rangle_{L^2(\partial D)}\\ &= \lim_{\varepsilon\to 0^+}\int_E\int_E S(z - \varepsilon\nu(z), \zeta)\rho(z)\overline{\rho(\zeta)}\,d\sigma(z)\,d\sigma(\zeta).\end{aligned}$$

Here $\nu(z)$ denotes the outward unit normal to ∂D at z. Thus condition (ii) of Theorem (1.1) can be reformulated as follows:

(ii′) *For every $\rho \in L^2(E)$ we have*

$$\lim_{\varepsilon\to 0^+}\int_E\int_E S(z - \varepsilon\nu(z), \zeta)\big(1 - f(z)\overline{f(\zeta)}\big)\rho(z)\overline{\rho(\zeta)}\,d\sigma(z)\,d\sigma(\zeta) \ge 0.$$

Finally, we remark that, in the one variable case, condition (ii) can also be formulated in terms of principal value integrals (c.f. FitzGerald [3]).

2. Proof of the main theorem. We begin with a uniqueness result.

(2.1) THEOREM. *Let D be a bounded domain in $\mathbf{C}^n$ with a C^2 boundary, and let E be a Borel set in ∂D with positive measure. Let $f \in \mathfrak{N}(D)$ and suppose that $f^*|_E \equiv 0$. Then $f \equiv 0$.*

Proof. Suppose $f \not\equiv 0$. Let $z_0 \in D$ with $f(z_0) \neq 0$. If $G(z, \zeta)$ is the Green's function for D, then $\rho = G(z_0, \cdot)$ is a characterizing function for D. For $t \in R$ and $c > 0$, set $\log_t c = \max\{\log c, t\}$ and $\log^- c = \min\{\log c, 0\}$. Since $\log_t |f|$ is subharmonic in D we have, for $\varepsilon > 0$ sufficiently small,

$$\log|f(z_0)| \leq \int_{\partial D_\varepsilon} \log_t |f(\zeta)| \left(-\frac{\partial}{\partial \nu_\zeta} G(z_0, \zeta)\right) d\sigma_\varepsilon(\zeta), \tag{3}$$

where $D_\varepsilon = \{\zeta \in D\colon\ G(z_0, \zeta) < -\varepsilon\}$. (Note that for $\zeta \in \partial D_\varepsilon$, $-\partial G(z_0, \zeta)/\partial \nu_\zeta$ is the Poisson kernel for D_ε). It is immediate from (3) that for $t \leq 0$ we have

$$\log|f(z_0)| \leq c_1 \int_{\partial D_\varepsilon} \log_t^- |f|\, d\sigma_\varepsilon + c_2 \int_{\partial D_\varepsilon} \log^+ |f|\, d\sigma_\varepsilon$$

where c_1 and c_2 are positive constants. Since f is in the Nevanlinna class, the last term is bounded, so we obtain

$$\int_{\partial D_\varepsilon} \log_t^- |f|\, d\sigma_\varepsilon \geq \text{const.} \tag{4}$$

As in §1, we let π_ε be the restriction to ∂D_ε of the normal projection onto ∂D, and set $f_\varepsilon = f \circ \pi_\varepsilon^{-1}$. Then we have

$$\int_{\partial D_\varepsilon} \log_t^- |f|\, d\sigma_\varepsilon \leq c_3 \int_{\partial D} \log_t^- |f_\varepsilon|\, d\sigma \tag{5}$$

for some positive constant c_3. It follows from (4) and (5) that for sufficiently small $\varepsilon > 0$ and for $t \leq 0$,

$$\int_{\partial D} \log_t^- |f_\varepsilon|\, d\sigma \geq c_4 > -\infty.$$

Letting $t \to -\infty$, we obtain from the Monotone Convergence Theorem that

$$\int_{\partial D} \log^- |f_\varepsilon|\, d\sigma \geq c_4.$$

But f_ε converges to f^* almost everywhere, so by Fatou's lemma

$$\int_{\partial D} \log^- |f^*|\, d\sigma \geq \limsup_{\varepsilon \to 0^+} \int_{\partial D} \log^- |f_\varepsilon|\, d\sigma \geq c_4 > -\infty.$$

It follows that f^* cannot vanish on a set of positive measure, so the proof is complete.

Let us now turn to the proof of Theorem (1.1). That the first condition implies the second is an immediate consequence of the following simple Hilbert space result.

(2.2) LEMMA. *Let P be the orthogonal projection from a Hilbert space H onto a closed subspace K of H and let C be a contraction operator on H which leaves K invariant. Then the operator $T = P - CPC^*$ is non-negative.*

Proof. Since T is self adjoint, it suffices to show that $\langle T\phi, \phi\rangle \geq 0$ for $\phi \in R(T)$. For such ϕ we have

$$\|PC^*\phi\|^2 = \langle PC^*\phi, PC^*\phi\rangle = \langle PC^*\phi, C^*\phi\rangle = \langle CPC^*\phi, \phi\rangle$$
$$\leq \|CPC^*\phi\| \, \|\phi\| \leq \|PC^*\phi\| \, \|\phi\|.$$

But $\phi \in R(T) \subset K$, so $\phi = P\phi$. Thus we have $\|PC^*\phi\| \leq \|P\phi\|$ from which the desired result follows immediately.

To show that condition (i) of Theorem (1.1) implies condition (ii), apply Lemma (2.2) with H, K, P, and C replaced by $L^2(\partial D)$, $H^2(\partial D)$, S, and multiplication by f^* respectively.

We now consider the reverse implication. We begin by showing that if condition (ii) is satisfied then there is an interpolating function in the Hardy class $H^2(D)$.

(2.3) LEMMA. *Let D be a bounded domain in $\mathbf{C}^n$ with C^2 boundary, E a Borel set in ∂D with positive measure, and let f be a measurable function on E such that $\|S\chi_E\bar{\rho}\bar{f}\|_2 \leq \|S\chi_E\bar{\rho}\|_2$ for every $\rho \in L^2(E)$. Then there is a unique $\tilde{f} \in H^2(D)$ with $\tilde{f}^*|_E = f$.*

Proof. The uniqueness assertion follows from Theorem (2.1). For the existence proof, we introduce a conjugation operator τ on $H^2(D)$. Let $\{\phi_j\}$ be any orthonormal basis for $H^2(D)$ with $\phi_0 \equiv (\sigma(\partial D))^{-1}$, and define $\tau(\Sigma\, a_j\phi_j) = \Sigma\, \bar{a}_j\phi_j$. (Of course τ depends on the choice of basis). Then τ is an isometry of $H^2(D)$ satisfying $\langle \tau\phi, \psi\rangle = \langle \tau\psi, \phi\rangle$ for any $\phi, \psi \in H^2(D)$. Set $M_0 = \{\tau S\chi_E\rho\colon\ \rho \in L^2(E)\}$, and define $C\colon M_0 \to H^2(D)$ by $C(\tau S\chi_E\rho) = \tau S\chi_E\underline{f}\rho$. Our hypothesis implies that C is a well defined operator of norm at most 1, so C has a unique continuous extension to $M = \overline{M}_0$, which we continue to denote by C. (In fact, it can be shown that $M = H^2(D)$, but this is not crucial here). Let $C^*\colon H^2(D) \to M$ be the Hilbert space adjoint, and set $\tilde{f} = \tau C^*1$. Then for any $\rho \in L^2(E)$ we have

$$\langle \tilde{f}^*, \rho\rangle_{L^2(E)} = \langle \tau C^*1, S\chi_E\rho\rangle_{H^2(D)} = \langle \tau S\chi_E\rho, C^*1\rangle$$
$$= \langle \tau S\chi_E\bar{f}\rho, 1\rangle = \langle 1, \chi_E\bar{f}\rho\rangle = \langle f, \rho\rangle_{L^2(E)},$$

so $\tilde{f}^*|_E = f$.

To complete the proof of Theorem (1.1) we must show that the interpolating function constructed above is in fact bounded by 1. We will require some technical lemmas.

(2.4) LEMMA. *Let D and D_0 be bounded domains in $\mathbf{C}^n$ with C^2 boundaries with $D_0 \subset D$. Then the restriction mapping $f \to f|_{D_0}$ is a continuous linear operator from $H^2(D)$ to $H^2(D_0)$.*

Proof. Let P and P_0 be the Poisson kernels for D and D_0 respectively and let $z_0 \in D_0$ be fixed. For $f \in H^2(D)$, set $h(z) = \int_{\partial D} |f(\zeta)|^2 P(z, \zeta)\, d\sigma(\zeta)$. Then

$$\begin{aligned}
\int_{\partial D_0} |f(\zeta)|^2\, d\sigma(\zeta) &\leq \int_{\partial D_0} h(\zeta)\, d\sigma(\zeta) \\
&\leq C_1 \int_{\partial D_0} h(\zeta) P_0(z_0, \zeta)\, d\sigma(\zeta) \\
&= C_1 \int_{\partial D} |f(\zeta)|^2 P(z_0, \zeta)\, d\sigma(\zeta) \\
&\leq C_1 C_2 \int_{\partial D} |f(\zeta)|^2\, d\sigma(\zeta)
\end{aligned}$$

where

$$C_1 = \left[\min\{P_0(z_0, \zeta) \colon \zeta \in \partial D_0\}\right]^{-1} \quad \text{and} \quad C_2 = \max\{P(z_0, \zeta) \colon \zeta \in \partial D\}.$$

(2.5) LEMMA. *Let D and D_0 be as in* (2.4) *and let $S(z, \zeta)$ be the Szegö kernel for D. For $u \in L^2(\partial D_0)$, $z \in D$, define*

$$(Tu)(z) = \int_{\partial D_0} u(\zeta) S(z, \zeta)\, d\sigma(\zeta).$$

Then T is a continuous linear operator from $L^2(\partial D_0)$ into $H^2(D)$.

Proof. Set $\Gamma_1 = \partial D_0 \cap \partial D$, $\Gamma_2 = \partial D_0 \backslash \partial D$, and write $T = T_1 + T_2$ where

$$(T_j u)(z) = \int_{\Gamma_j} u(\zeta) S(z, \zeta)\, d\sigma(\zeta), \qquad j = 1, 2.$$

We will show that each T_j is a continuous operator from $L^2(\partial D_0)$ into $H^2(D)$.

We can easily dispose of T_1. For $u \in L^2(\partial D_0)$ we have $\|T_1 u\| = \|S\chi_{\Gamma_1} u\| \leq \|u\|_{L^2(\Gamma_1)}$.

For T_2, let us first assume that u is a continuous function with compact support in Γ_2. One easily checks that the mapping $\zeta \to S_\zeta = S(\cdot, \zeta)$ is continuous from D into $H^2(D)$. Thus $\zeta \to u(\zeta)S_\zeta$ is a continuous $H^2(D)$ valued function with compact support in Γ_2. It follows that $\int_{\Gamma_2} u(\zeta)S_\zeta \, d\sigma(\zeta)$ converges in the norm of $H^2(D)$ to some function $\tilde{T}_2 u \in H^2(D)$. Moreover, for any $z \in D$ we have

$$(\tilde{T}_2 u)(z) = \langle \tilde{T}_2 u, S_z \rangle = \left\langle \int_{\Gamma_2} u(\zeta)S_\zeta \, d\sigma(\zeta), S_z \right\rangle$$

$$= \int_{\Gamma_2} u(\zeta)\langle S_\zeta, S_z \rangle \, d\sigma(\zeta) = \int_{\Gamma_2} u(\zeta)S(z, \zeta) \, d\sigma(\zeta).$$

Thus $\tilde{T}_2 u = T_2 u$, and so $T_2 u \in H^2(D)$ and

$$\|T_2 u\|^2 = \left\langle \int_{\Gamma_2} u(\zeta)S_\zeta \, d\sigma(\zeta), \int_{\Gamma_2} u(\eta)S_\eta \, d\sigma(\eta) \right\rangle$$

$$= \int_{\Gamma_2}\int_{\Gamma_2} u(\zeta)\overline{u(\eta)}\langle S_\zeta, S_\eta \rangle \, d\sigma(\zeta) \, d\sigma(\eta)$$

$$= \int_{\Gamma_2} (T_2 u)(\eta)\overline{u(\eta)} d\sigma(\eta) \leq \|T_2 u\|_{L^2(\Gamma_2)} \|u\|_{L^2(\Gamma_2)}$$

$$\leq \text{const} \|T_2 u\|_{H^2(D)} \|u\|_{L^2(\partial D_0)}.$$

The last inequality follows from Lemma (2.4). This yields the required estimate when u is a continuous function with compact support in Γ_2.

Now let $u \in L^2(\partial D_0)$ be arbitrary. Choose a sequence $\{u_j\}$ of continuous functions with compact support in Γ_2 such that $\lim \|u - u_j\|_{L^2(\Gamma_2)} = 0$. By the above estimate, $\|T_2 u_j\|_{H^2(D)}$ is bounded, so we may assume, after passing to a subsequence, that $\{T_2 u_j\}$ has a weak limit v in $H^2(D)$. But for $z \in D$ we have

$$v(z) = \langle v, S_z \rangle = \lim \langle T_2 u_j, S_z \rangle$$

$$= \lim \int_{\Gamma_2} u_j S(z, \zeta) \, d\sigma(\zeta) = (T_2 u)(z)$$

and so $\|T_2 u\| = \|v\| = \lim \|T_2 u_j\| \leq \text{const} \|u\|_{L^2(\Gamma_2)}$.

Combining the estimates for T_1 and T_2 yields the desired estimate for T.

(2.6) LEMMA. *Let D be a bounded domain in $\mathbf{C}^n$ with C^2 boundary, E an open subset of ∂D and let $f \in H^2(D)$ be such that $f^*|_E$ is bounded and $\|S\chi_E \bar{\rho} \bar{f}\| \leq \|S\chi_E \bar{\rho}\|$ for every $\rho \in L^2(E)$. Then the kernel $K(z, \zeta) = S(z, \zeta)\,(1 - f(z)\overline{f(\zeta)})$ is positive definite in $D \times D$.*

Proof. (cf. Donoghue **[2]** Theorem 4). Choose a proper subdomain D_0 of D with C^2 boundary such that $\partial D_0 \cap \partial D \subset E$, and such that $E' = \partial D_0 \cap \partial D$ has positive surface measure. By representing f as a Poisson integral in D, one easily verifies that f is bounded on D_0. We define an operator K: $L^2(\partial D_0) \to H^2(D_0)$ by

$$(Ku)(z) = \int_{\partial D_0} K(z, \zeta) u(\zeta)\, d\sigma(\zeta).$$

Note that K is a continuous operator by Lemmas (2.4) and (2.5). By identifying $H^2(D_0)$ with $H^2(\partial D_0)$, we may view K as a self adjoint operator on $L^2(\partial D_0)$. We will show that K is a positive operator. Since the range of K is contained in $H^2(\partial D_0)$ it suffices to show that $\langle Ku, u\rangle \geq 0$ whenever $u \in H^2(\partial D_0)$. Let R: $H^2(\partial D_0) \to L^2(E')$ denote the restriction mapping. By Theorem (2.1), R is one to one, so R^*: $L^2(E') \to H^2(\partial D_0)$ has dense range. Thus to verify that K is a positive operator, it suffices to show that $\langle KR^*\rho, R^*\rho\rangle_{L^2(\partial D_0)} \geq 0$ for every $\rho \in L^2(E')$. But one can easily check that

$$\langle KR^*\rho, R^*\rho\rangle_{L^2(\partial D_0)} = \| S\chi_{E'}\rho\|^2_{H^2(D)} - \| S\chi_{E'}\bar{f}\rho\|^2_{H^2(D)}$$

which is non-negative by hypothesis, so K is a positive operator on $L^2(\partial D_0)$.

Next we show that $K(z, \zeta)$ is a positive definite kernel on $(\partial D_0 \backslash \partial D) \times (\partial D_0 \backslash \partial D)$. Let $\{p_1, \ldots, p_N\}$ be arbitrary points in $\partial D_0 \backslash \partial D$. For $1 \leq j \leq N$, choose a family $\{\phi_j^\varepsilon, \varepsilon > 0\}$ of continuous, non-negative functions on $\partial D_0 \backslash \partial D$ such that $\int \phi_j^\varepsilon\, d\sigma = 1$ and such that the support of ϕ_j^ε is contained in $\{|z - p_j| < \varepsilon\}$. Then for any complex numbers $\{\alpha_1, \ldots, \alpha_N\}$ we have

$$\begin{aligned}
\sum K(p_i, p_j)\alpha_i\bar{\alpha}_j &= \sum \alpha_i\bar{\alpha}_j \lim_{\varepsilon\to 0^+} \int\!\!\int K(z, \zeta)\phi_i^\varepsilon(z)\phi_j^\varepsilon(\zeta)\, d\sigma(z)\, d\sigma(\zeta) \\
&= \lim_{\varepsilon\to 0^+} \int\!\!\int K(z, \zeta)\Big(\sum \alpha_i\phi_i^\varepsilon(z)\Big)\Big(\overline{\sum \alpha_i\phi^\varepsilon(\zeta)}\Big)\, d\sigma(z)\, d\sigma(\zeta) \\
&= \lim_{\varepsilon\to 0^+} \Big\langle K\Big(\sum \alpha_i\phi_i^\varepsilon\Big), \sum \alpha_i\phi_i^\varepsilon\Big\rangle
\end{aligned}$$

which is non-negative since K is a positive operator. Thus $K(z, \zeta)$ is a positive definite kernel on $(\partial D_0 \backslash \partial D) \times (\partial D_0 \backslash \partial D)$.

Finally, to conclude that $K(z, \zeta)$ is positive definite on $D \times D$ it is only necessary to observe that if we are given any finite set of points in D, then the domain D_0 in the above argument could be chosen so that the given points lie in the boundary of D_0.

With the aid of Lemma (2.6) it is now a simple matter to complete the proof of Theorem (2.1). If condition (ii) holds, then, by Lemma (2.3), there is an interpolating function $\tilde{f}$ in the Hardy class $H^2(D)$. But by

Lemma (2.6), the kernel $K(z, \zeta) = S(z, \zeta)(1 - \tilde{f}(z)\overline{\tilde{f}(\zeta)})$ is positive definite. In particular, $K(z, z) \geq 0$ for every $z \in D$. Since $S(z, z) > 0$, it follows that $|f(z)| \leq 1$ in D, and the proof is complete.

3. Concluding remarks. 1. Theorem (1.1) can be generalized to the case of product domains, provided that each factor is a bounded domain with C^2 boundary. In this case, ∂D must be replaced by the distinguished boundary $\partial_0 D$, σ is replaced by the product of the surface measures of the factors, and $S(z, \zeta)$ is the product of the Szegö kernels of the factors. The Hardy class H^p is identified with a subspace of $L^p(\partial_0 D)$, and the constructions in the above arguments are carried out in each factor.

2. In principle, it should be possible to recover the interpolating function from its values on E. An explicit formula has been obtained by Patil **[6]** in the case of the polydisc. It would be of interest to obtain analogous formulas for other domains.

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