PACIFIC JOURNAL OF MATHEMATICS
Vol. 106, No. 2, 1983

# A NOTE ON THE CARDINALITY OF INFINITE PARTIALLY ORDERED SETS

JOHN GINSBURG

**Let $P$ be an infinite partially ordered set with 0 and 1. A subset $B$ of $P$ is called a $\pi$-base for $P$ if for every element $x$ of $P$ with $0 < x < 1$ there exist elements $b$, $c$ in $B$ such that $0 < b \leq x \leq c < 1$. We let $\pi(P)$ denote the smallest cardinality of a $\pi$-base for $P$. We also let $h\pi(P) = \sup\{\pi(S) : S \subseteq P\}$. The width and depth of $P$ are defined as usual: $w(P) = \sup\{\kappa : P$ contains an antichain of cardinality $\kappa\}$; $d(P) = \sup\{\kappa : P$ contains a well-ordered or dually well-ordered subset of cardinality $\kappa\}$. We establish the following result:** THEOREM. $|P| \leq h\pi(P)^{d(P)}$. ***Various corollaries are obtained which imply and extend several known results on the cardinality of partially ordered sets, for example:*** COROLLARY. **(a)** $|P| \leq 2^{h\pi(P)}$. **(b)** $|P| \leq w(P)^{d(P)}$. **(c)** ***If B is a Boolean algebra then*** $|B| \leq 2^{w(B)}$.

1. **Preliminaries.** In this note several cardinality statements are established relating the cardinality of a partially ordered set to its width, depth, and $\pi$-weight. These results extend and imply several known results.

Our set-theoretic notation and terminology are standard and follow [**3**]. In particular the cardinality of a set $S$ is denoted by $|S|$ and a cardinal number is thought of as an initial ordinal. If $\alpha$ and $\beta$ are ordinals then $\alpha^\beta$ denotes the set of all mappings from $\beta$ into $\alpha$. If $\alpha$ and $\beta$ are cardinals then we also let $\alpha^\beta$ denote the cardinal exponentiation of $\alpha$ to the power $\beta$. If $\kappa$ is a cardinal then $\kappa^+$ denotes the first cardinal bigger than $\kappa$, and $\mathrm{cf}(\kappa)$ denotes the cofinality of $\kappa$—the least cardinal $\lambda$ such that $\kappa$ is the sum of $\lambda$ cardinals each of which is less than $\kappa$.

For additional information on partially ordered sets and the concepts considered here the reader is referred to [**5**]. Let $P$ be an infinite partially ordered set. The *width* of $P$, denoted by $w(P)$, is defined as

$$w(P) = \omega \cdot \sup\{\kappa : P \text{ contains an antichain of cardinality } \kappa\}.$$

We define

$$d^+(P) = \omega \cdot \sup\{\kappa : P \text{ contains a well-ordered subset of cardinality } \kappa\}$$

and

$$d^-(P) = \omega \cdot \sup\{\kappa : P \text{ contains a dually well-ordered subset of cardinality } \kappa\}.$$

The *depth* of $P$, denoted by $d(P)$, is defined by

$$d(P) = \omega \cdot \sup\{\kappa : P \text{ contains a well-ordered or dually well-ordered subset of cardinality } \kappa\}.$$

Note that $d(P) = d^-(P) \cdot d^+(P)$.

If $P$ has no largest element 1 then the expression $P - \{1\}$ is understood to just mean $P$ itself. A similar remark applies to $P - \{0\}$. Let $S$ be a subset of $P$. A subset $C$ of $S$ is said to be *cofinal* (*coinitial*) in $S$ if for every element $x$ of $S$ there exists an element $c$ in $C$ with $x \leq c$. ($c \leq x$). If $C$ is both cofinal and coinitial in $S$ we call $C$ a *$\pi$-base* for $S$. Following the notation of [**3**] we let $\pi_0(P) = \min\{\kappa : P - \{0\}$ has a coinitial subset of cardinality $\kappa\} \cdot \omega$ and $\pi_1(P) = \min\{\kappa : P - \{1\}$ has a cofinal subset of cardinality $\kappa\} \cdot \omega$, and $\pi(P) = \omega \cdot \min\{\kappa : P - \{0, 1\}$ has a $\pi$-base of cardinality $\kappa\}$. $\pi(P)$ is called the *$\pi$-weight* of $P$. Clearly $\pi(P) = \pi_0(P) \cdot \pi_1(P)$. We also define $h\pi_0(P) = \sup\{\pi_0(S) : S \subseteq P\}$, $h\pi_1(P) = \sup\{\pi_1(S) : S \subseteq P\}$, and $h\pi(P) = \sup\{\pi(S) : S \subseteq P\}$. Here, in considering $\pi(S)$ for subsets $S$ of $P$, we are of course considering $S$ as a partially ordered set in its own right in the induced order. $h\pi(P)$ is called the *hereditary $\pi$-weight* of $P$.

2. **Some relations involving cardinality, width, depth, and $\pi$-weight.** Throughout this section we assume that $P$ is an infinite partially ordered set with 0 and 1. This latter assumption is superfluous, since we can always add a 0 or 1 if necessary, and our results are valid for all infinite partially ordered sets. In the following proof of 2.1 we use a simple "tree-type argument" or ramification system as described in [**2**].

2.1. THEOREM. (a) $|P| \leq h\pi_1(P)^{d^-(P)}$. (b) $|P| \leq h\pi_0(P)^{d^+(P)}$.

*Proof.* We prove (a); the second statement follows by duality. We let $\kappa = h\pi_1(P)$ and let $\lambda = d^-(P)$. If $\phi \subset S \subseteq P$ then $S$ contains a cofinal subset of cardinality $\leq \kappa$. We choose one such subset $C(S)$ cofinal in $S$ and let $C(S) = \{p_S(\xi) : \xi < \kappa\}$ be a fixed enumeration of the elements of $C(S)$.

Now for every $\alpha < \lambda^+$ and for every $f \in \kappa^\alpha$ we will define a point $x_f$ of $P$ as follows: We let $x_\phi = 1$. Now, supppose $\alpha < \lambda^+$ and that for every $\beta < \alpha$ and for every $f \in \kappa^\beta$ we have defined $x_f$. Let $f \in \kappa^\alpha$. We will define $x_f$. If $\alpha$ is a limit ordinal we let $x_f = 1$. If $\alpha = \beta + 1$, let $S_f = \{p \in P : p < x_{f|\xi}$ for all $\xi \leq \beta\}$. If $S_f = \phi$, let $x_f = 1$. Otherwise let $x_f = p_{S_f}(f(\beta))$.

Thus for every $\alpha < \lambda^+$ and every $f$ in $\kappa^\alpha$ we have defined a point $x_f$. We now show that $P = \{x_f : f \in \bigcup\{\kappa^\alpha : \alpha < \lambda^+\}\}$. For suppose not. Then there exists an element $r$ in $P$ which is not equal to $x_f$ for any $f$ in $\kappa^\alpha$ for any $\alpha < \lambda^+$. In particular $r < 1$.

We now define inductively, for every $\alpha < \lambda^+$, a function $g_\alpha$ in $\kappa^\alpha$ as follows. Let $g_0 = \phi$. Now, let $\alpha < \lambda^+$ and suppose that for every $\beta < \alpha$ we have defined $g_\beta \in \kappa^\beta$ such that

1. $\gamma < \beta < \alpha \rightarrow g_\beta \mid \gamma = g_\gamma$,
2. $r < x_{g_\beta}$ for every $\beta < \alpha$, and
3. $\gamma < \beta < \beta + 1 < \alpha \rightarrow x_{g_{\beta+1}} < x_{g_{\gamma+1}}$.

We now define $g_\alpha \in \kappa^\alpha$. If $\alpha$ is a limit ordinal let $g_\alpha = \bigcup \{g_\gamma : \gamma < \alpha\}$. If $\alpha = \beta + 1$, let $T_\alpha = \{p \in P : p < x_{g_\gamma} \text{ for all } \gamma < \alpha\}$. Then $T_\alpha \neq \phi$ by (2) and so the set $C(T_\alpha)$ is defined. Since $r \in T_\alpha$ and $C(T_\alpha)$ is cofinal in $T_\alpha$, there exists an element $p_{T_\alpha}(\xi)$ in $C(T_\alpha)$ such that $r \leq p_{T_\alpha}(\xi)$. We define $g_\alpha$ as follows: $g_\alpha(\eta) = g_\beta(\eta)$ if $\eta < \beta$ and $g_\alpha(\beta) =$ the first ordinal $\xi < \kappa$ such that $r \leq p_{T_\alpha}(\xi)$. This defines $g_\alpha \in \kappa^\alpha$. We note that $p_{T_\alpha}(g_\alpha(\beta)) = x_{g_\alpha}$, as follows from the manner in which we defined the $x_f$'s. Therefore, by the choice of $r$ it follows that $r < x_{g_\alpha}$. Also note that since $p_{T_\alpha}(g_\alpha(\beta)) \in T_\alpha$, it follows that $x_{g_\alpha} < x_{g_\gamma}$ for all $\gamma < \alpha$. This completes the induction; the result is a sequence $\{g_\alpha : \alpha < \lambda^+\}$ which satisfies (1), (2), and (3) above for all $\gamma, \beta < \lambda^+$. But then $\{X_{g_{\alpha+1}} : \alpha < \lambda^+\}$ is a dually well-ordered subset of $P$ of cardinality $\lambda^+$. This is impossible since $d^-(P) = \lambda$. This proves our claim that $P = \{x_f : f \in \bigcup \{\kappa^\alpha : \alpha < \lambda^+\}\}$. Therefore

$$|P| \leq |\bigcup \{\kappa^\alpha : \alpha < \lambda^+\}| \leq \sum_{\alpha < \lambda^+} \kappa^{|\alpha|} \leq \lambda^+ \cdot \kappa^\lambda = \kappa^\lambda,$$

completing the proof of (a).

2.2. Corollary. $|P| \leq 2^{h\pi(P)}$.

*Proof.* First note that $d^+(P) \leq h\pi_1(P)$ (and dually that $d^-(P) \leq h\pi_0(P)$); for let $h\pi_1(P) = \kappa$. If $d^+(P) > \kappa$ then $P$ must contain a well-ordered subset of cardinality $\kappa^+$ and hence must contain a subset $S$ isomorphic to the ordinal $\kappa^+$. However any cofinal subset of $\kappa^+$ has cardinality $\kappa^+$ and so $\pi_1(S) = \kappa^+$, which contradicts the fact that $\pi_1(S) \leq h\pi_1(P) = \kappa$. Thus we have $d^+(P) \leq h\pi_1(P)$ and the dual. Therefore $d(P) \leq h\pi(P)$. Theorem 2.1 now implies that

$$|P| \leq h\pi(P)^{d(P)} \leq 2^{h\pi(P) \cdot d(P)} \leq 2^{h\pi(P) \cdot h\pi(P)} = 2^{h\pi(P)}$$

as desired.

2.3. Corollary. *If $P$ satisfies the descending chain condition then $|P| = h\pi_1(P)$. If $P$ satisfies the ascending chain condition then $|P| = h\pi_0(P)$.*

*Proof.* This is really a corollary of the proof of 2.1. We prove the first statement. If $P$ satisfies the descending chain condition then $P$ contains no decreasing $\omega$-sequence. In the proof of 2.1 we replace $\lambda^+$ by $\omega$ and we conclude that $P = \{x_f : f \in \bigcup\{\kappa^n : n \in \omega\}\}$. Therefore

$$|P| \leq \sum_{n<\omega} |\kappa^n| = \omega \cdot \kappa = \kappa = h\pi_1(P).$$

Since $h\pi_1(P) \leq |P|$ obviously holds in general, the corollary follows.

Theorem 2.1 can also be used to obtain a relationship between the cardinality of $P$ and the width of $P$ which was first obtained in [**4**] by a different method. First we establish a lemma.

2.4. Lemma. (a) $\pi_0(P) \leq w(P)^{d^-(P)}$. (b) $\pi_1(P) \leq w(P)^{d^+(P)}$.

*Proof*. We prove (b); the first statement is the dual. Thus let $\kappa = w(P)$ and let $\lambda = d^+(P)$. Now $P$ contains a cofinal subset $S$ which is *well-founded*; that is, a cofinal subset $S$ which contains no infinite decreasing sequence. (See [**5**].) Therefore every chain in $S$ is well-ordered, and so all chains in $S$ have cardinality $\leq \lambda$. Also all antichains of $S$ have cardinality $\leq \kappa$. Now, considering the partition relation $(\kappa^\lambda)^+ \to (\kappa^+, \lambda^+)$, (see [**2**]), and the partition of the pairs of elements of $S$ into incomparable pairs and comparable pairs, we conclude that $|S| \leq \kappa^\lambda$. Therefore $\pi_1(P) \leq |S| \leq \kappa^\lambda$, as desired.

As a corollary we now obtain the following result from [**4**].

2.5. Corollary. $|P| \leq w(P)^{d(P)}$.

*Proof*. Obviously if $S$ is any subset of $P$ then $w(S) \leq w(P)$, $d^-(S) \leq d^-(P)$, and $d^+(S) \leq d^+(P)$. Therefore the lemma implies that $\pi_0(S) \leq w(S)^{d^-(S)} \leq w(P)^{d^-(P)}$. Therefore $h\pi_0(P) = \sup\{\pi_0(S) : S \subseteq P\} \leq w(P)^{d^-(P)}$. Part (b) of Theorem 2.1 now implies that $|P| \leq h\pi_0(P)^{d^+(P)} \leq (w(P)^{d^-(P)})^{d^+(P)} = w(P)^{d(P)}$ as desired.

2.6. Corollary. $|P| \leq 2^{w(P) \cdot d(P)}$.

*Proof*. This follows immediately from 2.5.

2.7. Corollary. *Assume the generalized continuum hypothesis GCH. Let P be a partially ordered set. If $d(P) < cf(w(P))$ then $|P| = w(P)$. In particular, if $|P| = \kappa^+$ and if $d(P) < cf(w(P))$ then P contains an antichain of cardinality $\kappa^+$.*

*Proof.* Assuming GCH, we have $a^b = a$ whenever $b < cf(a)$. Therefore the assumptions imply by 2.5 that $|P| \leq w(P)$. Since always $w(P) \leq |P|$ the result follows. The second statement follows from the first and the definition of $w(P)$.

3. **Some comments and examples concerning the previous results.** We will now examine the sharpness of some of the preceding results and their connection with other results in the literature. Our comments pertain mainly to 2.2, 2.5, and 2.6.

First note that for any partially ordered set $P$, $w(P) \leq h\pi(P)$. For, if $S$ is an antichain then the only cofinal or coinitial subset of $S$ is $S$ itself. This relation, together with the fact that $d(P) \leq h\pi(P)$, mentioned in the proof of 2.2, are the only relations which hold in general involving these three cardinal numbers.

We observe that, in the statement of 2.2, $|P| \leq 2^{h\pi(P)}$, we cannot replace hereditary $\pi$-weight by $\pi$-weight. That is, $|P| \leq 2^{\pi(P)}$ does not hold in general. For example, if $Q$ is any partially ordered set of cardinality $\kappa$, then the ordinal sum $P = \omega + Q + \omega$ has $|P| = \kappa$ and $\pi(P) = \omega$. One class of partially ordered sets for which $|P| \leq 2^{\pi(P)}$ does hold is the class of Boolean algebras. (If $P$ is Boolean and if $B$ is a $\pi$-base for $P$, then for every non-zero element $x$ of $P$, we let $B_x = \{b \in B : b \leq x\}$. Note that if $x \neq y$ then $B_x \neq B_y$ and so $|P| \leq 2^{|B|}$.)

Familiar examples show that equality can be attained in the results of the preceding section. Regarding 2.5 and 2.6 we note that neither of the numbers $w(P)$ or $d(P)$ by itself in general limits the cardinality of $P$, as is shown in the case when $P$ is a chain or an antichain. For the important special class of Boolean algebras, $w(P)$ alone limits the cardinality of $P$. In fact, as is shown in [**1**], if $B$ is a Boolean algebra then $|B| \leq 2^{w(B)}$. (This follows from a stronger result proved in [**1**], that $\pi(B) \leq w(B)$ if $B$ is Boolean.) We note here that the result $|B| \leq 2^{w(B)}$ for Boolean algebras can also be deduced immediately from Corollary 2.6: $|P| \leq 2^{w(P)d(P)}$, because in any Boolean algebra we always have $d(B) \leq w(B)$. (If $\{x_\alpha : \alpha < \kappa\}$ is a well-ordered subset of $B$ with $\alpha < \beta \rightarrow x_\alpha < x_\beta$, then the elements $x_{\alpha+1} - x_\alpha$ for $\alpha < \kappa$ form an antichain.)

Regarding 2.5, we note that a weaker statement, namely $|P| \leq w(P)^{l(P)}$, can be proven directly from a partition relation. (Here $l(P)$ denotes the length of $P$; $l(P) = \sup\{\kappa : P$ contains a chain of cardinality $\kappa\}$.) In fact, the proof of 2.4 above really contains a proof of this fact using the partition relation $(\kappa^\lambda)^+ \rightarrow (\kappa^+, \lambda^+)$ from [**2**].

We also note that 2.6 includes as a special case the well-known result that, if $T$ is a totally ordered set then $|T| \leq 2^{d(T)}$.

In conclusion the author wishes to thank E. Milner for his comments concerning the above results. I would also like to thank the referee for the argument used to prove 2.4 above, much simplifying the original.

## REFERENCES

1. J. Baumgartner and P. Komjath, *Boolean algebras in which every chain and every antichain is countable*, to appear.
2. P. Erdos and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc., **62** (1956), 427–489.
3. I. Juhasz, *Cardinal Functions in Topology*, Math. Center Tracts, 34, Mathematical Center, Amsterdam, 1971.
4. G. Kurepa, *On the cardinal number of ordered sets and of symmetrical structures in dependence on the cardinal number of its chains and antichains*, Glasnik Mat.-Fiz. Astrnom. Drustvo, Mat. Fiz. Hrvatske Ser. II, **14** (1959), 183–203.
5. E. C. Milner and K. Prikry, *The cofinality of a partially ordered set*, to appear.

Received June 2, 1981 and in revised form April 13, 1982. This research was partially supported by a grant from NSERC of Canada.

UNIVERSITY OF WINNIPEG
WINNIPEG, MANITOBA, CANADA R3B 2E9