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CRAWLEY'S PROBLEM ON THE UNIQUE ω-ELONGATION OF p-GROUPS IS UNDECIDABLE

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Let G be an abelian p-group with $p^{\omega}G = 0$. Crawley has raised the following question: If all groups A with $p^{\omega}A$ cyclic of order p and $A/p^{\omega}A \cong G$ are mutually isomorphic, is G necessarily a direct sum of cyclic groups? We show this question to be independent of the axioms of set theory. Specifically, we prove that MA + $\neg$CH implies a negative answer for some G of cardinality $\aleph_1$; whereas, if $V = L$ is assumed, then every such G of cardinality $\aleph_1$ must be a direct sum of cyclic groups.

1. Introduction. All groups considered in this article are additively written p-primary abelian groups. If G is such a p-group, then $p^nG = \{p^nx : x \in G\}$ for $n < \omega$ and $p^{\omega}G = \bigcap_{n<\omega} p^nG$. We call G separable if $p^{\omega}G = 0$ and $\aleph_1$-separable if $p^{\omega}G = 0$ and every countable subset is contained in a countable direct summand. By a subsocle of G we mean a subgroup of the socle $G[p] = \{x \in G : px = 0\}$. We shall view the p-group G as a topological group endowed with the p-adic topology (the p^nG's form a neighborhood basis at 0) and its socle $G[p]$ as a topological vector space in the induced topology. A particularly prominent role in our considerations will be played by dense subsocles P of codimension one (that is, P is a dense subspace of $G[p]$ and $G[p]/P \cong Z(p)$, the cyclic group of order p). We shall write "Σ-cyclic" as an abbreviation for "a direct sum of cyclic subgroups."

It has been proved by Crawley [**2**] and by Hill and Megibben [**8**] that if the p-group G is Σ-cyclic, then any two p-groups A and B with $p^{\omega}A \cong p^{\omega}B$ and $A/p^{\omega}A \cong G \cong B/p^{\omega}B$ are necessarily isomorphic. That, conversely, Σ-cyclic groups are characterized as precisely the separable p-groups G satisfying this unique ω-elongation property was later established by Nunke [**12**] and Warfield [**14**]. But Crawly had previously raised the question of a somewhat stronger converse: If G is a separable p-group with the property that all groups A with $p^{\omega}A \cong Z(p)$ and $A/p^{\omega}A \cong G$ are mutually isomorphic, is G necessarily Σ-cyclic? We find it convenient to use the term "Crawley group" for such p-groups G. The conjecture that the Crawley groups are precisely the Σ-cyclic p-groups appears, in view of the Nunke-Warfield theorem, quite promising; and, assuming CH, Warfield [**14**] succeeded in showing that every Crawley group with countable basic subgroup is in fact Σ-cyclic. On the otherhand, the Nunke-Warfield result strongly uses the fact that $p^{\omega}A$ is allowed to be uncountable and, mindful of the analogous impact that countability considerations have on the Baer and Whitehead problems (see [4]), we should be alert to the possibility that we might be dealing here with a problem that cannot be resolved in ordinary set theory. Such indeed is the case. Precisely, we shall show that Martin's Axiom and the denial of the Continuum Hypothesis (MA + $\neg$CH) lead to the existence of a Crawley group of cardinality $\aleph_1$ which is not Σ-cyclic; whereas, Gödel's Axiom of Constructibility ($V = L$) implies that all Crawley groups of cardinalilty $\aleph_1$ are Σ-cyclic.

Unlike most of the independence and consistency results obtained heretofore for abelian groups (see for example, [4] and [11]), the Crawley Problem has no natural homological formulation, that is, although it certainly deals with extensions, the problem is not equivalent to the vanishing of some $\mathrm{Ext}(B, C)$. There is, nevertheless, an extremely useful translation of the Crawley Problem to an appropriate question about the internal structure of the group G. Indeed the following criterion is an easy consequence of the main theorem of [13]:

RICHMAN'S CRITERION: The separable p-group G is a Crawley group if and only if $\mathrm{Aut}\, G$, the automorphism group of G, acts transitively on the dense subsocles of codimension one.

2. Crawley's problem and $V = L$. Using a standard variant of Jensen's $\diamondsuit$-principle, a known consequence of $V = L$, we shall prove the following result.

THEOREM 2.1. ($V = L$) *A Crawley group of cardinality* $\aleph_1$ *is* Σ*-cyclic.*

First we need to recall certain definitions. By an ω_1-filtration of the group A we mean a well-ordered family $\{A_\alpha\}_{\alpha<\omega_1}$ of countable subgroups of A such that $A_\alpha \subseteq A_{\alpha+1}$ for all α, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is a limit ordinal and $A = \bigcup_{\alpha<\omega_1} A_\alpha$. A cub is a subset of ω_1 which is closed and unbounded in the order topology of ω_1 and a subset E of ω_1 is said to be stationary if it has nontrivial intersection with each cub. Equivalently, E is stationary if it meets the range of every strictly increasing, continuous function $f\colon \omega_1 \to \omega_1$. The fundamental combinatorial result we require is the following observation of Jensen's [10]:

LEMMA 2.2. ($V = L$) *If* $\{G_\alpha\}_{\alpha<\omega_1}$ *is an* ω_1*-filtration of* G *and if* E *is a stationary subset of* ω_1, *then there is a family of maps* $f_\alpha\colon G_\alpha \to G_\alpha$ ($\alpha \in E$) *such that for each map* $g\colon G \to G$, $\{\alpha\colon g \mid G_\alpha = f_\alpha\}$ *is also stationary in* ω_1.

We shall prove Theorem 2.1 by using 2.2 to argue that Richman's Criterion must fail for any separble p-group G of cardinality $\aleph_1$ which is not Σ-cyclic. The crucial link between all these seemingly disparate notions is contained in our next result.

LEMMA 2.3. *Let G be a separable p-group which is not Σ-cyclic and suppose P is a dense subsocle of codimension one. Then for any ω_1-filtration $\{P_\alpha\}_{\alpha<\omega_1}$ of P, $E = \{\alpha$: α is a limit and P_α is not a closed subset of $P\}$ is a stationary subset of ω_1.*

Proof. Since the limit ordinals in ω_1 form a cub and the intersection of a cub with a stationary set is itself stationary, it suffices to show that the set E_0 consisting of all $\alpha < \omega_1$ with P_α not closed in P is stationary. Let us suppose to the contrary that E_0 fails to be stationary. Then there is a strictly increasing, continuous function f: $\omega_1 \to \omega_1$ having range disjoint from E_0. If we now set $T_\alpha = P_{f(\alpha)}$ or all $\alpha < \omega_1$, then it is clear that $\{T_\alpha\}_{\alpha<\omega_1}$ is an ω_1-filtration and, moreover, each T_α is closed in P by choice of f. To see what this implies, it is convenient to view P as an object in the category $\mathcal{V}$ of valued vector spaces (in the sense of Fuchs [**6**]) with valuation induced by the height function on G. Then since each $T_{\alpha+1}/T_\alpha$ is countable and T_α is closed in $T_{\alpha+1}$, theorems 1 and 2 in [**6**] imply that we have a direct decomposition $T_{\alpha+1} = F_\alpha \oplus T_\alpha$ in the category $\mathcal{V}$ where F_α is free as a valued vector space. It then follows that $P = \bigoplus_{\alpha<\omega_1} F_\alpha$ in $\mathcal{V}$ and hence P is free as a valued vector space; that is, in the terminology of [**9**], P is a summable subsocle. Since P is a dense subsocle, there is a pure subgroup H of G such that $H[p] = P$ [**5**, Theorem 66.3]. Consequently, the version of the Kulikov Criterion given by Charles [**1**, Théorème 1] implies that H is Σ-cyclic. But G/H is countable since P has codimension one in $G[p]$ and, by a standard argument, $G = K \oplus C$ where K is a summand of H and C is countable. This, however, yields the contradiction that G itself is Σ-cyclic by Prüfer's Theorem [**5**, Theorem 17.3].

We are now ready to prove Theorem 2.1. Let G be a separable p-group of cardinality $\aleph_1$ which is not Σ-cyclic and let P be a fixed dense subsocle of codimension one. Take $z \in G[p]\setminus P$ and select an ω_1-filtration $\{P_\alpha\}_{\alpha<\omega_1}$ such that z lies in the closure of P_ω and $P_{\alpha+1} = P_\alpha + \langle z_\alpha \rangle$ for all α. We then construct inductively an ω_1-filtration $\{G_\alpha\}_{\alpha<\omega_1}$ of G such that each G_α is maximal in G with respect to $G_\alpha[p] = P_\alpha + \langle z \rangle$. Now take E to be the stationary subset of ω_1 described in Lemma 2.3 and let f_α: $G_\alpha \to G_\alpha$ $(\alpha \in E)$ be a family of maps satisfying Lemma 2.2. To show that Richman's Criterion fails for G, we need to find a dense subsocle Q of codimension one such that $\theta(P) \neq Q$ for all automorphism θ of G. We accomplish this by constructing inductively a family of subsocles $\{Q_\alpha\}_{\alpha<\omega_1}$ satisfying the following three conditions:

(1) $Q_n = P_n$ for $n < \omega$, $Q_\alpha \subseteq Q_{\alpha+1}$ and $z \notin Q_\alpha$ for all α and $Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta$ for limit ordinals α.

(2) If $\alpha \in E$, $f_\alpha(P_\alpha) = Q_\alpha$ and f_α is the restriction to G_α of some $\phi \in \operatorname{Aut} G$ with $z \notin \phi(P)$, then $Q_{\alpha+1} = Q_\alpha + \langle y_\alpha - z \rangle$ where $y_\alpha = \phi(x_\alpha)$ for some $x_\alpha \in P$ in the closure of P_α but jot in P_α itself.

(3) $P_\beta \subseteq Q_\beta + \langle z \rangle$ except when $\beta = \alpha + 1$ with α satisfying condition (2).

Notice that the assumption that $z \notin \phi(P)$ in (2) guarantees that $z \notin Q_{\alpha+1} = Q_\alpha + \langle y_\alpha - z \rangle$ so that (2) does not conflict with (1). Also note that if β is a limit ordinal and (3) is satisfied for all smaller ordinals, then the choice of Q_β dictated by (1) will automatically satisfy (3). Therefore the only possible difficulty that could occur in the inductive construction of the Q_α's is with condition (3) for successor ordinals $\beta = \alpha + 1$ with α not satisfying (2). But it is easy to see that there is no real problem here either, since one need only enlarge from Q_α to Q_β using appropriate z_γ's when needed.

The Q_α's having been constructed, we take $Q = \bigcup_{\alpha<\omega_1} Q_\alpha$ and observe that (3) insures that Q has codimension one since $z \notin Q$ and $G[p] = P + \langle z \rangle = Q + \langle z \rangle$. Since z is in the closure of $Q_\omega = P_\omega$, the last equation shows that Q is a dense subsocle. Finally, assume by way of contradiction that θ is an automorphism of G such that $\theta(P) = Q$. Then a simple back-and-forth argument establishes the fact that $\{\alpha: \theta(P_\alpha) = Q_\alpha\}$ is a cub in ω_1. But by Lemma 2.2, $\{\alpha: \theta \mid G_\alpha = f_\alpha\}$ is stationary in ω_1 and therefore there is an ordinal β such that $\theta(P_\beta) = Q_\beta$ and $\theta \mid G_\beta = f_\beta$. Since also $z \notin Q = \theta(P)$, β is an ordinal satisfying condition (2). Let x_β and ϕ be as in the statement of (2) and choose a sequence $\{x_n\}_{n<\omega}$ in P_β such that $x_\beta = \lim_{n\to\infty} x_n$. Then since automorphisms are continuous relative to the p-adic topology,

$$\begin{aligned} y_\beta = \phi(x_\beta) &= \lim_{n\to\infty} \phi(x_n) \\ &= \lim_{n\to\infty} f_\beta(x_n) = \lim_{n\to\infty} \theta(x_n) = \theta(x_\beta) \end{aligned}$$

is in $\theta(P) = Q$. This, however, is a contradiction since clearly $y_\beta \notin Q$ by the choice of $Q_{\beta+1}$ in (2).

It would, of course, have been more satisfactory if we had been able to prove that $V = L$ implies all Crawley groups are Σ-cyclic regardless of their cardinality, and it would be rather surprising if this were not the case. There, however, appear to be formidable difficulties in removing the cardinality restriction from Theorem 2.2. First, one would evidently need to prove Lemma 2.3 for all regular cardinals; and secondly, to push an induction through the singular cardinals, one would probably be required to show in general that pure subgroups of Crawley groups are once again Crawley groups.

3. Crawley's problem and Martin's axiom. It is somewhat easier to prove that MA + $\neg$CH implies there are Crawley groups of cardinality $\aleph_1$ which are not Σ-cyclic. The reason for this is the fact that we can show the existence of such a group is a consequence of a result which is suggested by the corresponding theorem for Whitehead's Problem (see [3]), and which can be proved by a slight refinement of the argument used in the proof of that theorem.

Specifically, we can prove the following:

THEOREM 3.1. (MA + $\neg$CH) *If G is an $\aleph_1$-separable p-group of cardinality $\aleph_1$, then* $\mathrm{Pext}(G, S) = 0$ *for all countable groups S.*

Since, as indicated above, the proof of Theorem 3.1 is similar to that of Theorem 7.2 in [3], we shall delay sketching the proof until we have shown the relevance of the theorem to Crawley's Problem. As there do indeed exist $\aleph_1$-separable p-groups of cardinality $\aleph_1$ which fail to be Σ-cyclic [5, Theorem 75.1], the following theorem (proved in ordinary ZFC set theory) shows that MA + $\neg$CH yields a negative answer to Crawley's Problem.

THEOREM 3.2. *If G is an $\aleph_1$-separable p-group such that* $\mathrm{Pext}(G, S) = 0$ *for all countable p-groups S, then G is a Crawley group.*

Proof. Let P and Q be dense subsocles of G having codimension one in $G[p]$. Choose a in $G[p]\setminus P$ and b in $G[p]\setminus Q$ together with sequences $\{a_n\}_{n<\omega}$ in P and $\{b_n\}_{n<\omega}$ in Q such that $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Since G is $\aleph_1$-separable, there is a countable direct summand C containing all the a's and b's. The crux of the proof is showing that there exist pure subgroups H and K such that $H[p] = P$, $K[p] = Q$ and direct decompositions $G = C \oplus M = C \oplus L$ with $M \subseteq H$ and $L \subseteq K$. Indeed assuming that this can be done, observe that $H = (H \cap C) \oplus M$, $K = (K \cap C) \oplus L$ and $C/H \cap C \cong G/H \cong Z(p^\infty) \cong G/K \cong C/K \cap C$. Then by Theorem 1 in [7] there is an automorphism ϕ of C such that $\phi(H \cap C) = K \cap C$. But then, since $M \cong G/C \cong L$, there must exist an automorphism θ of G with $\theta \mid C = \phi$ and $\theta(M) = L$, that is, $\theta(H) = K$ and hence $\theta(P) = Q$.

Thus it remains only to establish the existence of H, K and the desired direct decompositions. Since C is countable and thus Σ-cyclic, it is easy to see that each of its closed subsocles supports a direct summand. Therefore we have a direct decomposition $C = C_1 \oplus B_1$ where $C_1[p]$ is the closure of $C \cap P$. Next choose a basic subgroup A of C_1 with $A[p] = C \cap P$ and a pure subgroup H of G such that $H \supseteq A$ and $H[p] = P$. Then

$(H \cap C_1)[p] = P \cap C_1 = A[p]$ and an easy inductive argument establishes that $H \cap C_1 = A$. Now fix a direct decomposition $G = C_1 \oplus N$ and let π be the corresponding projection of G onto N. Using the facts that C_1/A is divisible and H is pure in G, it is readily shown that $\pi(H)$ is also pure in G. Then consider the induced short exact sequence $A \hookrightarrow H \twoheadrightarrow \pi(H)$. Since $P\text{ext}(G, A) = 0$ and $\pi(H)$ is pure in G, $P\text{ext}(\pi(H), A)$ also vanishes, that is, the short exact sequence splits and we have a direct decomposition $H = A \oplus J$. Since A is basic in C_1 and $G[p] \subseteq C_1 + P$, one can prove inductively that $G[p^n] \subseteq C_1 + H$ for all n consequently that $G = C_1 \oplus J$. But then $C = C_1 \oplus (C \cap J)$ and since C is a direct summand of G, it follows that $G = C \oplus M$ where $M \subseteq J \subseteq H$. Indeed if $G = C \oplus D$, then $M = J \cap (C_1 \oplus D)$. Of course, the same reasoning applies to Q to yield a pure subgroup K having Q as its socle and a direct decomposition $G = C \oplus L$ with $L \subseteq K$.

Remark. In reference to 3.2, it is noteworthy that $V = L$ implies that the Σ-cyclics are the only separable p-groups G enjoying the property that $P\text{ext}(G, S) = 0$ for all countable p-groups S.

Finally, we must indicate how Theorem 3.1 is proved. Suppose then that G is an $\aleph_1$-separable p-group of cardinality $\aleph_1$ and consider a short exact sequence $S \hookrightarrow K \overset{\pi}{\twoheadrightarrow} G$ where S is a countable pure subgroup of K. We, of course, need to use Martin's Axiom to find a homomorphism $\psi: G \to K$ such that $\pi\psi = 1_G$, the identity map of G. As in the proofs of Theorem 7.2 in [**3**] and Theorem 3.2 in [**4**], we consider the poset P of finite approximations to ψ; that is, P consists of all homomorphisms $\phi: T \to K$ with T a finite direct summand of G and $\pi\phi = 1_T$. It is clear that, for each $g \in G$, $D_g = \{\phi \in P : g \text{ is in the domain of } \phi\}$ is a dense subset of P and that the existence of a filter (subnet) in P which meets each D_g will yield the desired ψ. The only difficulty then is to show that P satisfies the ccc (countable antichain condition) so that Martin's Axiom is applicable. In other words, given any uncountable subset P' of P we need to find distinct elements ϕ_1 and ϕ_2 of P' such that some ϕ in P extends both ϕ_1 and ϕ_2. But by the same reasoning used in [**3**] and [**4**], it suffices to prove that there is a pure Σ-cyclic subgroup A of G such that A contains the domains of uncountably many members of P'. Since we, however, are dealing with torsion groups rather than homogeneous torsion-free groups, the proof of the existence of such an A involves a slightly more delicate argument than that given for the corresponding result in [**3**].

Let $P' = \{\phi_\alpha\}_{\alpha<\omega_1}$ be an uncountable subset of P with T_α the domain of ϕ_α and $\phi_\alpha \neq \phi_\beta$ when $\alpha \neq \beta$. Since countable subgroups of G are necessarily Σ-cyclic, we may assume that no countable pure subgroup of G contains uncountably many of the T_α's. Also since each T_α is finite, by reducing if necessary to an appropriate uncountable subset of P', we may further assume that all the T_α's have the same number of elements. This then allows us to select a subgroup T of G such that T is contained in uncountably many of the T_α's, but no subgroup properly containing T enjoys this property. Again without loss of generality, we may assume that $T \subseteq T_\alpha$ for all α. We now wish to construct inductively a family $\{A_\alpha\}_{\alpha<\omega_1}$ of countable pure subgroups of G and a strictly increasing function $f\colon \omega_1 \to \omega_1$ such that the following three conditions hold:

(1) $T \subseteq A_0$, $A_\alpha \subseteq A_{\alpha+1}$ for all α, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is a limit ordinal.

(2) $T_{f(\alpha)} \subseteq A_{\alpha+1}$ for all α.

(3) $A_{\alpha+1}/A_\alpha$ is Σ-cyclic for all α.

If such a construction is possible, the purity of A_α will imply that $A_{\alpha+1} = A_\alpha \oplus L_\alpha$ [**5**, Theorem 30.2] and then $A = \bigcup_{\alpha<\omega_1} A_\alpha = A_0 \oplus (\bigoplus_{\alpha<\omega_1} L_\alpha)$ will be a pure Σ-cyclic subgroup of G containing all the $T_{f(\alpha)}$'s. Suppose then that $\beta < \omega_1$ and that the A_α and $f(\alpha)$ with the requisite properties have been defined for all $\alpha < \beta$, except that $f(\gamma)$ remains undefined if $\beta = \gamma + 1$. If β is a limit ordinal, then the choice of A_β dictated by (1) satisfies all the required conditions and we need not define $f(\beta)$ until we construct $A_{\beta+1}$. We may therefore assume that β is a nonlimit, say, $\beta = \gamma + 1$. Since G is $\aleph_1$-separable, we have a direct decomposition $G = C \oplus K$ where C is a countable subgroup containing A_γ. Let $\delta = \sup_{\alpha<\gamma} f(\alpha)$ and enlarge to a basic subgroup $B = A_\gamma \oplus H$ of C. Now there are uncountably many α's larger than δ and we claim that there is furthermore an $\alpha > \delta$ such that $(T_\alpha + B/B) \cap (C/B)$ is the zero subgroup of G/B. Indeed if this is not the case, then for each $\alpha > \delta$ we have an $x_\alpha \in T_\alpha \cap C$ with $x_\alpha \notin B$. But C is countable and therefore there is a fixed $c \in C$ such that $x_\alpha = c$ for uncountably many α's. By choice of T, however, $T + \langle c \rangle = T \subseteq B$, contrary to $x_\alpha \notin B$. Then we take $f(\gamma) = \mu$ where μ is an ordinal such that $\mu > \delta$ and $(T_\mu + B/B) \cap (C/B) = 0$. Now if $x + B$ is a nonzero element of $T_\mu + B/B$, we write $x = c + k$ where $c \in C$, $k \in K$ and $k \neq 0$. Observe that since C/B is divisible and $K + B/B \cong K$ is separable, $x + B = (c + B) + (k + B)$ has finite height in $G/B = (C/B) \oplus (K + B/B)$. But $T_\mu + B/B$ is a finite group and therefore by [**5**, Corollary 27, 8] there is a subgroup $A_\beta = A_{\gamma+1}$ of G containing $T_\mu + B$ such that A_β/B is a finite direct summand of G/B. Since B is a pure subgroup of G, it follows that A_β is pure and $A_\beta = A_{\gamma+1} = L \oplus B = L \oplus A_\gamma \oplus H$ where L is finite. We conclude by observing that $T_{f(\gamma)} \subseteq A_{\gamma+1}$ and that $A_{\gamma+1}/A_\gamma \cong L \oplus H$ is Σ-cyclic.

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