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INTEGRAL INVARIANTS OF FUNCTIONS AND L^p ISOMETRIES ON GROUPS

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For $p \in (0, \infty)$ and not an even integer it is proved that every isometric multiplier on an invariant subspace of $L^p(G)$ is a translation operator.

1. Introduction. Let f and g be real-valued measurable functions on $\mathbf{R}$ which satisfy

$$\int_{\mathbf{R}} \left| \sum \alpha_j f(t_j + x) \right|^p dx = \int_{\mathbf{R}} \left| \sum \alpha_j g(t_j + x) \right|^p dx < \infty$$

for arbitrary finite sets of real numbers $\{\alpha_j\}$ and $\{t_j\}$. In [**3**], M. Kanter showed that if $p \in (0, \infty)$ is not an even integer then for some $\varepsilon = \pm 1$ and some $t_0 \in \mathbf{R}$, $g(x) = \varepsilon f(t_0 + x)$ a.e. When rephrased in the language of multipliers, Kanter's theorem becomes: let F be the closed linear span in $L^p(\mathbf{R})$ of the translates of f. Suppose $p \in (0, \infty)$ is not an even integer and that R: $F \to L^p(\mathbf{R})$ is an isometry which commutes with translations. Then for some $\varepsilon = \pm 1$ and some $t_0 \in \mathbf{R}$

$$Rf(x) = \varepsilon f(t_0 + x).$$

A related theorem was proved by R. S. Strichartz [**8**] in the case of a locally compact group. Namely, Strichartz showed that if $p \in [1, \infty)$ and $p \neq 2$ then each invertible isometric multiplier on $L^p(G)$ is a translation operator. Since the space F in Kanter's theorem need not equal $L^p(\mathbf{R})$ it is clear that Kanter's theorem does not follow from that of Strichartz. Also since Strichartz's theorem only requires that $p \neq 2$ and Strichartz's group is arbitrary it is clear that his results do not follow from those of Kanter.

The main result in this paper is an extension of Kanter's theorem to an arbitrary locally compact group G. The restriction that p is not an even integer is still needed but we will see that the proof also contains new information for $p \neq 2$.

Concerning the restriction on p, Strichartz's result is known to be false if $p = 2$. Katznelson [**4**] showed that Kanter's theorem fails if p is an even integer. Precisely what does happen for $p = 2n \geq 4$ is not yet understood but is related to the work of R. L. Adler and A. G. Konheim on higher order autocorrelation functions on abelian groups [**1**]. In §5 we give a modest extension of the Adler-Konheim results to nonabelian groups.

Our results are also complemented by the work [6, 7] of D. M. Oberlin on non-isometric multipliers on subspaces of $L^p(G)$ when G is compact and abelian. For $E \subset \hat{G}$, he studies multipliers on the space $L_E^p \subset L^p(G)$ of functions whose Fourier transforms vanish off E. He shows that when G is infinite and $1 \leq p < 2$ or $p = 2n \geq 4$ that there are sets E for which not every multiplier on L_E^p extends to a multiplier on $L^p(G)$.

2. Statement of results. Let G be a locally compact Hausdorff group with left invariant Haar measure dm defined on the Borel σ-field $\mathfrak{B}$ of G. $\Delta(x)$ will denote the modular function on G which is defined by $\Delta(x)m(Bx) = m(B)$ for each $B \in \mathfrak{B}$. For $0 < p < \infty$, $L^p(G) = L^p(G, \mathfrak{B}, m)$ will denote either the real or the complex L^p space with the norm $\|f\|_p^p = \int_G |f(x)|^p \, dm(x)$. The left and right translation operators will be denoted as $\lambda_g f(x) \equiv f(gx)$ and $\rho_g f(x) \equiv f(xg)$. A linear space F of functions on G is called left-invariant (resp. right-invariant) if $\lambda_g F = F$ (resp. $\rho_g F = F$) for each $g \in G$. If F is both left-invariant and right-invariant, it is bi-invariant. If $F \subseteq L^p(G)$ is left-invariant (resp. right-invariant) and R: $F \to L^p(G)$ is a bounded operator it is called a right multiplier (resp. left multiplier) provided R commutes with each left (resp. right) translation operator. If F is bi-invariant and R commutes with both left and right translations, R is called a central multiplier.

We now state our main results.

THEOREM 1. *Suppose $p \in (0, \infty)$ is not an even integer.*

(a) *If $F \subseteq L^p(G)$ is left-invariant and R: $F \to L^p(G)$ is an isometric right multiplier then R has the form*

$$Rf = c[\Delta(h^{-1})]^{1/p} \rho_h f, \qquad f \in F,$$

for some c with $|c| = 1$ and some $h \in G$.

(b) *If $F \subseteq L^p(G)$ is right-invariant and R is an isometric left multiplier then R has the form*

$$Rf = c\lambda_h f$$

for some c with $|c| = 1$ and some $h \in G$.

THEOREM 2. *If $p \in (0, \infty)$ is not an even integer, then $R\colon F \to L^p(G)$ is an isometric central multiplier iff R has the form*

$$Rf = c\lambda_k f$$

for some c with $|c| = 1$ and some $k \in G$ satisfying

$$f(gkx) = f(kgx) \quad a.e. \tag{2.1}$$

for each $f \in F$ and each $g \in G$.

THEOREM 3. (*See Kanter* [3].) *Suppose $p \in (0, \infty)$ is not an even integer and that $f_1, f_2 \in L^p(G)$ are real (resp. complex) valued. If the identity*

$$\left\| \sum \alpha_j \lambda_{t_j} f_1 \right\|_p^p = \left\| \sum \alpha_j \lambda_{t_j} f_2 \right\|_p^p \tag{2.2}$$

holds for all finite sets $\{t_1, \ldots, t_n\} \subseteq G$ and $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbf{R}$ (resp. $\mathbf{C}$) then there is a c with $|c| = 1$ and an $h \in G$ with

$$f_2(x) = c\left[\Delta(h^{-1})\right]^{1/p} \rho_h f_1(x) \quad a.e.\ [m].$$

In case $p = 2k$ these results fail but something remains. The computation of the norms $\|\sum \alpha_j \lambda_{t_j} f\|_{2k}^{2k}$ is equivalent to computing the functions

$$r_{k,k}(f)(t_1, \ldots, t_k; s_1, \ldots, s_k) \equiv \int_G \prod_{i=1}^k f(t_i x) \prod_{j=1}^k \bar{f}(s_j x)\, dx. \tag{2.3}$$

(Here $\bar{f}$ denotes complex conjugation.)

It follows from the examples of Katznelson [4] that $r_{k,k}(f)$ does not in general determine f. On the other hand, Adler and Konheim [1] have shown that if $f \in L^1(G)$ is real-valued and G is abelian then the sequence $\{r_k(f)\}_1^\infty$ of so-called kth order autocorrelation functions

$$f_k(f)(t_1, \ldots, t_k) = \int_G f(x) f(t_1 x) \cdots f(t_k x)\, dm(x)$$

determine f up to a translation. Our modest result here is

THEOREM 4. *Let f_1, $f_2 \in L^p(G)$, $1 \le p \le \infty$. Then for $k \ge p/2$ the integrals defining the functions*

$$r_{k,k}(f_i)(t_1, \ldots, t_k; s_1, \ldots, s_k), \qquad i = 1, 2,$$

converge a.e. Moreover, for each integer $N \ge p/2$, if

$$r_{jN,jN}(f_1) = r_{jN,jN}(f_2) \quad a.e.\ for\ all\ j = 1, 2, \ldots \tag{2.4}$$

then for some c with $|c| = 1$ and some $h \in G$,

$$f_2(x) = c\rho_h f_1(x) \quad a.e.$$

3. Proofs of Theorems 2 and 3. Theorems 2 and 3 are elementary corollaries of Theorem 1.

Proof of Theorem 2. By part (b) of Theorem 1 we know that R has the form $c\lambda_k$ for some c with $|c| = 1$ and some $k \in G$. Each such operator is a left multiplier. The condition that λ_k commutes with each λ_g on F is condition (2.1).

Incidentally, the set of elements $k \in G$ satisfying (2.1) is a closed normal subgroup of G. Thus if G is simple there are no bi-invariant subspaces $F \subseteq L^p(G)$ with $\dim F \geq 2$ which admit non-trivial isometric central multipliers.

Proof of Theorem 3. Let F be the linear span of the translates $\{\lambda_t f_1: t \in G\}$ of f_1. Define the operator $R: F \to L^p(G)$ by

$$R\left(\sum \alpha_j \lambda_{t_j} f_1\right) \equiv \sum \alpha_j \lambda_{t_j} f_2.$$

From (2.2) we see R is an isometry. From the definition of R, R is a right multiplier and hence by Theorem 1, $R = c[\Delta(h^{-1})]^{1/p}\rho_h$ and hence

$$f_2(x) = c\left[\Delta(h^{-1})\right]^{1/p}\rho_h f_1(x).$$

The same proof and Theorem 2 immediately give a two-sided version of Theorem 3 which we state as a

COROLLARY 2.1. *Suppose $p \in (0, \infty)$ is not an even integer and that f_1, $f_2 \in L^p(G)$ satisfy*

$$\int \left|\sum \alpha_j f_1(t_j x s_j)\right|^p dm(x) = \int \left|\sum \alpha_j f_2(t_j x s_j)\right|^p dm(x) < \infty$$

for all $\{\alpha_1, \ldots, \alpha_n\}$ and $\{t_1, s_1, \ldots, t_n, s_n\} \subset G$. Then for some c with $|c| = 1$ and some k satisfying

$$f_1(gkx) = f_1(kgx) \quad \textit{for a.a. } g, x$$

we have

$$f_2(x) = cf_1(kx).$$

4. Proof of Theorem 1. The proof will be given first in the σ-finite case. At the end this condition will be removed. The proof breaks into several steps which we outline here.

(a) The general results of Hardin [2] are applied to extend R to an isometric right multiplier $\tilde{R}$ on a (possibly) larger subspace $\tilde{F} \supseteq F$ of $L^p(G)$, which is explicitly described.

(b) Group theoretic arguments are then given to establish the existence of a compact subgroup $K \subseteq G$ and of a fuction $j(x)$ with $|j(x)| = 1$ a.e. which satisfy

$$\tilde{F} = j(x) \cdot L^p(G/K) \subseteq L^p(G). \tag{4.1}$$

(c) Arguments analogous to those of Strichartz in [8] are applied to characterize isometric multipliers on spaces $\tilde{F}$ of the form (4.1). The results are then lifted to $L^p(G)$.

Part (a) *Preliminary results* [2].

Let $(X, \mathcal{B}, m)$ denote a σ-finite measure space and suppose $p \in (0, \infty)$ is not an even integer. If $\mathcal{F} \subseteq \mathcal{B}$ is a sub-σ-field we write $\mathfrak{M}(\mathcal{F}) = \mathfrak{M}(X, \mathcal{F}, m)$ for the algebra of all (equivalence classes of) $\mathcal{F}$-measurable functions. If F is a space of functions on X we write $\mathfrak{M}(\mathcal{F}) \cdot F$ for the minimal vector space containing F and closed under multiplication by functions in $\mathfrak{M}(\mathcal{F})$. $\mathfrak{M}(\mathcal{F}) \cdot F$ is an $\mathfrak{M}(\mathcal{F})$-module.

If Φ: $\mathfrak{M}(\mathcal{F}_1) \leftrightarrow \mathfrak{M}(\mathcal{F}_2)$ is an algebra isomorphism there is a unique non-singular σ-field isomorphism ϕ: $\mathcal{F}_1 \to \mathcal{F}_2$ with $\Phi 1_E(x) = 1_{\phi(E)}(x)$. Here 1_E denotes the indicator function of the set E. The restriction of Φ is an isometry of $L^\infty(X, \mathcal{F}_1 m)$ onto $L^\infty(X, \mathcal{F}_2, m)$.

We denote by $l(y)$ a measure preserving automorphism of $(X, \mathcal{F}, m)$ and let $Lf(x) = f(l(x))$ be the associated function transformation.

Let F_1 be a closed subspace of $L^p(X, \mathcal{B}, m)$ and let R: $F_1 \to L^p(X, \mathcal{B}, m)$ be a linear isometry. The range $F_2 = RF_1$ of R is a closed subspace of $L^p(X, \mathcal{B}, m)$. We say that L commutes with R if $LF_1 = F_1$ and $LR = RL$. Note that if L and R commute then $LF_2 = F_2$.

A function $f_i \in F_i$ is said to have full support in F_i if $m\{x\colon f_i(x) = 0$ and $g(x) \neq 0\} = 0$ for each $g \in F_i$. The "ratio" σ-field generated by ratios $g_i(x)/f_i(x)$ of functions with full support in F_i is written as $\mathcal{F}_i$

The following proposition summarizes in a convenient form the results from [2] which we require.

PROPOSITION 4.1. (*Hardin* [2].) *Let* R: $F_1 \to F_2$ *be an isometry.*

(i) *Functions with full support are dense in* F_i *and if* $f(x) \in F_1$ *has full support in* F_1 *then* $g(x) = Rf(x)$ *has full support in* F_2.

(ii) *If* $m\{x\colon f(x) = 0\} = 0$ *for each function of full support then there exists an algebra isomorphism*

$$\Phi\colon \mathfrak{M}(\mathcal{F}_1) \leftrightarrow \mathfrak{M}(\mathcal{F}_2)$$

such that the operator defined by

$$\tilde{R}(m\cdot f)(x) \equiv \Phi m(x)\cdot Rf(x) \tag{4.2}$$

is an L^p isometry of

$$\tilde{F}_1 \equiv [\mathfrak{M}(\mathfrak{F}_1)\cdot F_1] \cap L^p(X, \mathfrak{B}, m)$$

onto

$$\tilde{F}_2 \equiv [\mathfrak{M}(\mathfrak{F}_2)\cdot F_2] \cap L^p(X, \mathfrak{B}, m).$$

(iii) *Moreover, if $Lf(x) = f(l(x))$ and the operator L commutes with R then*

$$l(\mathfrak{F}_i) = \mathfrak{F}_i,$$

or equivalently,

$$L(\mathfrak{M}(\mathfrak{F}_i)) = \mathfrak{M}(\mathfrak{F}_i),$$

and L commutes with both $\tilde{R}$ and Φ.

Note. It follows from (4.2) that for $m \in \mathfrak{M}(\mathfrak{F}_1)$ and $f \in \mathfrak{M}(\mathfrak{F}_1)\cdot F_1$ we have

$$\tilde{R}(m\cdot f) = \Phi(m)\cdot \tilde{R}(f). \tag{4.3}$$

Part (b) *Identification of* $\mathfrak{M}(\mathfrak{F})$.

Applying the results above to the case of a group G for which m is σ-finite, a closed left-invariant space $F \neq \{0\}$ of $L^p(G, \mathfrak{B}, m)$ and $L = \lambda_g$ we observe that $f \in F$ has full support iff $f(x) \neq 0$ a.e. The ratio σ-field of the Borel field and is left-invariant in that

$$E \in \mathfrak{F} \quad \text{and} \quad g \in G \quad \text{implies } gE \in \mathfrak{F}. \tag{4.4}$$

The next proposition characterizes sub-σ-fields of $\mathfrak{B}$ which satisfy (4.4). If $K \subseteq G$ is a closed subgroup of G we will write G/K for the coset space with elements $[g] = gK$. G/K inherits a topology, a Borel field $\mathfrak{B}_K$ and, if K is compact, an invariant measure dm_K. The Borel sets of G/K can be identified with those Borel sets A of G for which $A\cdot K = A$ and for such A, $m(A) = m_K(A)$. The Borel (resp. continuous) functions on G/K can be identified with the Borel (resp. continuous) functions on G which are constant on cosets.

PROPOSITION 4.2. *For each $\mathfrak{F} \subseteq \mathfrak{B}$ satisfying* (4.4) *there is a unique closed subgroup $K \subset G$ with*

$$\mathfrak{M}(\mathfrak{F}) = \mathfrak{M}(G/K, \mathfrak{B}_K).$$

The proof of Proposition 4.2 rests on two lemmas. If $K \subset G$ is a closed subgroup we write $C(G/K)$ for the algebra of continuous functions on G/K and $C_0(G/K)$ for those vanishing at infinity if G/K is not compact.

LEMMA 4.3. *Let $\{0\} \neq \mathcal{A} \subseteq C(G)$ be an algebra satisfying $\lambda_g \mathcal{A} = \mathcal{A}$ for each g, and closed under complex conjugation in the complex case. The set*

$$K = \{k \in G: \rho_k f = f \text{ for each } f \in \mathcal{A}\}$$

is a closed subgroup of G. $\mathcal{A}$ can be identified with a subalgebra of $C(G/K)$, and in the topology of uniform convergence on compact sets $\mathcal{A}$ is dense in $C(G/K)$.

If $\mathcal{A} \subseteq C_0(G)$, K is compact and $\mathcal{A}$ is uniformly dense in $C_0(G/K)$.

An analogous statement holds if $\mathcal{A}$ is right-invariant, and if $\mathcal{A}$ is bi-invariant, then K is normal.

Proof. For $f \in \mathcal{A}$, $K_f \equiv \{k \in G: \rho_k f = f\}$ is a closed subgroup and thus $K = \cap\{K_f: f \in \mathcal{A}\}$ is a closed subgroup. Each $f \in \mathcal{A}$ is constant on the cosets of K and thus we can identify $\mathcal{A} \subseteq C(G/K)$.

If $f \in \mathcal{A} \cap C_0(G)$ and $f(x_0) = c \neq 0$ then $x_0 K \subseteq B = \{x: |f(x)| \geq |c|\}$. Since B is compact, $K \subseteq x_0^{-1}B$ is compact.

The density of $\mathcal{A}$ in $C(G/K)$ (or in $C_0(G/K)$ if $\mathcal{A} \subseteq C_0(G)$) follows from the Stone-Weierstrass theorem.

If $\mathcal{A}$ is bi-invariant then for $f \in \mathcal{A}$, $g \in G$ and $k \in K$, $\rho_{gkg^{-1}} f = \rho_g(\rho_k(\rho_{g^{-1}} f)) = \rho_g \rho_{g^{-1}} f = f$ which shows K is normal.

LEMMA 4.4. *Let $\mathcal{F} \subseteq \mathcal{B}$ satisfy (4.4) and let $\mathcal{A}$ be the subalgebra of $\mathcal{M}(\mathcal{F})$ consisting of all continuous $\mathcal{F}$-measurable functions. Let $0 < g(x) \in L^1(G, \mathcal{B}, m)$ and let $d\mu$ be the restriction of the finite measure $g(x)dm(x)$ to the σ-field $\mathcal{F}$. Then in the topology of convergence in μ-measure, $\mathcal{A}$ is dense in $\mathcal{M}(\mathcal{F})$.*

Proof. Let $f \in L^\infty(G, \mathcal{F}, \mu)$ and let $h(x)$ be a continuous function with compact support in G. We claim $h * f(x) = \int h(t) f(t^{-1}x)\, dm(t)$ is in $L^\infty(G, \mathcal{F}, \mu)$. To see this note that the map $t \to \lambda_t f(x)$ is continuous as a function from G into $L^1(G, \mathcal{F}, \mu)$. Riemann sums can thus be found which approximate $h * f(x)$ in $L^1(G, \mathcal{F}, \mu)$. Thus $h * f(x) \in \mathcal{M}(\mathcal{F})$. But $h * f$ is bounded and continuous so $h * f \in \mathcal{A}$. Letting h vary over an approximate identity we can find a sequence h_n so that $h_n * f \to f$ in μ-measure as $n \to \infty$. Thus f is in the closure of $\mathcal{A}$. Since $L^\infty(G, \mathcal{F}, \mu)$ is dense in $\mathcal{M}(\mathcal{F})$ the result follows.

Combining the two lemmas with the observation that for any finite Borel measure $\tilde{\mu}$ on G/K the continuous functions $C(G/K)$ are dense (in the topology of convergence in $\tilde{\mu}$-measure) among the Borel functions on G/K, Proposition 4.2 follows.

The next lemma uses the fact that $F \subseteq L^p(G, \mathcal{B}, m)$ with $0 < p < \infty$ to deduce that the group K is compact.

LEMMA 4.5. *Let* $\{0\} \neq F \subseteq L^p(G, \mathcal{B}, m)$ *be closed and left invariant and let* $\mathcal{F}$ *be the ratio σ-field. Then the group K in Proposition* 4.2 *is compact and for each* $f \in F$, $|f(x)| \in \mathfrak{M}(\mathcal{F})$.

Proof. Let $f \in F$ have full support and set $r_t(x) = |f(tx)/f(x)|^p$. Then $r_t(x)$ is jointly measurable and for t fixed $r_t \in \mathfrak{M}(\mathcal{F})$. Thus for $k \in K$ $r_t(x) = r_t(xk)$ for a.a. x. Now

$$|f(x)|^p \int r_t(x)\, dm(t) = \int |f(tx)|^p\, dm(t).$$

But

$$\int |f(tx)|^p\, dm(t) = \Delta(x) \int |f(t)|^p\, dm(t) = c\Delta(x),$$

with $c = \int |f(t)|^p\, dm(t)$. Setting $\tilde{r}(x) = (\int r_t(x)\, dm(t))^{-1}$ gives

$$|f(x)|^p = c\Delta(x)\tilde{r}(x), \quad \text{for a.a. } x. \tag{4.5}$$

But $\tilde{r}(x) = \tilde{r}(xk)$ for a.a. x if $k \in K$ so

$$\begin{aligned} |f(xk)|^p &= c\Delta(xk)\tilde{r}(xk) = c\Delta(x)\Delta(k)\tilde{r}(x) \\ &= |f(x)|^p \Delta(k) \quad \text{for a.a. } x. \end{aligned} \tag{4.6}$$

K is now seen to be compact because if this were not the case (4.6) would contradict the integrability of $|f(x)|^p$. From $\Delta(kl) = \Delta(k)\Delta(l)$ and the compactness of K it now follows that $\Delta(k) \equiv 1$ on K and hence

$$|f(xk)|^p = |f(x)|^p \quad \text{for a.a. } x \text{ if } k \in K.$$

By Proposition 4.2, $|f(x)| \in \mathfrak{M}(\mathcal{F})$ whenever f has full support in F. Since such f are dense in F we see that $|f| \in \mathfrak{M}(\mathcal{F})$ for all $f \in F$.

The space $\mathfrak{M}(\mathcal{F}) \cdot F$ can now be completely described. Let $f \in F$ have full support. Define $j(x) = f(x)/|f(x)|$. Then $\mathfrak{M}(\mathcal{F}) \cdot F$ agrees with the space of functions of the form $j(x) \cdot r(x)$ with $r(x) \in \mathfrak{M}(\mathcal{F})$. Since

$$\int_G |j(x)r(x)|^p\, dm(x) = \int_{G/K} |r(x)|^p\, dm_K(x)$$

we see that

(4.7) $\tilde{F} = [\mathfrak{M}(\mathfrak{F})\cdot F] \cap L^p(G, \mathfrak{B}, m) \simeq j(x)\cdot L^p(G/K, \mathfrak{B}_K, m_K).$

One further note is that for each $t \in G, j(tx)/j(x) \in \mathfrak{M}(\mathfrak{F})$ and thus $j(txk)/j(xk) = j(tx)/j(x)$ for a.a. x if $k \in K$. Thus $j(txk)/j(tx) = j(xk)/j(x)$ for a.a. x which shows that the function $x \to j(xk)/j(x)$ is essentially equal to some constant $w(k)$. A little algebra shows that $|w(k)| = 1$ and $w(k_1k_2) = w(k_1)w(k_2)$. Thus $w(k)$ is a measurable character of K. Further the equation $j(xk) = j(x)w(k)$ shows that

(4.8) $\tilde{F} = \{f \in L^p(G, \mathfrak{B}, m)$: for $k \in K, f(x) = f(xk)w(k)$ for a.a. $x\}$.

Thus each space $\tilde{F}$ is indexed by a compact group $K \subseteq G$ and a character w of K.

Part (c) *Identification of isometric multipliers on* $j \cdot L^p(G/K)$.

Parts (a) and (b) show that when p is not an even integer each isometric left multiplier R: $F_1 \to F_2$ in $L^p(G)$ extends to an isometric left multiplier

$$\tilde{R}: j_1 \cdot L^p(G/K_1) \xrightarrow{\text{onto}} j_2 \cdot L^p(G/K_2),$$

of the special form

$$\tilde{R}(j_1 \cdot r)(x) = q(x)\cdot \Phi r(x). \tag{4.9}$$

Here $j_i = f_i/|f_i|$ where $f_i \in F_i$ has full support, Φ is an algebra isomorphism of $\mathfrak{M}(G/K_1)$ onto $\mathfrak{M}(G/K_2)$, and $q = Rf_1 \cdot \Phi(1/|f_1|)$ ($q = \tilde{R}j_1$ if m is finite).

Now we can drop the assumptions on p since for $p \neq 2$ the Banach-Lamperti result [5] on L^p isometries implies that each isometry from $j_1 \cdot L^p(G/K_1)$ onto $j_2 \cdot L^p(G/K_2)$ has this form and both the function $q(x)$ and the isomorphism Φ are unique.

Our result here is

PROPOSITION 4.6. *Suppose $p \neq 2$ is fixed and that K_1 and K_2 are compact subgroups of G and $j_1(x)$ and $j_2(x)$ are two measurable functions with $|j_1(x)| = |j_2(x)| = 1$ a.e. for which the spaces $\tilde{F}_1 = j_1 \cdot L^p(G/K_1)$ and $\tilde{F}_2 = j_2 \cdot L^p(G/K_2)$ are left-invariant. Then each isometric multiplier*

$$\tilde{R}: \tilde{F}_1 \xrightarrow{\text{onto}} \tilde{F}_2$$

has the form

$$\tilde{R} = c[\Delta(h^{-1})]^{1/p}\rho_h$$

where $|c| = 1$ *and* $h \in G$ *satisfies*

$$h^{-1}K_2 h = K_1. \tag{4.10}$$

REMARK. Before starting the proof we note that Proposition 4.6 will complete the proof of Theorem 1, part (a). The proof of part (b) is completely analogous with the only difference being that the term $\Delta(h^{-1})$ is missing because dm is left-invariant.

Proof of Proposition 4.6. First we need the observation that Φ must commute with each λ_g. This follows from (4.9), for if $m \in L^\infty(G/K_1)$ and $f \in L^p(G/K_1)$ has full support, then (4.9) gives

$$\tilde{R}(j_1 mf) = q\Phi(mf) = q\Phi(f)\Phi(m) = \tilde{R}(j_1 f)\Phi(m)$$

and so

$$\lambda_g\Phi(m) = \lambda_g\tilde{R}(j_1 mf)/\lambda_g\tilde{R}(j_1 f) = \tilde{R}\lambda_g(j_1 mf)/\tilde{R}\lambda_g(j_1 f) = \Phi\lambda_g(m).$$

Thus the isomorphisms $\lambda_g\Phi$ and $\Phi\lambda_g$ are equal.

The next lemma is basic.

LEMMA 4.7. *The map* Φ *when restricted to* $C_0(G/K_1)$ *is an algebra isomorphism of* $C_0(G/K_1)$ *onto* $C_0(G/K_2)$.

Proof. Let $\phi\colon \mathfrak{B}_{K_1} \leftrightarrow \mathfrak{B}_{K_2}$ be the σ-field isomorphism corresponding to Φ and let m_i denote the G-invariant measure on G/K_i. For $A \in \mathfrak{B}_{K_2}$ with finite measure we set $B = \phi^{-1}(A)$ and note

$$\begin{aligned}
\int_{G/K_2} |q(x)|^p 1_A(x)\, dm_2(x) &= \int_{G/K_1} |j_1(x)1_B(x)|^p\, dm_1(x) \\
&= m_1(B) = m_1(g^{-1}B) \\
&= \int_{G/K_2} |q(x)|^p 1_{\phi(g^{-1}B)}(x)\, dm_2(x) \\
&= \int_{G/K_2} |q(x)|^p 1_{g^{-1}A}(x)\, dm_2(x).
\end{aligned}$$

This shows that $\int_{G/K_2} |q(x)|^p 1_A(x)\, dm_2(x)$ is an invariant measure on G/K_2 and by the uniqueness of Haar measure the function $|q(x)|$ is constant.

Thus Φ defines an invertible bounded transformation of $L^1(G/K_1)$ onto $L^1(G/K_2)$. Since $\lambda_g\Phi = \Phi\lambda_g$ we have

$$\Phi(f * h) = f * \Phi(h), \tag{4.11}$$

for each $f \in L^1(G)$ and $h \in L^1(G/K_1)$. If, in addition $f, h \in C_0(G)$, (4.11) shows that $\Phi(f * h)$ is uniformly continuous. Since it is also in L^1 it must be in $C_0(G/K_2)$. Now let $\{f_\alpha\}$ be an approximate identity. Then $f_\alpha * h \to h$ uniformly. Since Φ is an L^∞ isometry $\Phi(f_\alpha * h) \to \Phi(h)$ uniformly and $\Phi(h) \in C_0(G/K_2)$. Since $C_0(G/K_1) \cap L^1(G/K_1)$ is dense in $C_0(G/K_1)$ we see that

$$\Phi\colon C_0(G/K_1) \to C_0(G/K_2).$$

Similarly, $\Phi^{-1}\colon C_0(G/K_2) \to C_0(G/K_1)$ and the result follows.

Each algebra isomorphism of $C_0(G/K_1)$ onto $C_0(G/K_2)$ has the form $\Phi f(x) = f(\psi(x))$ where $\psi\colon G/K_2 \to G/K_1$ is a homeomorphism onto G/K_1. From $\lambda_g \Phi = \Phi \lambda_g$, it follows that

$$g\psi(x) = \psi(gx).$$

Let $k_2 \in K_2$ and consider the coset K_2. Then

$$\psi(K_2) = hK_1 \quad \text{for some } h \in G.$$

But

$$k_2 h K_1 = k_2 \psi(K_2) = \psi(k_2 K_2) = hK_1$$

and

$$h^{-1} K_2 h \subseteq K_1. \tag{4.12}$$

Now let $k_1 \in K_1$. Then

$$\psi(h^{-1}K_2) = K_1 = k_1 K_1 = \psi(k_1 h^{-1} K_2).$$

Since ψ is one-to-one,

$$h^{-1}K_2 = k_1 h^{-1} K_2$$

and

$$hK_1 h^{-1} \subseteq K_2. \tag{4.13}$$

Together (4.12) and (4.13) give the desired

$$h^{-1}K_2 h = K_1.$$

Hence

$$\Phi f(x \cdot K_2) = f(x \cdot h \cdot K_1) = f(x \cdot K_2 \cdot h)$$

and Φ is the restriction of ρ_h to $\mathfrak{M}(G/K_2) \subseteq \mathfrak{M}(G)$.

It follows that

$$S = [\Delta(h)]^{1/p} \rho_{h^{-1}} \tilde{R}$$

is an isometric multiplier of $\tilde{F}_1 = j_1 \cdot L^p(G/K_1)$ onto $j_1' \cdot L^p(G/K_1)$ where

$$j_1'(x) = [\Delta(h)]^{1/p} q(xh^{-1}).$$

But $Sj_1(x)f(x) = j_1'(x)f(x), f \in L^p(G/K_1)$. Since j_1 and j_1' are both Borel measurable on G/K_1 and $|j_1(x)| \equiv 1$ it follows from the fact that S is an isometry that $|j_1'(x)| \equiv 1$. S commutes with λ_g so $j_1'(x)/j_1(x)$ is equal to a.e. to a constant c with $|c| = 1$ and

$$\tilde{R}(j_1 \cdot f)(x) = c[\Delta(h^{-1})]^{1/p}(j_1 \cdot f)(xh).$$

The proof is complete.

Part (d) *The non σ-finite case.*

We observe that if $E = \{f_1, \ldots, f_n\} \subseteq F$ is a finite subset of F which contains a non-zero element then the set A_E consisting of all pairs (c, g) with $|c| = 1$ and $g \in G$ for which

$$Rf_i(x) = c[\Delta(g)]^{-1/p} \rho_g f_i(x), \qquad f_i \in E, \tag{4.14}$$

is compact. As in the proof that K is compact in Lemma 4.5, this follows from the integrability of $|f_i(x)|^p$.

Now for E fixed there is a σ-finite closed subgroup $G_0 \subseteq G$ such that the function $\Sigma |f_i(x)| + |Rf(x)|$ vanishes a.e. on $G - G_0$. Letting F_0 be the closed linear span in $L^p(G_0, \mathfrak{B}_0, m_0)$ of the functions $\lambda_g f_i$ with $f_i \in E$ and $g \in G_0$ we see that the restriction of R to F_0 defines an isometric multiplier on F_0 to $L^p(G_0, \mathfrak{B}_0, m_0)$. Applying the result for σ-finite groups gives the existence of a pair (c, g) satisfying (4.14) Thus A_E is non-empty. Since A_E is also compact $A = \cap\{A_E \colon E \subseteq F\}$ is non-empty. If $(c, g) \in A$ we have

$$R = c[\Delta(g)]^{-1/p} \rho g$$

and the proof of Theorem 1 is complete.

5. Proof of Theorem 4. The machinery used in proving Theorem 1 applies here also.

We first show that $r_{k,k}(f)$ is defined a.e. for $2k \geq p$. Let ϕ_i and ψ_j be continuous functions with compact support. Then for $f \in L^p$ with $1 \leq p < \infty$ the functions $\phi_i * f(x) = \int \phi_i(t) f(t^{-1}x)\, dm(t)$ and $\bar{\psi}_j * \bar{f}(x)$ are in L^r if $p \leq r \leq \infty$ and $C_0(G)$.

If the ϕ_j's and ψ_j's are non-negative and if $2k \geq p$ we have by Hölder's inequality that

$$\infty > \int_G \prod_1^k \phi_j * |f|(x) \prod_1^k \psi_j * |f|(x)\, dm(x)$$

$$\geq \int \cdots \int \prod \phi_j(t_j)\psi_j(s_j) \prod \left|f(t_j^{-1}x)\right| \left|f(s^{-1}x)\right| dm^{2k+1}(x, s, t).$$

Thus the integrals defining the functions $r_{k,k}(f)$ converge absolutely a.e.

Now let $f \in L^p$ and $N \geq p/2$ be fixed and bring in the space $\mathfrak{M}(N, f)$ spanned by all functions of the form

$$\prod_{j=1}^{nN} \phi_j * f(x) \bar{\psi}_j * \overline{f(x)}, \qquad n = 1, 2, \ldots. \tag{5.1}$$

For $f \not\equiv 0$, $\mathfrak{M}(N, f)$ is a left-invariant non-trivial subalgebra of $C_0(G)$. By Lemma 4.3, $\mathfrak{M}(N, f)$ is uniformly dense in $C_0(G/K)$ for some compact subgroup K.

Suppose that (2.4) holds and consider the map Φ: $\mathfrak{M}(N, f_1) \to \mathfrak{M}(N, f_2)$ which is given on the generators (5.1) by

$$\Phi \prod_1^{nN} \phi_j * f_1(x) \bar{\psi}_j * \bar{f}_1(x) = \prod_1^{nN} \phi_j * f_2(x) \cdot \bar{\psi} * \bar{f}_2(x),$$

and then extended by linearity.

Observe that Φ is well-defined since by (2.4) it is an L^2-isometry. Moreover, Φ is an algebra isomorphism of $\mathfrak{M}(N, f_1)$ onto $\mathfrak{M}(N, f_2)$ commuting with left translations. As such it is continuous in the sup norm and extends to an isometric isomorphism of $C_0(G/K_1)$ onto $C_0(G/K_2)$ which commutes with left translations. As in the proof of Proposition 4.6 this implies Φ is the restriction of ρ_h to $C_0(G/K_1)$ for some h with

$$h^{-1}K_2 h = K_1.$$

In particular

$$\prod_1^N \phi_j * f_2(x) \cdot \bar{\psi}_j * \bar{f}_2(x) = \Phi\left[\prod_1^N \phi_j * f_1 \cdot \bar{\psi}_j * \bar{f}_1\right](x)$$

$$\equiv \prod_1^N \phi_j * f_1(xh) \cdot \bar{\psi}_j * \bar{f}_1(xh).$$

Choosing $\phi_j = \psi_k = \phi$ gives

$$\left|\int \phi(t) f_2(t^{-1}xh)\, dm(t)\right|^{2N}$$

for all x and ϕ. Thus

$$\left|\int \phi(t) f_2(t)\, dm(t)\right| = \left|\int \phi(t) f_1(th)\, dm(t)\right| \tag{5.2}$$

holds for any ϕ in L^q, $(1/p + 1/q = 1)$. The annihilators in L^q of the two functions f_2 and $\rho_h f_1$ thus agree and hence $f_2 = c\rho_h f_1$ for some c. That $|c| = 1$ follows from the fact that (5.2) implies $\|f_2\|_p = \|\rho_h f_1\|_p$.

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