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NONSMOOTH ANALYSIS ON PARTIALLY ORDERED VECTOR SPACES: PART 1 — CONVEX CASE

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Convex analysis provides the tools to extend results of differential calculus to nonsmooth real valued functions. The purpose of this article is to study those extensions for convex vector valued mappings. We study their continuity properties, develop a subdifferential calculus and a duality theory, similar to the one existing for real valued functions. We conclude with some useful deconvexification results.

1. Introduction. During the last two decades, many essential contributions to the theory of extremal problems were made. Beginning with the development of convex analysis by Brønsted [1], Rockafellar [29], [31], [32], [33] and Moreau [20], [21], we have the first important generalizations in optimization theory. By substituting the smoothness assumption with the convexity one and using the rich functional analytic theory of convex sets (see [5]), they were able to obtain a complete, interesting in its own sake, theory, which is known today as "Convex Analysis". Its usage in optimization turned out to be extremely fruitful and produced a harmonious duality theory for convex problems. These achievements are exemplified by the work of Brønsted [1], Ekeland-Temam [6], Ioffe-Levin [11], Ioffe-Tichomirov [12], [13], Pshenichnyi [26], Rockafellar [29], [30], [33], [34], Valadier [37], [38] and others. All this work on the one hand extended the range of treatable problems and, on the other hand, introduced new concepts and techniques, useful in solving old problems also.

The next natural step in this program was to try to get rid of the convexity assumption too. The first important contribution in that direction was the work of Clarke [2], [3]. Following Clarke, Hiriart-Urruty [9], [10], Ioffe-Tichomirov [13], Lebourg [17] and Penot [23] have significantly contributed to this or neighboring areas.

In the last few years, there has been a growing interest in vector optimization problems (see [7], [14], [15], [16], [19], [27], [28], [38], [39], [40], [41], and [42]). For those problems, although we have a Hahn-Banach type theorem due to L. Kantorovich, we do not have any functional separability results. So the arguments that led to the development of the analytic foundations in the scalar valued case, break down. So we have to find new ways to approach the problem.

In this work, we undertake this task. In the present paper, which is the first in a series of two papers, we deal with the convex case. Having as our starting point the paper of Valadier **[38]**, we generalize several of its results and also obtain new ones. So we develop a duality theory for convex operators, a subdifferential calculus and even go further and introduce some deconvexification methods, like quasidifferentiability and quasiconvexity and obtain new results there, too.

In the second paper **[22]**, which is a continuation of this one, we pass to the nonconvex theory and we develop an analog of Clarke's calculus of generalized gradients for vector valued mappings.

In both papers, our approach is topological, as opposed to that of Kutatelazde **[14]**, **[15]**, **[16]**, which is completely algebraic in nature.

We hope that this work will be helpful in developing a unified approach to necessary conditions of vector valued and multiobjective optimization problems. In fact, this is the topic of a forthcoming paper by the author.

For the necessary background from the theory of ordered vector spaces, we refer the reader to **[24]**, **[35]**, and **[36]**.

Finally we constantly assume that all mappings involved are proper, i.e. they do not take the value $-\infty$.

2. Preliminary material. In this section, which is preparatory in nature, we present the necessary definitions and notational conventions that we will be using in the sequence and also prove some general facts about ordered topological vector spaces for future reference.

In what follows, X and Y will be, in general, Hausdorff locally convex vector spaces and K_Y^+ will be a convex cone of positive elements that makes Y a partially ordered topological vector space. We adjoin to Y a greatest element $+\infty$ and extend the vector space operations in the natural way. Therefore we have automatically adjoined also a smallest element $-\infty$. We will denote the augmented space $Y \cup \{\pm\infty\}$ by $\overline{Y}$. We will always assume that Y is order complete. By that we mean that every nonempty subset of Y that is majorized in Y has a supremum. This is, in fact, equivalent to saying that every minorized subset of Y has an infimum.

If $C \subset Y$ then the full hull $[C]$ of C is defined to be the set

$$\begin{aligned}[C] &= \{z \in Y\colon x \leq z \leq y \text{ for } x, y \in C\} \\ &= (C + K_Y^+) \cap (C - K_Y^+).\end{aligned}$$

If $C = [C]$ then C is said to be full (or order convex).

We say that Y or (K_Y^+) is normal for the topology τ_Y if there is a local basis of the origin consisting of full sets. In the sequence, we will often assume that Y is normal, since normality is one of the most fruitful, but also natural, topological restrictions on the order structure of an ordered topological vector space. For further characterization of normality, see **[24]** and **[35]**.

If Y is an o.t.v.s., we say that a sequence $\{x_n\}_{n\in N}$ of elements order converges (o-converges) to x if and only if there are sequences $\{p_n\}_{n\in N}$ and $\{q_n\}_{n\in N}$ such that (α) $p_n \le x_n \le q_n$ and (β) $p_n \uparrow x$ and $q_n \downarrow x$, where by $p_n \uparrow x$ we mean that $\bigvee_{n\in N} p_n = x$ and by $q_n \downarrow x$ that $\bigwedge_{n\in N} q_n = x$. This, in vector lattice, is equivalent to saying that there is a sequence y_n decreasing to zero such that $|x_n - x| \le y_n$. In the $L^p(\Omega, \Sigma, Y)$ spaces $(1 \le p \le \infty)$ o-convergence is equivalent to μ-a.e. convergence.

There are some other notions of convergence that we will be using. We mention them briefly. For further details, we refer the reader to **[24]** and **[35]**.

(1) A sequence $\{x_n\}_{n\in N}$ *o^*-converges to* x if and only if from any subsequence $\{x_{n_k}\}_{k\in N}$ we can extract another subsequence $\{x_{n_{k_l}}\}_{l\in N}$ which converges to x.

(2) A sequence $\{x_n\}_{n\in N}$ *converges relatively uniformly to* x, denoted by

$$x_n \xrightarrow{ru} x$$

if and only if there is an element $z \in K_Y^+$ such that $|x_n - x| \le \lambda_n z$, where $\lambda_n \in R^+$ and $\lambda_n \downarrow 0$. Similarly we can have relative uniform $*$-convergence.

It is easy to see from the above definitions that $x_n \xrightarrow{ru} x$ implies that $x_n \xrightarrow{o} x$. The converse, however, is true only in a special class of linear lattices.

A vector lattice is said to have the *diagonal property* if whenever $\{x_{nm}\}_{n,m\in N} \subseteq Y$ and

(1) $$x_{nm} \xrightarrow{o} x_n \quad \forall n \in N \quad \text{and}$$

(2) $$x_n \xrightarrow{o} x$$

then there is a diagonal subsequence $\{x_{nm_n}\}_{n,m\in N}$ which order converges to x.

An obvious example are the $L^p(\Omega, \Sigma, Y)$ $(1 \le p \le \infty)$ spaces.

Now in an order complete vector lattice with the diagonal property order convergence and relative uniform convergence are equivalent (see **[24]**).

An element e of an ordered vector space Y is said to be an *order unit* if and only if for each $x \in Y$ there is a $\lambda > 0$ s.t. $x \leq \lambda e$.

Also in an ordered vector space, the *order topology* τ_0 is the finest locally convex topology for which every order bounded set is bounded.

Finally a Banach lattice is an ordered Banach space in which the following monotonicity relation holds: $|x| \leq |y| \rightarrow \|x\| \leq \|y\|$.

In the sequence, we give some easy results that we will need on several occasions in the next sections.

First, it is very easy to see that if Y is as above, then we have that $y_n \xrightarrow{o} y$ and $y'_n \xrightarrow{o} y'$ imply that $y_n \vee y'_n \xrightarrow{o} y \vee y'$.

Also the following fact is useful:

LEMMA 2.1. *If K_Y^+ is closed then if $y_n \leq y'_n\ \forall n \in N$ and*

$$y_n \xrightarrow{\tau_Y} y,\ y'_n \xrightarrow{\tau_Y} y'$$

then $y \leq y'$.

Proof. Since by hypothesis, $y_n \leq y'_n\ \ \forall n \in N$, we have that $y'_n - y_n \in K_Y^+\ \forall n \in N$. Also

$$y_n^* - y_n \xrightarrow{\tau_Y} y' - y$$

and since K_Y^+ is closed, we conclude that $y \leq y'$. □

Finally, we have the following result.

LEMMA 2.2. *If Y is a Banach lattice and $x_n \xrightarrow{ru} x$ then $x_n \xrightarrow{s} x$.*

Proof. Since by hypothesis $x_n \xrightarrow{ru} x$, we have that $|x_n - x| \leq \lambda_n z$ for some $z \in K_Y^+$ and for $\lambda_n \downarrow 0$, $\lambda_n \in R^+$. Fix $n \in N$. Then $x_n - x \in [-\lambda_n z, \lambda_n z]$. Since Y is normal, the order interval $[-\lambda_n z, \lambda_n z]$ is bounded. So there is $\mu > 0$ such that $[-\lambda_n z/\mu, \lambda_n z/\mu] \in U$ where U is the unit ball in Y. Hence, for $\varepsilon > 0$, we have $[-\varepsilon\lambda_n z/\mu, \varepsilon\lambda_n z/\mu] \in U$. Since $\lambda_n \downarrow 0$, we know that there is $m_0 \in N$ such that for λ_m with $m \geq m_0$ we have

$$\begin{aligned}
[-\lambda_m z, \lambda_m z] &\subset \left[-\frac{\varepsilon\lambda_n z}{\mu}, \frac{\varepsilon\lambda_n z}{\mu}\right] \subset \varepsilon U \\
&\rightarrow x_m - x \in \varepsilon U \quad \text{for } m \geq m_0 \\
&\rightarrow \|x_m - x\| \leq \varepsilon \\
&\rightarrow x_m \xrightarrow{s} x.
\end{aligned}$$

□

We close this section with a notational clarification. By $L(X, Y)$ we denote all linear operators from X to Y and by $\mathfrak{L}(X, Y)$ we denote all continuous linear operators from X to Y.

3. Convex mappings; continuity and subdifferentiality.

DEFINITION 3.1. We call a mapping $f\colon X \to \overline{Y}$ *midconvex* if and only if $f(\frac{1}{2}x + \frac{1}{2}z) \le \frac{1}{2}f(x) + \frac{1}{2}f(z)$ for all $x, z \in X$. We call a mapping $f\colon X \to \overline{Y}$ *convex* if and only if $f(\lambda x + (1-\lambda)z) \le \lambda f(x) + (1-\lambda)f(z)$ for all $\lambda \in [0,1]$ $x, z \in X$.

Its *effective domain*, denoted by $\operatorname{dom} f = \{x \in X\colon f(x) < +\infty\}$.

LEMMA 3.1. *If f; $X \to \overline{Y}$ is midconvex and finite and there is $x_0 \in X$ such that f is majorized in a neighborhood of x_0 then f is locally o-bounded.*

Proof. By hypothesis there is a neighborhood $U(x_0) = x_0 + U$ of x_0 (U = neighborhood of the origin in X) such that $f(z') \le y$ for all $z' \in U(x_0) = x_0 + U$.

By considering if necessary the mapping $\hat{f}(z) = f(z') = f(x_0 + z)$ we can assume without loss of generality that $x_0 = 0$. Let U be a symmetric and convex neighborhood of the origin. Then if $z \in U \to -z \in U$. So by midconvexity, we have that

$$\begin{aligned} f(\tfrac{1}{2}z + \tfrac{1}{2}(-z)) = f(0) &\le \tfrac{1}{2}f(z) + \tfrac{1}{2}f(-z) \\ &\to -\tfrac{1}{2}f(-z) + f(0) \le \tfrac{1}{2}f(z) \to -\tfrac{1}{2}y + f(0) \le \tfrac{1}{2}f(z) \\ &\to -y + 2f(0) \le f(z). \end{aligned}$$

This is true for any $z \in U$. So we conclude that f is also minorized in U and therefore it is order bounded in U. Now let x be any point of X.

Let $w = 2x$ and consider the neighborhood $V = \frac{1}{2}U$ of 0. Then $x + V$ is a neighborhood of x and $x + V = \frac{1}{2}w + \frac{1}{2}U$

$$f(\tfrac{1}{2}w + \tfrac{1}{2}z) \le \tfrac{1}{2}f(w) + \tfrac{1}{2}f(z) \le \tfrac{1}{2}f(w) + \tfrac{1}{2}y \quad \forall z \in U.$$

So f is majorized in $x + V$. Repeating the initial argument, we see that f is also minorized in $x + V$. So it is o-bounded in that neighborhood of x. Since x was an arbitrary point, the Lemma follows. □

COROLLARY 1. *If $f\colon X \to \overline{Y}$ is convex and majorized in a neighborhood of x_0 then f is locally o-bounded at every point of* $\operatorname{int}\operatorname{dom} f$.

COROLLARY 2. *If, in addition, Y is normal then f is τ_Y-locally bounded (τ_Y = the topology of Y).*

Proof. By the normality of Y every other bounded set is τ_Y-bounded (see [24] and [35]). □

We will use the above material to obtain some continuity results about convex mappings. The first of these appears also in Valadier [38] and we include it here for completion and because we added a second part to it. Furthermore, our proof is more transparent with the use of the previous results.

THEOREM 3.1. *If Y is normal and $f\colon X \to \overline{Y}$ is convex then the following two statements are equivalent*

(1) *f is majorized in a neighborhood U of $\hat{x} \in X$*

(2) *Y has an order bounded neighborhood of the origin* int dom $f \neq \varnothing$ *and f is continuous there.*

Proof. (1) → (2). Statement (1) implies that there is a neighborhood U' of $\hat{x} \to X$ such that for all $z \in U\, f(z) \leq y$.

By considering if necessary $U - \hat{x}$ and $f(x + \hat{x}) - f(\hat{x})$ instead of U and f respectively, we see that we can assume without loss of generality that $\hat{x} = 0$ and $f(\hat{x}) = f(0) = 0$.

By Lemma 3.1, we know that f is also minorized in U, i.e. there is a $-y' \in Y$ such that $-y' \leq f(z)$ for all $z \in U$. Now consider the order interval $[-y', y]$ and let V be any neighborhood of the origin in Y. Let $1 \geq \varepsilon \geq 0$ such that $\varepsilon[-y', y] = [-\varepsilon y', \varepsilon y] \subset V$. This is possible since Y is assumed to be normal and so every o-bounded set is bounded (see [24]).

Let $U_\varepsilon = \varepsilon U$ and take $z' \in U_\varepsilon$. Then we have from the convexity of f that

$$(1)\quad f(z') = f(\varepsilon z) = f(\varepsilon z + (1-\varepsilon)0) \leq \varepsilon f(z) + (1-\varepsilon) f(0) = \varepsilon f(z)$$
$$\to f(z') \leq \varepsilon f(z) \leq \varepsilon y.$$

Also since $f(0) = 0$ by the convexity of f, we have

$$f(z') \geq -f(-z') \geq \varepsilon f(-z) \geq -\varepsilon y'$$

since $-z \in U$ by symmetry. So we have

$$(2)\qquad f(z') \geq -\varepsilon y' \quad \text{for all } z' \in U_\varepsilon.$$

From (1) and (2) above we conclude that

$$f(\varepsilon U) \subset [-\varepsilon y', \varepsilon y] \subset V \to f \text{ is continuous at } \hat{x}.$$

Since by Lemma 3.1 we know that f is locally o-bounded on every $x \in \inf \operatorname{dom} f$ we deduce that $f|_{\operatorname{int}\operatorname{dom} f}$ is continuous.

(2) → (1). Let f be continuous at $\hat{x}$. Let V be the order bounded neighborhood of the origin in Y. Then $V \subseteq [-y', y]$ where $y = \sup V$ and $-y' = \inf V$.

Now $f(\hat{x}) + V$ is a neighborhood of $f(\hat{x})$.

By continuity there is a neighborhood U of $\hat{x}$ such that

$$
\begin{aligned}
f(U) &\subseteq f(\hat{x}) + V \subseteq f(\hat{x}) + [-y', y] \\
&\to f \text{ is locally bounded at } \hat{x} \\
&\to f \text{ is locally bounded at every point of } \operatorname{int}\operatorname{dom} f. \qquad \square
\end{aligned}
$$

Before stating the next theorem, we will introduce a new class of mappings that will play a central role in the second part of this work which deals with the nonconvex case (see [22]). Assume now that X is a Banach space and Y a Banach lattice.

DEFINITION 3.2. $f\colon X \to \overline{Y}$ is said to be *locally o-Lipschitz* if and only if for every bounded open set U of X there is a $y \in K_Y^+$ such that

$$|f(x) - f(z)| \leq y\|x - z\| \quad \forall x, z \in U.$$

THEOREM 3.2. *If $f\colon X \to \overline{Y}$ is a convex mapping majorized in a neighborhood of $x_0 \in$* int dom *f then f is locally o-Lipschitz in* int dom *f.*

Proof. By Lemma 3.1, we know that f is locally bounded on the interior of its domain.

So let $\varepsilon > 0$ be such that for $z \in U_\varepsilon(x_0) = \{x\colon \|x - x_0\| < \varepsilon\}$ we have that $|f(z)| \leq y$. We will show that in $U_{\varepsilon/2}(x_0)$ we have $|f(z) - f(x)| \leq y'\|z - x\|$ where $y' = 4y/\varepsilon$.

Suppose that there are $z_1, z_2 \in U_{\varepsilon/2}(x_0)$ such that $f(z_2) - f(z_1) \nleq y'\|z_2 - z_1\|$.

Let $z_3 = z_2 + \lambda(z_2 - z_1)$ for $\lambda > 0$. Since $z_2, z_1 \in U_{\varepsilon/2}(x_0)$, it is possible to pick $\lambda > 0$ such that $z_3 \in U_\varepsilon(x_0)$ and $\|z_3 - z_2\| = \lambda\|z_2 - z_1\| = \varepsilon/2$.

Let $\phi(\lambda) = f(z_2 + \lambda(z_2 - z_1))$.

Then we have that $\phi(\lambda) = f(z_3)$, $\phi(0) = f(z_2)$, $\phi(-1) = f(z_1)$ and it is easy to see that ϕ is a convex mapping defined on R. Using the properties of such mappings we have that

$$\frac{\phi(0)-\phi(-1)}{1} \leq \frac{\phi(\lambda)-\phi(0)}{\lambda}$$
$$\to \frac{\phi(0)-\phi(-1)}{\|z_2-z_1\|} \leq \frac{\phi(\lambda)-\phi(0)}{\lambda\|z_2-z_1\|}$$
$$\to \frac{\phi(0)-\phi(-1)}{\|z_2-z_1\|} \leq \frac{\phi(\lambda)-\phi(0)}{\|z_3-z_1\|}.$$

Since by hypothesis $f(z_2)-f(z_1) \nleq y'\|z_2-z_1\|$ then we have

$$\frac{f(z_3)-f(z_2)}{\|z_3-z_2\|} \nleq y' = 2y/(\varepsilon/2).$$

But recall that z_3 was chosen so that $\|z_3-z_2\| = \varepsilon/2$. Hence we get that

$$f(z_3)-f(z_2) \nleq 2y. \tag{1}$$

But by hypothesis, $z_3, z_2 \in U_\varepsilon(x_0) \to f(z_3) \leq y$ and

$$-f(z_2)-y \to f(z_3)-f(z_2) \leq 2y. \tag{2}$$

From **(1)** and **(2)** above, we get a contradiction. So there do not exist $z_3, z_2 \in U_{\varepsilon/2}(x_0)$ such that

$$f(z_2)-f(z_1) \nleq y'\|z_2-z_1\|.$$

Now suppose that there exist $z_1', z_2' \in U_{\varepsilon/2}(x_0)$ such that

$$f(z_2')-f(z_1') \ngeq -y'\|z_2'-z_1'\|.$$

But then $f(z_1')-f(z_2') \nleq y'\|z_2'-z_1'\|$ contradicts the first part of the proof.

So we conclude that for all $z, x \in U_{\varepsilon/2}(x_0)$ we have

$$-y'\|z-x\| \leq f(z)-f(x) \leq y'\|z-x\|$$
$$\to |f(z)-f(x)| \leq y'\|z-x\|$$
$$\to f \text{ is locally } o\text{-Lipschitz as claimed.} \qquad \square$$

REMARK. We will have a more detailed study of locally o-Lipschitz mappings in the second paper. At this point, we just mention that local-o-Lipschitzness implies local-norm Lipschitzness. This is very easy to see. For $x, z \in U$ $|f(x)-f(z)| \leq y\|x-z\| \to \|f(x)-f(z)\| \leq \|y\|\,\|x-z\|$ f is locally-norm Lipschitz.

Now we will introduce the notion of the subdifferential for convex mappings and we will have a straightforward generalization of Valadier's

Theorem 6 (see **[38]**), which nevertheless will turn out to be very useful in the sequence.

DEFINITION 3.3. Let $f: X \to \overline{Y}$ be a convex mapping. We call the *algebraic subdifferential of f at* x_0 to be the set

$$\partial^\alpha f(x_0) = \{A \in L(X, Y): A(x - x_0) \leq f(x) - f(x_0)\ \forall x \in \operatorname{dom} f\}.$$

Similarly we call the (*topological*) *subdifferential of f at* x_0 to be the set

$$\partial f(x_0) = \{A \in \mathfrak{L}(X, Y): A(x - x_0) \leq f(x) - f(x_0)\ \forall x \in \operatorname{dom} f\}.$$

The elements $\partial^\alpha f(x_0)$ and $\partial f(x_0)$ are called the *algebraic subgradients* and the *subgradients of f at* x_0 respectively.

The next result is an easy generalization of Theorem 6 of Valadier **[38]**.

LEMMA 3.2. *If Y is normal and f is continuous at* $\hat{x} \in \operatorname{dom} f$ *then* $\partial^\alpha f(x) = \partial f(x)$ *for all* $x \in X$.

Proof. It is clear that in general $\partial^\alpha f(x) \supseteq \partial f(x)$ and for all $x \in \operatorname{int} \operatorname{dom} f\, \partial^\alpha f(x) \neq \varnothing$ (by Valadier's Proposition 4 (see **[38]**) and the Hahn-Banach-Kantorovich (see **[24]**, **[35]**)).

Now let $A \in \partial^\alpha f(x)$. We will show that A is continuous at $\hat{x}$ and then by linearity $A \in \mathfrak{L}(X, Y)$, which will give us the desired equality.

For that, we pick a neighborhood V of the origin in Y such that V is symmetric and full. Since Y is a topological vector space there is a neighborhood V_1 of the origin such that $V_1 + V_1 \subseteq V$ and V_1 is absorbing. This implies that there is no $\lambda > 0$ such that $\lambda[A(x) - f(x)] \in V_1 \to A(x) - f(x) \in V_1/\lambda$.

Also for $V_1' = V_1 + f(\hat{x})$ there is U a neighborhood of 0 such that for $U' = U + \hat{x}$ we have from the continuity of f at $\hat{x}$ that $f(U') \subseteq V'/\lambda$.

Now let z be any point in U'. We have

$$\begin{aligned} A(x) &= A(z + x - x) = A(x) + A(z - x) \leq A(x) + f(z) - f(x) \\ &= [A(x) - f(x)] + f(z). \end{aligned}$$

But $[A(x) - f(x)] \in V_1/\lambda$ and $f(z) \in (V_1 + f(\hat{x}))/\lambda = V_1'/\lambda$. So

$$\text{(1)} \qquad \begin{aligned} A(z) &\leq [A(x) - f(x)] + f(z) \in \frac{1}{\lambda}[V_1 + V_1 + f(\hat{x})] \\ &\subseteq \frac{1}{\lambda}[V + f(\hat{x})] = \frac{1}{\lambda}V'. \end{aligned}$$

Also since U' is symmetric $-y' \in U'$. Then

$$A(-z) = A(-z + x - x) = A(x) + A(-z - x) \leq A(x) + f(-z) - f(x)$$
$$\to A(z) \geq -[A(x) - f(x)] - f(-z).$$

But since $[A(x) - f(x)] \in V_1/\lambda$, by the symmetry of V_1 we have that $-[A(x) - f(x)] \in V_1/\lambda$ and similarly $-f(-z) \in V_1'/\lambda$.

So

$$(2) \qquad -[A(x) - f(x)] - f(-z) \in \frac{1}{\lambda} V_1 + \frac{1}{\lambda} V_1 + \frac{1}{\lambda} f(\hat{x})$$
$$\subseteq \frac{1}{\lambda}(V + f(\hat{x})) = \frac{1}{\lambda} V'$$
$$\to A(z) \geq -[A(x) - f(x)] - f(z) \in \frac{1}{\lambda} V'.$$

From (**1**) and (**2**) above we deduce that

$$A(z) \in [-[A(x) - f(x)] - f(z), A(x) - f(x) + f(z)].$$

Since the two end points of the above order interval are in V'/λ and V' is full by the normality assumption on Y, we get that $A(z) \in V'/\lambda \to A(U') \subseteq V'/\lambda \to A(\lambda U') \subseteq V' \to A$ is continuous at $\hat{x} \to A \in \mathcal{L}(X, Y) \to \partial^\alpha f(x) = \partial f(x)\ \forall x \in X$. □

Remark. From this result, we also conclude that $\partial f(x) \neq \varnothing$ for all $x \in \operatorname{int} \operatorname{dom} f$.

We will conclude this section with some results concerning the o-directional derivative of integral operators. The results that we obtain here will be used in the second paper to get a formula for the generalized gradient of an integral operator (see [**22**]). Similar results in the real valued case were obtained by Ioffe-Levin [**11**]. Although with a small additional effort the results hold for nonseparable Banach spaces too, in order to avoid unnecessary technical complications, we will constantly assume that our spaces are separable. For the definition and properties of the Bochner integral, that we will use in the sequence, the reader can refer to Hille-Phillips [**8**].

So let (Ω, Σ, μ) be a positive measure space, X a separable Banach space and Y a separable Banach lattice. As is always the case in this paper Y is order complete and we adjoin to it the elements $\{\pm\infty\}$

Let $f: \Omega \times X \to \overline{Y}$ be a mapping satisfying

(i) f_ω are convex and proper for μ-almost all $\omega \in \Omega$

(ii) $f(\cdot, x)$ are weakly (Pettis) measurable for every $x \in X$.

Notice that since Y is separable then (ii) implies that $f(\cdot, x)$ is strongly (Bochner) measurable for every $x \in X$.

Now let U be an open subset of X and let $x_0 \in U$. Assume that $\omega \to | f_\omega(x) |$ is an $L_1(\Omega; Y)$ mapping for every $x \in U$ and $y^*(f_\omega(x)) \in L_1(\Omega, R)$ for every $x \in U$. Define

$$f(x) = \int_\Omega f_\omega(x)\, d\mu(\omega).$$

We start with some auxiliary general results.

LEMMA 3.3. *If* $g_1, g_2 \in L_1(\Omega, V)$, $g_1 \geq g_2$ *μ-a.e. and* $y^*(g_i) \in L_1(\Omega, R)$, $i = 1, 2\ \forall y^* \in Y^*$ *then*

$$\int_\Omega g_1(\omega)\, d\mu(\omega) \geq \int_\Omega g_2(\omega)\, d\mu(\omega).$$

Proof. Consider

$$\int_\Omega g_1(\omega)\, d\mu(\omega) - \int_\Omega g_2(\omega)\, d\mu(\omega) = \int_\Omega [g_1(\omega) - g_2(\omega)]\, d\mu(\omega).$$

Let $y^* \in (K_Y^+)^*$. Then we have

$$y^*\left[\int_\Omega (g_1(\omega) - g_2(\omega))\, d\mu(\omega)\right] = \int_\Omega y^*(g_1(\omega) - g_2(\omega))\, d\mu(\omega).$$

But since by hypothesis $g_1 \geq g_2$ μ-a.e., we have that $y^*(g_1(\omega) - g_2(\omega)) \geq 0$ μ-a.e. and because μ is a positive measure we get that

$$\int_\Omega y^*(g_1(\omega) - g_2(\omega))\, d\mu(\omega) = y^*\left[\int_\Omega (g_1(\omega) - g_2(\omega))\, d\mu(\omega)\right] \geq 0.$$

Since this is true for every $y^* \in K_Y^*$ we get that

$$\int (g_1(\omega) - g_2(\omega))\, d\mu(\omega) \geq 0 \to \int_\Omega g_1(\mu)\, d\mu(\omega)$$
$$\geq \int_\Omega g_2(\omega)\, d\mu(\omega). \qquad \square$$

Now we can prove the following proposition which is also interesting in its own sake. (For the definition of $f'(x_0; h)$, see §4 and **[38]**.)

PROPOSITION 3.1. $f'(x_0; h) = \int_\Omega f'_\omega(x_0; h)\, d\mu(\omega)$.

Proof. First let us check that the integral $\int_\Omega f'(x_0; h)\, d\mu(\omega)$ is well defined.

By the convexity of $f_\omega(\cdot)$,

$$f_\omega(x_0) - f_\omega(x_0 - h) \leq \frac{f_\omega(x_0 + \lambda h) - f_\omega(x_0)}{\lambda}$$

for $\lambda \in R^+$ and

$$f'_\omega(x_0; h) = \inf_{\lambda > 0} \frac{f_\omega(x_0 + \lambda h) - f_\omega(x_0)}{\lambda}.$$

So by Valadier's Lemma 8 (see **[38]**) we conclude that

$$f'_\omega(x_0; h) = w\text{-}\lim_{\lambda \downarrow 0} \frac{f_\omega(x_0 + \lambda h) - f_\omega(x_0)}{\lambda}$$

$$\to f'_\omega(x_0; h) \text{ is strongly measurable.}$$

Now assume without loss of generality that $x_0 \pm h \in U$. Then

$$\begin{aligned} f_\omega(x_0) - f_\omega(x_0 - h) \leq f'_\omega(x_0; h) &\leq f_\omega(x_0 + h) - f'_\omega(x_0) \\ \to |f'_\omega(x_0; h)| &= f'_\omega(x_0; h) \vee (-f'_\omega(x_0; h)) \\ &\leq (f_\omega(x_0 + h) - f_\omega(x_0)) \vee (f_\omega(x_0 - h) - f_\omega(x_0)). \end{aligned}$$

But

$$\begin{aligned} &(f_\omega(x_0 + h) - f_\omega(x_0)) \vee (f_\omega(x_0 - h) - f_\omega(x_0)) \\ &\quad = \frac{1}{2}\big[f_\omega(x_0 + h) - f_\omega(x_0) + f_\omega(x_0 - h) - f_\omega(x_0) \\ &\qquad\qquad + |f_\omega(x_0 + h) - f_\omega(x_0 - h)|\big] \\ &\quad \leq \frac{1}{2}\big[|f_\omega(x_0 + h) + f_\omega(x_0 - h)| \\ &\qquad\qquad + |f_\omega(x_0 + h) - f_\omega(x_0 - h)| + 2|f_\omega(x_0)|\big] \\ &\quad = |f_\omega(x_0 + h)| \vee |f_\omega(x_0 - h)| + |f_\omega(x_0)| \\ &\qquad\qquad \text{(using the properties of } |\cdot| \text{ (see [36]))} \\ &\quad \leq |f_\omega(x_0 + h)| + |f_\omega(x_0 - h)| + |f_\omega(x_0)|. \end{aligned}$$

Since Y is a Banach lattice we conclude that

$$\|f'_\omega(x_0; h)\| \leq \|\, |f_\omega(x_0 + h)|\, \| + \|\, |f_\omega(x_0 - h)|\, \| + \|\, |f_\omega(x_0)|\, \|$$

$$\to \|f'_\omega(x_0; h)\| \text{ is integrable.}$$

This, together with the fact that it is measurable, imply that $f'_\omega(x_0; h)$ is B-integrable. So the integral is indeed well defined.

Then we have for $f(\cdot)$

$$f'(x_0; h) = \inf_{\lambda>0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda} = o\text{-}\lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda h) - f(x_0)}{\lambda}$$
$$= o\text{-}\lim_{\lambda_n \downarrow 0} \int_\Omega \frac{f_\omega(x_0 + \lambda_n h) - f_\omega(x_0)}{\lambda_n} d\mu(\omega).$$

Using our initial observations and Valadier's Lemma 8 (see [**38**]) we deduce that

$$f'(x_0; h) = w\text{-}\lim_{\lambda_n \downarrow 0} \int_\Omega \frac{f_\omega(x_0 + \lambda_n h) - f_\omega(x_0)}{\lambda_n} d\mu(\omega).$$

So for every $y^* \in Y^*$, we have

$$(f'(x_0; h), y^*) = \lim_{n\to\infty} \left(\int_\Omega \frac{f_\omega(x_0 + \lambda_n h) - f_\omega(x_0)}{\lambda_n} d\mu(\omega), y^* \right)$$
$$= \lim_{n\to\infty} \left(\int_\Omega \left(\frac{f_\omega(x_0 + \lambda_n h) - f_\omega(x_0)}{\lambda_n}, y^* \right) d\mu(\omega) \right).$$

Now if $y^* \in (K_Y^+)^*$ by the monotone convergence theorem, we have that

$$(\mathbf{1}) \quad (f'(x_0; h), y^*) = \int_\Omega \lim_n \left(\frac{f_\omega(x_0 + \lambda_n h) - f_\omega(x_0)}{\lambda_n}, y^* \right) d\mu(\omega)$$
$$= \int_\Omega (f'_\omega(x_0; h), y^*) \, d\mu(\omega)$$
$$= y^* \left[\int_\Omega f'_\omega(x_0; h) \, d\mu(\omega) \right].$$

But since by hypotehsis K_Y^+ is normal, $(K_Y^+)^*$ is generating, i.e. $(K_Y^+)^* = Y^*$ (Krein's Theorem, see [**35**]). So for any $y^* \in Y^*$, there are $y_1^*, y_2^* \in (K_Y^+)^*$ such that $y^* = y_1^* - y_2^*$. Using that in (**1**), we conclude that

$$(f'(x_0; h), y^*) = \left(\int_\Omega f'_\omega(x_0; h) \, d\mu(\omega), y^* \right) \quad \forall y^* \in Y^*$$

$$f'(x_0; h) = \int_\Omega f'_\omega(x_0; h) \, d\mu(\omega). \qquad \square$$

REMARK. Clearly the result of Proposition 3.1 holds for any $x \in U$.

Now define the operators Φ, F: $X \to L_1(\Omega, \overline{Y})$ by the formulas

$$[\Phi(h)](\omega) = f'_\omega(x_0; h) \text{ and } F(x_0) = f(\cdot, x_0).$$

Make the following additional assumptions

"$f_\omega(\cdot)$ is continuous at $x_0 \in U$ for μ-almost all $\omega \in \Omega$".

Then we have the following result about this operator.

PROPOSITION 3.2. *$F'(x; h) = \Phi(h)$ and Φ is demicontinuous at* 0.

Proof. In $L_1(\Omega, \overline{Y})$ we consider the ordering induced by Y i.e. $\phi \geq \psi$ if and only if $\phi(\omega) \geq \psi(\omega)$ μ-a.e. on Y.

Then by the definition of the directional derivative, we have

$$\begin{aligned} F'(x_0; h) &= \inf_{\lambda>0} \frac{F(x_0 + \lambda h) - F(x_0)}{\lambda} \\ &= \inf_{\lambda>0} \frac{f(\cdot, x_0 + \lambda h) - f(\cdot; x_0)}{\lambda} \\ &= f'(\cdot, x_0; h) = \Phi(h). \end{aligned}$$

So we deduce that Φ is a sublinear operator.

Let $h_n \xrightarrow{w} 0$ in X. Then for every $\omega \in \Omega$, we have

$$f_\omega(x_0) - f_\omega(x_0 - h_n) \leq f'_\omega(x_0; h) \leq f_\omega(x_0 + h_n) - f(x_0).$$

But

$$\varliminf_{n\to\infty} (f'_\omega(x_0; h_n), y^*) \geq \varliminf_{n\to\infty} (f_\omega(x_0) - f_\omega(x_0 - h_n), y^*)$$

for

$$\text{(1)} \qquad y^* \in (K_Y^+)^* \to \varliminf_{n\to\infty} (f'_\omega(x_0; h_n), y^*) \geq 0.$$

Similarly again for $y^* \in (K_Y^+)^*$ we get that

$$\text{(2)} \qquad \varlimsup_{n\to\infty} (f_\omega(x_0 + h_n) - f_\omega(x_0), y^*) \geq \varlimsup_{n\to\infty} (f'_\omega(x_0, h_n), y^*)$$
$$\to 0 \geq \overline{\lim}(f'_\omega(x_0; h_n), y^*).$$

From inequalities (1) and (2) above we get that

$$\lim_{n\to\infty} (f'_\omega(x_0; h), y^*) = 0 \quad \forall y^* \in (K_Y^+)^*.$$

From Krein's Theorem, $(K_Y^+)^*$ is generating. Therefore we get that

$$f'_\omega(x_0; h_n) \xrightarrow{w} 0 \ \mu\text{-a.e.} \to \Phi(h_n) \xrightarrow{w} 0 = \Phi(0) \quad \text{for any } h_n \to 0.$$

So indeed Φ is demicontinuous as claimed. □

Now if $f_\omega(x) \le \phi(\omega)$ μ-a.e. for all x in a neighborhood of x_0 and $\phi \in L_1(\Omega, Y)$, then it is easy to see from the continuity of $f_\omega(\cdot)$ at x_0 and the Lebesgue dominated convergence theorem for Bochner integrals, that $F(\cdot)$ is also continuous at x_0.

So we have that $\partial^\alpha F(x_0) = \partial F(x_0)$ (by Valadier's Theorem 6) and $\partial^\alpha F(x_0) = \partial\Phi(0)$ (by Valadier's Lemma 4).

Therefore we conclude that $\partial F(x_0) = \partial\Phi(0)$.

Hence, the following result is true for $Z = L_1(\Omega, Y)$.

PROPOSITION 3.3. *If $A \in L(X, Z)$ is such that $A(h) \le \Phi(h)$ $\forall h \in X$ then $A \in \partial\Phi(0)$ i.e. $A \in \mathfrak{L}(X, Z)$.*

Proof. The fact that $A(h) \le \Phi(h)$ $\forall h \in X$ implies that $A \in \partial^\alpha F(x_0) = \partial^\alpha\Phi(0) = \partial\Phi(0)$. So $A \in \mathfrak{L}(X, Z)$. □

Finally, using Valadier's Theorem 6 and Corollary 7 (see [**38**]), together with our previous observations and results, we get the following Proposition.

PROPOSITION 3.4. *$\partial F(x_0) = \partial\Phi(0)$ is equicontinuous, convex and compact subset of $\mathfrak{L}(X, Z)$ with the weak operator topology. Also*

$$F'(x_0; h) = \Phi(h) = \max\{R(h) \colon R \in \partial F(x_0) = \partial\Phi(0)\}.$$

4. The subdifferential calculus. First we recall some useful notions from Geometric Functional Analysis.

DEFINITION 4.1. Let D be a subset of a vector space Y. The *algebraic interior of D*, denoted by $D^{\alpha i}$, is the set of points x_0 such that for each $x \in X$ there is a $\lambda_0 > 0$ such that for $\lambda \in [-\lambda_0, \lambda_0]$ $\lambda x + (1 - \lambda)x_0 \in D$ or which is equivalent that $D - x_0$ is an absorbing set.

Intuitively this means that we can move from x_0 towards any point in X along a ray by staying in the beginning for a while in the set D.

The relative algebraic interior of D denoted by $D^{r\alpha i}$ is the algebraic interior of D relative to aff D.

If $f\colon X \to \overline{Y}$ is a convex mapping, then for every $x_0 \in (\operatorname{dom} f)^{\alpha i}$ the o-directional derivative exists in every direction and is given by

$$\begin{aligned} f'(x_0; h) &= \inf_{\lambda>0} \frac{f(x_0+\lambda h) - f(x_0)}{\lambda} \\ &= o\text{-}\lim_{\lambda \downarrow 0} \frac{f(x_0+\lambda h) - f(x_0)}{\lambda} \end{aligned}$$

(see **[38]**). Again, we point out that this doesn't imply convergence in the topology of Y since in general order convergence and topological convergence are disjoint notions. For more details about this issue, we refer to **[24]**, **[35]**, and **[38]**.

In the sequence, we will generalize several formulas of the subdifferential calculus. Results in this direction were obtained by Kutatelazde **[14]**, **[15]** and **[16]**, but from a purely algebraic point of view. Our emphasis however is topological and so our approach relies more on the work of Valadier **[38]**.

We start with a generalization of the Moreau-Rockafellar formula.

THEOREM 4.1. *If Y is normal $f_1, f_2\colon X \to \overline{Y}$ are convex mappings with f_1 continuous at $x_0 \in \operatorname{dom} f_2$ then*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x) \quad \forall x \in X.$$

Proof. It is easy to see, directly from the definitions, that in general we have $\partial(f_1 + f_2) \supseteq \partial f_1(x) + \partial f_2(x)$ for every $x \in X$. Let $A \in \partial(f_1 + f_2)(x)$. Then $A(h) \le f_1(x+h) + f_2(x+h) - f_1(x) - f_2(x)$. Define $q(h) = f_1(x+h) - f_1(x)$ and $p(h) = A(h) - f_2(x+h) - f_2(x)$. Clearly by their definitions, q is convex and p is concave. Since

$$\begin{aligned} (\operatorname{dom} f_1)^{\alpha i} \cap \operatorname{dom} f_2 \supseteq \{x\}_0 \ne \varnothing &\to (\operatorname{dom} q)^{\alpha i} \cap \operatorname{dom} p \ne \varnothing \\ &\to 0 \in (\operatorname{dom} q - \operatorname{dom} p)^{\alpha i}. \end{aligned}$$

So by applying Zowe's sandwich theorem (see **[42]**), we know that there is $A_1 \in L(X, Y)$ such that

$$\begin{aligned} & p(h) \le A_1(h) \le q(h) \\ & \qquad \to A(h) - f_2(x+h) - f_2(x) \le A_1(h) \le f_1(x+h) - f_1(x). \end{aligned}$$

So $A_1 \in \partial^\alpha f_1(x) = \partial f_1(x)$ and setting $A(h) - A_1(h) = A_2(h)$, we have that $A_2(h) \le f_2(x+h) - f_2(x)$ $\forall h \in X$. So $A_2 \in \partial^\alpha f_2(x)$, while $A \in \partial^\alpha(f_1 + f_2)(x)$.

Hence finally $A \in \partial(f_1 + f_2)(x)$ and $A_2 \in \partial f_2(x)$ and so we have that $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ $\forall x \in X$. $\square$

Now we will get formulas for the subdifferential of composite mappings.

The first concerns the composition of a convex mapping with an affine continuous operator.

The second is about the composition of two convex mappings.

For that purpose let Z be an o-complete ordered topological vector space.

Let $A \in \mathfrak{L}(X, Y)$ such that $\operatorname{Im} A = Y$ (surjective) and for some $y \in Y$ consider the affine continuous mapping of $\alpha(x) = Ax + y$.

Let $f\colon Y \to \overline{Z}$ be a convex mapping and consider the mapping $f \circ \alpha$: $X \to \overline{Z}$.

Then the following result is a generalization of Goldstein's formula for real valued functions.

THEOREM 4.2. *If f and α are as above then*

$$\partial(f \circ \alpha)(x) = \bigcup_{R \in \partial f(\alpha(x))} R \circ A \quad \forall x \in X.$$

Proof. Following the definitions, it is easy to see that, in general, we have that $\partial(f \circ \alpha)(x) \supseteq \bigcup_{R \in \partial f(\alpha(x))} R \circ A$. So we need to show inclusion in the opposite direction. For that purpose, let $B \in \partial(f \circ \alpha)(x)$. Then by definition we have that

$$\begin{aligned} B(h) &\leq (f \circ \alpha)(x+h) - (f \circ \alpha)(x) \quad \forall h \in X \\ &\to B(h) \leq f(A(x+h)+y) - f(Ax+y). \end{aligned}$$

Now if $h \in \ker A$, we deduce that

$$B(h) \leq f(Ax+y) - f(Ax+y) = 0$$

and since $-h \in \ker A$ similarly we get that $B(-h) \leq 0 \to B(h) \geq 0$. So $B(h) = 0$ and therefore we conclude that $\ker B \supseteq \ker A$. So we have $A \in \mathfrak{L}(X, Y)$, $B \in \mathfrak{L}(X, Z)$, $\ker B \supseteq \ker A$ and by hypothesis $\operatorname{Im}(A) = Y$. By the factorization theorem, we know that there exists $\Phi \in \mathfrak{L}(Y, Z)$ such that $\Phi \circ A = B$.

We claim that in fact $\Phi \in \partial f(\alpha(x))$. If we show this, then clearly we are finished with the proof of the theorem.

Let $w \in Y$. Then there is $h \in X$ such that $Ah = w$.

Hence $\Phi(w) = \Phi(Ah)$. Since $B = \Phi \circ A \in \partial(f \circ \alpha)(x)$, we get that

$$\begin{aligned} A(h) = \Phi(Ah) &\leq (f \circ \alpha)(x+h) - (f \circ \alpha)(x) \\ &= f(\alpha(x+h)) - f(\alpha(x)) \\ &= f(A(x+h)+y) - f(Ax+y) \\ &= f(Ax+w+y) - f(Ax+y). \end{aligned}$$

Because this is true $\forall w \in Y$, we deduce that

$$\Phi \in \partial f(Ax+y) \to \Phi \in \partial f(\alpha(x)).$$

So the theorem follows. □

Before going on to our next subdifferential formula for the composition of two mappings, we need some auxiliary material.

Suppose that Z is an order complete locally convex o.t.v.s. and both Y and Z are weakly sequentially complete with K_Y^+ and K_Z^+ closed.

Let $f\colon X \to \overline{Y}$ be a convex mapping and $g\colon Y \to \overline{Z}$ be a convex increasing mapping with $g(\pm\infty) = \pm\infty$.

It is a trivial exercise to check that $g \circ f$ is in fact convex.

We start with a chain rule for directional derivatives.

Lemma 4.1. *If $x \in (\operatorname{dom} f)^{\alpha i}$ and $g(f(x)+h_1) - g(f(x)+h_2) \leq p(h_1 - h_2)$ where $p\colon Y \to Z$ is sublinear and continuous at 0 then $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$.*

Proof. By definition we have that

$$\begin{aligned} (g \circ f)'(x; h) &= \operatorname*{o\text{-}lim}_{\lambda \downarrow 0} \frac{(g \circ f)(x+\lambda h) - (g \circ f)(x)}{\lambda} \\ &= \operatorname*{o\text{-}lim}_{\lambda \downarrow 0} \frac{g(f(x+\lambda h)) - g(f(x))}{\lambda}. \end{aligned}$$

Because of our assumptions on the spaces Y and Z, we have that

$$(g \circ f)'(x; h) = \operatorname*{w\text{-}lim}_{\lambda \downarrow 0} \frac{g(f(x+\lambda h)) - g(f(x))}{\lambda}.$$

But by the definition of the directional derivative we have that

$$f(x+\lambda h) = f(x) + \lambda f'(x; h) + O(\lambda)$$

where $O(\lambda)/\lambda \downarrow 0$ as $\lambda \downarrow 0 \to O(\lambda)/\lambda \overset{w}{\to} 0$ as $\lambda \downarrow 0$. Hence we have that

$$
\begin{aligned}
(g \circ f)'(x; h) &= \underset{\lambda \downarrow 0}{w\text{-lim}} \frac{g(f(x) + \lambda f'(x; h) + O(\lambda)) - g(f(x))}{\lambda} \\
&= \underset{\lambda \downarrow 0}{w\text{-lim}} \frac{g(f(x) + \lambda f'(x; h) + O(\lambda)) - g(f(x) + \lambda f'(x; h))}{\lambda} \\
&\quad + \frac{g(f(x) + \lambda f'(x; h)) - g(f(x))}{\lambda}.
\end{aligned}
$$

But *from our hypothesis about* g, we have that the first quotient tends to zero while the second one tends to $g'(f(x); f'(x; h))$. So the chain rule follows. □

We will also need the following result.

LEMMA 4.2. *If* g: $Y \to \overline{Z}$ *as before then* $g'(x; \cdot)$ *is an increasing sublinear operator.*

Proof. We have to show that if $y_1, y_2 \in Y$ and $y_1 \geq y_2$, then we have $g'(x; y_1) \geq g'(z; y_2)$.

But $x + \lambda y_1 \geq x + \lambda y_2$ and from the monotonicity of g we get that $g(x + \lambda y_1) \geq g(x + \lambda y_2)$.

Hence

$$
\begin{aligned}
\frac{g(x + \lambda y_1)}{\lambda} - \frac{g(x)}{\lambda} &\geq \frac{g(x + \lambda y_2)}{\lambda} - \frac{g(x)}{\lambda} \quad \text{for all } \lambda > 0 \\
&\to \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{g(x + \lambda y_1) - g(x)}{\lambda} \\
&\geq \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{g(x + \lambda y_2) - g(x)}{\lambda} \\
&\to g'(x; y_1) \geq g'(x; y_2)
\end{aligned}
$$

which is what we wanted. □

In the sequence we will need the Mazur-Orlicz Theorem, which for completeness we state here. For a proof of it, look in the original paper **[18]** or in Peressini **[24]**.

THEOREM (*Mazur-Orlicz*). *Let $p\colon X \to Y$ be a sublinear operator.*

Let ϕ and u be mappings defined on an arbitrary set S with values in X and Y respectively. In order to exist $A \in L(X, Y)$ such that

$$A(x) \le p(x) \quad \text{and} \quad A(\phi(s)) \ge u(s).$$

It is necessary and sufficient that for any $\{s_k\}_{k=1}^n \{\lambda_k\}_{k=1}^n > 0$ we have the inequality

$$\sum_{k=1}^n \lambda_k u(s_k) \le p\left(\sum_{k=1}^n \lambda_k \phi(s_k)\right).$$

Now we are ready to state and prove the second subdifferential formula about composite mappings which can be considered as a kind of chain rule for the subdifferential calculus.

For that purpose, let X be a Banach space and Y an order complete Banach lattice.

THEOREM 4.3. *If f, g are as before and for an $x_0 \in (\operatorname{dom} f)^{\alpha i}$ f is continuous at x_0 and g is completely continuous at $f(x_0)$ then*

$$\partial(g \circ f)(x) = \bigcup_{A \in \partial g(f(x))} \partial(A \circ f)(x) \; \forall x \in (\operatorname{dom} f)^{\alpha i}.$$

Proof. First let $T \in \bigcup_{A \in \partial g(f(x))} \partial(A \circ f)(x)$. Then by definition we have that there is $A \in \partial g(f(x))$ such that

$$T(z) - T(x) \le A(f(z)) - A(f(x)) \quad \forall z \in \operatorname{dom} f.$$

Also, since $A \in \partial g(f(x))$, we have

$$\begin{aligned} A(w) - A(f(x)) \le g(w) - g(f(x)) \quad & \forall w \in \operatorname{dom} g \\ & \to A(f(z)) - A(f(x)) \le g(f(z)) - g(f(x)) \\ & \to T(z) - T(x) \le g(f(z)) - g(f(x)) \quad \forall z \in \operatorname{dom} f \\ & \to T \in \partial(g \circ f)(x). \end{aligned}$$

So we conclude that, in general, we have that

$$\partial(g \circ f)(x) \supseteq \bigcup_{A \in \partial g(f(x))} \partial(A \circ f)(x).$$

In the sequence, we will try to show that opposite inclusion also holds.

So let $R \in \partial(g \circ f)(x)$. By Valadier's Proposition 4 (see [**38**]), we know that for $x \in (\operatorname{dom} f)^{\alpha i}$

$$R(h) \le (g \circ f)'(x; h) \quad \forall h \in X.$$

But by Lemma 4.1, we know that

$$(g \circ f)'(x; h) = g'(f(x); f'(x; h)).$$

So $R(h) \leq g'(f(x); f'(x; h))\ \forall h \in X$.

The problem now is to find a linear operator A from X to Y such that

$$\textbf{(1)} \qquad A(d) \leq g'(f(x); d) \quad \forall d \in Y$$

and

$$\textbf{(2)} \qquad R(z) - R(x) \leq A(f(x) - f(x)) \quad \forall z \in X.$$

At this point, we employ the Mazur-Orlicz Theorem. According to that, such an operator exists, if and only if, $\forall \{z_k\}_{k=1}^n \subset \operatorname{dom} f$ and $\forall \{\lambda_k\}_{k=1}^n \subset R^+$ we have

$$\sum_{k=1}^n \lambda_k R(z - z_k) \leq g'\left(f(x) \sum_{k=1}^n \lambda_k (f(z_k) - f(x))\right).$$

But

$$\begin{aligned} R\left(\sum_{k=1}^n \lambda_k (z_k - z)\right) &\leq g'\left(f(x); f'\left(x; \sum_{k=1}^n \lambda_k (z_k - x)\right)\right) \\ &\leq g'\left(f(x); \sum_{k=1}^n \lambda_k f'(x; z_k - x)\right) \quad \text{(by Lemma 4.2)} \\ &\leq g'\left(f(x); \sum_{k=1}^n \lambda_k (f(z_k) - f(x))\right) \end{aligned}$$

(the last inequality comes from Lemma 4.2 and the definition of f').

This is what we wanted.

So there is an $A \in L(Y, Z)$. By **(1)** we see that $A \in \partial g^\alpha(f(x))$ and $\partial^\alpha g(f(x)) = \partial g(f(x))$.

Therefore we conclude that $R \in \bigcup_{A \in \partial g(f(x))} \partial(A \circ f)(x)$ for $x \in (\operatorname{dom} f)^{ai}$ and so the claim of the Theorem follows. □

The next formula will be for the subdifferential of the supremum of two convex mappings.

So again let X be a Banach space and let Y be a weakly sequentially complete Banach lattice. This means that Y is a (KB)-space (the terminology is due to Vulikh (see **[34]**)) i.e. Y is an order ideal of its bidual.

DEFINITION 4.1. By $\Lambda(Y)$ we denote the following set of operators

$$\Lambda(Y) = \{m \in L^+(Y) : m \leq \mathrm{Id}_Y\}.$$

These are called *multiplier operators*.

DEFINITION 4.2. A linear operator $A\colon Y \to Y$ is said to be o-continuous if and only if

$$y_n \xrightarrow{o} y \quad \text{implies that } Ay_n \xrightarrow{o} Ay.$$

We have the following easy result characterizing the multiplier operators.

LEMMA 4.3. *Multipliers are o-continuous operators.*

Proof. Let C be an order bounded subset of Y. Then it is easy to see that for any $m \in \Lambda(Y)$, we have

$$m(\sup C) \geq \sup m(C).$$

Similarly, since $0 \leq m \leq \mathrm{Id}_Y$ and so $(\mathrm{Id}_Y - m) \in \Lambda(Y)$, we also have that $(\mathrm{Id}_Y - m)(\sup C) \geq \sup(\mathrm{Id}_Y - m)(C)$. Furthermore

$$\begin{aligned} \sup C &= m(\sup C) + (\mathrm{Id}_Y - m)(\sup C) \\ &\geq \sup(m(C)) + \sup(\mathrm{Id}_Y - m)(C) = \sup C. \end{aligned}$$

Hence we conclude that $\sup m(C) = m(\sup C)$.

Now let $y_n \xrightarrow{o} y$. By the definition of o-convergence, there are $\{q_n\}_{n\in N}$ $\{p_n\}_{n\in N} \subseteq Y$ such that $\forall n \in N$ $p_n \leq y_n \leq q_n$ and $p_n \uparrow y$, $q_n \downarrow y$. Since $m \in L^+(Y)$, we have $m(p_n) \leq m(y_n) \leq m(q_n)$. From the first part of the proof, $m(p_n) \uparrow y$ and $m(q_n) \downarrow y$. Hence $m(y_n) \xrightarrow{o} m(y)$ and so m is o-continuous. □

LEMMA 4.4. *If $f\colon X \to \overline{Y}$ is continuous or o-continuous at x then for any $m \in \Lambda(Y)$ $m \circ f$ is also continuous at x.*

Proof. First assume that f is continuous at x. Then if $x_n \xrightarrow{s} x$, we have that $f(x_n) \xrightarrow{s} f(x)$. But in a *KB* space s (= strong) and o^* convergence are equivalent. So $f(x_n) \xrightarrow{o^*} f(x)$ which means that for some subsequence of $\{x_n\}_{n\in N}$ we have that $f(x_n) \xrightarrow{o} f(x)$. By Lemma 4.3, we get that $m(f(x_n)) \xrightarrow{o} (f(x)) \to m(f(x_n)) \xrightarrow{s} m(f(x))$.

Now let f be o-continuous at x. Then if $x_n \xrightarrow{s} x$, we have that

$$f(x_n) \xrightarrow{o} f(x) \to m(f(x_n)) \xrightarrow{o} m(f(x)) \text{ (by Lemma 4.3)}$$

$$\to m(f(x_n)) \xrightarrow{s} m(f(x)) \to m \circ f \text{ is continuous.}$$ □

LEMMA 4.5. *If f_1: $X \to \overline{Y}$ is convex and continuous at x; f_2: $X \to \overline{Y}$ is convex and o-continuous at x then*

$$(f_1 \vee f_2)(x) = f_1(z) \vee f_2(z) \quad \forall z \in X$$

is convex and continuous at x.

Proof. Let $x_n \xrightarrow{s} x$. Then by the continuity of f_1, $f_1(x_n) \xrightarrow{s} f_1(x)$ and by the o-continuity of f_2, $f_2(x_n) \xrightarrow{o} f_2(x)$.

Now if $f_1(x_n) \xrightarrow{s} f_1(x) \to f_1(x_n) \xrightarrow{o^*} f_2(x)$ (since Y is a KB-space) and so there is $\{x_{n_{k_l}}\}_{l\in N} \subseteq \{x_n\}_{n\in N}$ s.t. $f_1(x_{n_{k_l}}) \xrightarrow{o} f_1(x)$. By the o-continuity of the lattice operations, we have that

$$f_1(x_{n_{k_l}}) \vee f_2(x_{n_{k_l}}) \xrightarrow{o} f_1(x) \vee f_2(x),$$

$$\to f_1(x_{n_{k_l}}) \vee f_2(x_{n_{k_l}}) \xrightarrow{s} f_1(x) \vee f_2(x).$$

So by passing from the beginning, if this is necessary, to a subsequence of x_n, we conclude that f is continuous at x. Finally, it is trivial to check the convexity of $f_1 \vee f_2$. □

Now we are in the position to formulate and prove the following Theorem about the subdifferential of the supremum of two convex mappings.

THEOREM 4.4. *If f_1 and f_2 are as in the previous Lemma with* dom f_1 *and* dom f_2 *in general position then*

$$\partial(f_1 \vee f_2)(z) = \bigcup_{m\in\Lambda(Y)} \{(\partial(m \circ f_1)(z) + \partial((\mathrm{Id}_Y - m) \circ f_2(z)))\} \quad \forall z \in X.$$

Proof. By Kutatelazde's result (see [15]), we have

$$\partial^\alpha(f_1 \vee f_2) = \bigcup_{m\in\Lambda(Y)} \left\{\partial^\alpha(m \circ f_1) + \partial^\alpha\left((\mathrm{Id}_y - m) \circ f_2\right)\right\}.$$

Combining Lemmas 3.2, 4.4 and 4.5, we can see that

$$\partial^\alpha(f_1 \vee f_2)(z) = \partial(f_1 \vee f_2)(z),$$

$$\partial^\alpha(m \circ f_1)(z) = \partial(m \circ f_1)(z)$$

and

$$\partial^{\alpha}((\mathrm{Id}_Y - m) \circ f_2(z) = \partial(\mathrm{Id}_Y - m) \circ f_2)(z) \quad \forall z \in X.$$

So the formula of the Theorem follows. □

Also based on another result of Kutatelazde (see **[15]**), we obtain the next Theorem.

THEOREM 4.5. *If f_1 and f_2 are as above with $x \in (\operatorname{dom} f_2)^{\alpha i}$ then*

$$\partial(f_1 \vee f_2)(z) = \bigcup_{m \in \Lambda(Y)} \{m \circ \partial f_1(z) + (\mathrm{Id}_Y - m) \circ \partial f_2(z)\}$$

for every $z \in (\operatorname{dom} f_1 \cap \operatorname{dom} f_2)^{\alpha i}$.

Proof. Again this follows by a corresponding algebraic result of Kutatelazde (see **[15]**) and Lemmas 3.2, 4.4 and 4.5. □

We will continue our study of the subdifferential calculus of vector valued convex mappings by relating the subdifferential with the usual Gateaux differential when the latter exists. In the process of doing that, we get a result which is the natural completion of Valadier's Theorem 6 (see **[38]**).

So we have for $x \in (\operatorname{dom} f)^{\alpha i}$

$$\sup\left\{\frac{f(x+\lambda h) - f(z)}{\lambda} : \lambda < 0\right\} = \sup\left\{\frac{f(x+\lambda(-h)) - f(x)}{-\lambda} : \lambda > 0\right\}$$

$$= -\inf\left\{\frac{f(x+\lambda(-h)) - f(x)}{\lambda} : \lambda > 0\right\} = -f'(x; -h).$$

Now since $0 = f'(x; 0) = f'(x; h - h) \leq f'(x; h) + f'(x; -h)$ we have that for every $h \in X$ $-f'(x; -h) \leq f'(x; h)$.

Next let $A \in \partial f(x)$. Then we know that

$$\begin{aligned} A(h) &\leq f'(x; h) \quad \forall h \in X \\ &\leftrightarrow A(-h) \leq f'(x; -h) \\ &\leftrightarrow -A(-h) \geq -f'(x; -h) \\ &\leftrightarrow A(h) \geq -f'(x; -h). \end{aligned}$$

So for $x \in (\operatorname{dom} f)^{\alpha i}$ we have that $A \in \partial f(x) \leftrightarrow A(h) \geq -f'(x; -h)$. The next proposition is the natural completion of Valadier's Theorem 6 (see **[38]**).

PROPOSITION 4.1. *If Y is normal and $f\colon X \to \overline{Y}$ is convex and continuous at x then* $-f'(x; -h) = \min\{A(h); A \in \partial f(x)\}$.

Proof. By Valadier's Theorem 6, we know that

$$\begin{aligned} f'(x; h) &= \max\{A(h): A \in \partial f(x)\} \\ &\to f'(x; -h) = \max\{A(-h): A \in \partial f(x)\} \\ &\to -f'(x; -h) = -\max\{A(-h): A \in \partial f(x)\} \\ &\to -f'(x; -h) = \min\{A(h); A \in \partial f(x)\}. \qquad \square \end{aligned}$$

This observation will help us in achieving our next goal which is to relate the subdifferential and the G-differential.

First we have to introduce some auxiliary material.

Assume that Y is normal and K_Y^+ is closed.

Recall if (τ_Y)-$\lim_{\lambda \to 0}[(f(x + \lambda h) - f(x))/\lambda]$ exists for all $h \in X$ and belongs to $\mathfrak{L}(X, Y)$, then f is said to be Gateaux differentiable at x and the limit is called the Gateaux or G-differential and it is denoted by $f_G'(x; h)$.

We already know that

$$p_n = \frac{f(x + \lambda_n' h) - f(x)}{\lambda_n'} \quad \text{for } \lambda_n' < 0 \text{ increases as } \lambda_n' \uparrow 0$$

and

$$q_n = \frac{f(x + \lambda_n'' h) - f(x)}{\lambda_n''} \quad \text{for } \lambda_n'' > 0 \text{ decreases as } \lambda_n'' \downarrow 0.$$

Combine $\{\lambda_n'\}_{n \in N}$ and $\{\lambda_n''\}_{n \in N}$ in one sequence $\{\lambda_n\}_{n \in N}$ and let

$$r_n = \frac{f(x + \lambda_n h) - f(x)}{\lambda_n}.$$

By repeating terms, when this is necessary, we have

$$\text{(1)} \qquad p_n \le r_n \le q_n \quad \forall n \in N.$$

Since f is a G-differentiable we know that $p_n \xrightarrow{\tau_w} f_-' = f_G'$ and $q_n' \xrightarrow{\tau_w} f_+' = f_G'$. But by Peressini's Corollary 3.2 (see [**24**], p. 91), we have that

$$\text{(2)}' \qquad p_n \uparrow f_G' \quad \text{and} \quad q_n \downarrow f_G'.$$

Relations (**1**) and (**2**) above imply that $r_n \xrightarrow{o} f_G'$.

So we have proved the following result.

PROPOSITION 4.2. *If f, X, Y as before then*

$$f_G'(x; h) = \underset{\lambda \to 0}{o\text{-lim}} \frac{f(x+\lambda h) - f(x)}{\lambda}.$$

In the next result we achieve what we wanted. For a similar result, but under more restrictive assumptions, see Zowe [**39**].

In the sequence, let Y be a normal and weakly sequentially complete Banach lattice.

Then the following result is true.

THEOREM 4.6. *If $f\colon X \to \overline{Y}$ is convex and continuous at x then f is G-differentiable if and only if $\partial f(x)$ is a singleton.*

Proof. Let f be G-differentiable at x.

Then we deduce that

$$f_+'(x; h) = f_-'(x; h) \to f'(x; h) = -f'(x; -h)$$

(again by the result of Peressini (see [**24**], p. 91))

$$\to \max\{A(h)\colon A \in \partial f(x)\} = \min\{A(h)\colon A \in \partial f(x)\} \quad \forall h \in X$$

$\to \partial f(x)$ is indeed a singleton.

Now let $\partial f(x)$ be a singleton.

Then

$$\max\{A(h)\colon A \in \partial f(x)\} = \min\{A(h)\colon A \in \partial f(x)\} \quad \forall h \in X$$
$$f'(x; h) = -f'(x; -h).$$

From the assumptions on Y, we deduce that this is a KB space. So we have

$$\begin{aligned} f'(x; h) &= \underset{\lambda \uparrow 0}{o\text{-lim}} \frac{f(x+\lambda h) - f(x)}{\lambda} \\ &= (\tau_Y)\text{-}\lim_{\lambda \downarrow 0} \frac{f(x+\lambda h) - f(x)}{\lambda} = f_+'(x; h) \end{aligned}$$

and

$$\begin{aligned} -f'(x; h) &= \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{f(x+\lambda h) - f(x)}{\lambda} \\ &= (\tau_Y)\text{-}\lim_{\lambda \uparrow 0} \frac{f(x+\lambda h) - f(x)}{\lambda} = f_-'(x; h). \end{aligned}$$

Therefore $f_+'(x; h) = f_-'(x; h)$ $\forall h \in X$, f is G-differentiable. □

Finally, we close our study of the subdifferential calculus with a brief look at the subdifferential of the composition of a multiplier and a convex mapping. In the light of Theorems 4.4 and 4.5, this is a very useful result to have. So let X be a t.v.s. and Y an order complete o.t.v.s.

THEOREM 4.7. *If* $\alpha \in \Lambda(Y) \cap \mathfrak{L}(Y)$ *and* f: $X \to \overline{Y}$ *is convex then* $\alpha \circ \partial f(x) \subseteq \partial(\alpha \circ f)(x)$ *and if* α *is invertible equality holds.*

Proof. First observe that since α is linear monotone and f is convex, then $\alpha \circ f$ is convex.

Now let $A \in \partial f(x)$. By definition we have

$$\begin{aligned} A(z-x) &\leq f(z) - f(x) \quad \forall z \in \operatorname{dom} f \\ &\to \alpha(A(z-x) \leq \alpha(f(z) - f(x)) \quad (\text{since } \alpha \text{ is monotonic}) \\ &\to (\alpha \circ A)(z-x) \leq (\alpha \circ f)(z) - (\alpha \circ f)(x) \\ &\qquad\qquad \forall z \in \operatorname{dom} f = \operatorname{dom}(\alpha \circ f). \end{aligned}$$

So we get the first part of the Theorem, namely that

$$\alpha \circ \partial f(x) \subseteq \partial(\alpha \circ f)(x).$$

Now assume, in addition, that α is invertible.

Let $B \in \partial(\alpha \circ f)(x)$. Then we have

$$\begin{aligned} B(z-x) &\leq (\alpha \circ f)(z) - (\alpha \circ f)(x) \quad \forall z \in \operatorname{dom} f \\ &\to B(z-x) \leq \alpha(f(z) - f(x)) \\ &\to \alpha^{-1}B(z-x) \leq (\alpha^{-1} \circ \alpha)(f(z) - f(x)) \end{aligned}$$

(since α^{-1} is monotone, too. See Kutatelazde [**15**]).

$$\to (\alpha^{-1} \circ B)(z-x) \leq f(z) - f(x).$$

So $\alpha^{-1} \circ B \in \partial f(x) \to B \in \alpha \circ \partial f(x)$. Hence we conclude that $\alpha \circ \partial f(x) = \partial(\alpha \circ f)(x)$. □

In the hypothesis of Theorem 4.7, we had that $\alpha \in \mathfrak{L}(Y)$. So it would be nice to have conditions under which this is in fact true. The next Proposition answers this question. First we need a definition.

DEFINITION 4.3. An o.t.v.s. which is a vector lattice is a *topological vector lattice* if there is a neighborhood basis of solid sets. If it is locally convex, then it is called a *locally convex lattice*.

PROPOSITION 4.3. *If $\alpha \in \Lambda(Y)$ and Y is a topological vector lattice then $\alpha \in \mathfrak{L}(Y)$.*

Proof. Let V be a solid neighborhood of the origin in Y. Then for $x \in V$, we have that $\alpha(x) \leq x$ since $\alpha \leq \mathrm{Id}_Y$. But

$$\begin{aligned}\alpha(x) \leq x \text{ and } \alpha(-x) \leq -x &\to \alpha(x) \vee \alpha(-x) \leq x \vee (-x)\\ &\to \alpha(x) \vee (-\alpha(x)) \leq x \vee (-x)\\ &\to |\alpha(x)| \leq |x| \to \alpha(x) \in V \quad \forall x \in V.\end{aligned}$$

So $\alpha(V) \subseteq V \to \alpha \in \mathfrak{L}(X, Y)$. □

As an epilog to this section, we include a result which is not directly connected to the subdifferential calculus but which nevertheless gives us a better understanding of the multiplier operators which, as we saw, play a key role in obtaining subdifferential formulas.

First, another definition.

DEFINITION 4.4. If X, Y are o.t.v.s. which are lattices and $A\colon X \to Y$ a linear operator which preserves the lattice operations, then A is called a *Lattice or Riesz homomorphism*.

PROPOSITION 4.4. *If $A \in L^+(X, Y)$ is bijective then A is a lattice isomorphism if and only if A^{-1} is positive.*

Proof. Necessity: Obvious. *Sufficiency*: Suppose that $A^{-1} \geq 0$. We know that since $A \geq 0$,

$$\text{(1)} \qquad \begin{aligned}&A(x \vee z) \geq A(x) \text{ and } A(x \vee z) \geq A(z)\\ &\to A(x \vee z) \geq A(x) \vee A(z).\end{aligned}$$

Also $A(x) \vee A(x) \geq A(x)$ and $A(x) \vee A(z) \geq A(z)$. Using the fact that $A^{-1} \geq 0$, we get

$$\text{(2)} \qquad \left.\begin{aligned}A^{-1}(A(x) \vee A(x)) \geq x\\ A^{-1}(A(x) \vee A(z)) \geq z\end{aligned}\right\} \to A^{-1}(A(x) \vee A(z)) \geq x \vee z$$

$$\to A(x) \vee A(z) \geq A(x \vee z).$$

From (1) and (2) above, we deduce that

$$A(x \vee z) = A(x) \vee A(z).$$

So indeed A is a lattice isomorphism. □

COROLLARY. *If $\alpha \in \Lambda(Y)$ is bijective then α is a lattice isomorphism.*

This concludes the fourth section. In the next section, we will develop a duality theory for convex mappings.

5. Lower semicontinuity and duality theory of convex mappings. In this section, we will extend the Young-Fenchel theory of conjugate convex functions to convex operators.

Let X be a just countable locally convex t.v.s. Let Y be a locally convex lattice which is normal. Assume that $\mathrm{int}(K_Y^+) = (\mathring{K}_Y^+) \neq \varnothing$. Then we know (see [24]) that Y is normable. As in the previous sections, adjoin to Y a greatest element $+\infty$ and a smallest element $-\infty$ and denote the augmented space by $\overline{Y}$. For $\overline{Y}$ an open set will be of the form $U \cup \{+\infty\}$ or $U \cup \{-\infty\}$ where U is open in Y. Clearly with the induced topology $Y \subseteq \overline{Y}$ is the initial o.t.v.s.

We will denote the topology of X by τ_X and that of Y by τ_Y.

DEFINITION 5.5. Let $f\colon X \to \overline{Y}$ be a mapping.

If f is finite at $x \in X$, then it is said to be *lower semicontinuous at x* (abbreviated by l.s.c.) if and only if for every $y \in (\mathring{K}_Y^+)$ there is a neighborhood U of x_0 in X such that for every $z \in U, f(z) + y - f(x_0) \in (\mathring{K}_Y^+)$. Since our mapping is allowed to take also the value $+\infty$ we complete the definition by saying that f is l.s.c. at $x \notin \mathrm{dom}\, f$ if and only if for every $y \in (\mathring{K}_Y^+)$ there is a neighborhood U of x such that for all $z \in U, f(z) - y \in (\mathring{K}_Y^+)$ (this actually is equivalent to saying that dom f is closed in X). Finally if $f(x_0) = -\infty$, then f is l.s.c. at x_0.

In the sequence we will study in more detail this new class of mappings.

We start with the following characterization of l.s.c. mappings.

LEMMA 5.1. *If f is l.s.c. at $x_0 \in X$ then $\underline{\lim}_{x_n \to x_0} f(x_n) \geq f(x_0)$.*

Proof. If $f(x_0) = +\infty$ this is obvious from Definition 5.1. So let $f(x_0) < +\infty$ i.e. $x_0 \in \mathrm{dom}\, f$. Let $x_n \to x_0$ and $\lambda_n \downarrow 0$ so that $\lambda_n y = y_n \downarrow 0$ for $y \in (\mathring{K}_Y^+)$ by the Definition 5.1 there is a neighborhood U of x such that for all $z \in U$ $f(x_0) < f(z) + y_{n1}$. Also for $n \geq n_2$, $x_n \in U$ since $x_n \to x_0$ and so $f(x_0) < f(x_n) + y$ for $n \geq n_2$.

Using the definition of the superior limit, we have

$$\begin{aligned}\underline{\lim}_{n\to\infty} f(x_n) &= \bigvee_{n_2\in N} \bigwedge_{n\geq n_2} f(x_n) \\ &\geq \bigvee_{n_2\in N} f(x_0)-y_{n1} = \bigvee_{n_1\in N} f(x_0)-y_{n1} = f(x_0).\end{aligned}$$

So indeed $\underline{\lim}_{x_n\to x_0} f(x_n)\geq f(x_0)$. □

Now assume that for any $y\in Y$ the order intervals that contain y in their interior form a local basis of neighborhoods of y, and for $y\in(\mathring{K}_Y^+)$ the order intervals $[y,+\infty]$ form a filter of neighborhoods of "infinity" (in the extended topology).

Using this assumption, which in the sequence we will call assumption H, we have the following partial converse of Lemma 5.1.

LEMMA 5.2. *If K_Y^+ is complete and Y satisfies assumption H above then $\underline{\lim}_{x_n\to x_0} f(x_n)\geq f(x_0)$ $\forall x_n\to x_0$ implies that f is l.s.c. at x_0.*

Proof. First consider the case $f(x_0)=+\infty$. Then $\underline{\lim}_{x_n\to x_0} f(x_n)\geq f(x_0)$ $=+\infty$ means that $\underline{\lim}_{n\to\infty} f(x_n)=+\infty$. Now suppose that f was not l.s.c. at x_0. Then there is $y\in(K_Y^+)$ such that for every $U\in\mathfrak{F}(x_0)$ (filter of neighborhoods of x_0) there is $x\in U$ such that $f(x)\not\geq y$. So we form a sequence $\{x_n\}_{n\in N}$ such that $x_n\to x_0$. By hypothesis then $\underline{\lim}_{n\to\infty} f(x_n)\geq$ $f(x_0)$ while $f(x_n)\not\geq y$ $\forall n\in N$. Observe that by hypothesis H, the set $\{x$: $f(x)\geq y\}=[y,\infty]^c$ is closed in the extended topology. Now let $\beta_n=$ $\bigwedge_{n'\geq n} f(x_{n'})\geq y$. Then $\beta_n=\underline{\lim}_{n\to\infty} f(x_n)$ and since, from our assumptions on Y, $\tau_y=\tau_0=$ order topology, then by a result in [24] (see p. 160), $\beta_n\to^{\tau_Y=\tau_0}\underline{\lim}_{n\to\infty} f(x_n)\to$ by the closedness of the set $\{x$: $f(x)\not\geq y\}$ we conclude that $\underline{\lim}_{n\to\infty} f(x_n)$ belongs to that set which is a contradiction since $\underline{\lim}_{n\to\infty} f(x_n)=+\infty$. So f is l.s.c. at x_0.

Now let $f(x_0)<+\infty$ i.e. $x_0\in\operatorname{dom} f$. Again we proceed to prove the Lemma by contradiction.

So suppose that the claim of the Lemma is not true. Then there is a $y\in(\mathring{K}_Y^+)$ such that for every $U\in\mathfrak{F}(x_0)$ there is $x\in U$ such that $f(x)+y\not\geq f(x_0)$. So we can form a sequence $\{x_n\}_{n\in N}$ such that $x_n\to x_0$.

By hypothesis then

$$\underline{\lim}_{n\to\infty} f(x_n)\geq f(x_0)$$

$$\underline{\lim}_{n\to\infty} f(x_n)+y>f(x_0)\quad \text{for } y\in\left(\mathring{K}_Y^+\right).$$

Let $y_n = f(x_n) + y - f(x_0)$. We have that $\forall n \in N y_n \notin (\mathring{K}_Y^+)$ and $\underline{\lim}_{n\to\infty} y_n \geq y$ where $y \in (\mathring{K}_Y^+)$. This, with the help of our hypothesis, will give us the desired contradiction. Let $\beta_n = \bigwedge_{n' \geq n} y_n$. Then $\beta_n \uparrow \underline{\lim}_{n\to\infty} y_n = y_0$. If $y_0 = +\infty$ then, since $\beta_n \xrightarrow{\tau_0} \underline{\lim}_{n\to\infty} y_n$, by hypothesis H there will be an n_0 such that, for $n \geq n_0$, $\beta_n \in (\mathring{K}_Y^+)$. If $y_0 < +\infty$ then $y_0 \in (\mathring{K}_Y^+)$. But then $(\beta_n, 2y_0)$ form a local basis of y_0. So for some n_0, we will have for $n \geq n_0$ that $\beta_n \in (\mathring{K}_Y^+)$. Hence in any case $\beta_n \in (\mathring{K}_Y^+)$ for n-large enough. Therefore, since $\beta_n \leq y_n = f(x_n) + y - f(x_0)$ we get that $y_n \in (\mathring{K}_Y^+)$ for n large enough $\to f(x_n) + y - f(x_0) \in (\mathring{K}_Y^+)$ for n large enough which is a contradiction.

So f is l.s.c. at x_0. □

We can have the same result with a different set of hypotheses. So again we assume that for $y \in (\mathring{K}_Y^+)$, the order intervals $[y, +\infty]$ form a filler of neighborhoods (in the extended topology) of "infinity". Call this assumption H_∞.

LEMMA 5.2′. *If H_∞ holds, Y has the diagonal property and K_Y^+ is closed then $\underline{\lim}_{n\to\infty} f(x_n) \geq f(x_0)$ for every $x_n \to x_0$ implies that f is l.s.c. at x_0.*

Proof. First consider the case $f(x_0) = +\infty$. Again we continue by contradiction. We form $x_n \to x_0$ such that $f(x_n) \not\geq y$ for some $y \in (\mathring{K}_Y^+)$. Let β_n be as before. Then $\beta_n \leq f(x_n)$ and $\beta_n \uparrow \underline{\lim}_{n\to\infty} f(x_n) = +\infty \to \beta_n \xrightarrow{ru} \underline{\lim}\, f(x_n)$ (by the diagonal property) $\to \beta_n \xrightarrow{\tau_Y} \underline{\lim}_{n\to\infty} f(x_n)$ (by the normality of Y which we assumed once and for all in the beginning of this section). So by H_∞ there is an n_0 such that, for $n \geq n_0$, $\beta_n \in (y, \infty] \to f(x_n) \in (y, \infty]$ for all $n \geq n_0$, a contradiction.

So f is l.s.c. at x_0.

Now consider the case where $f(x_0) < +\infty$ i.e. $x_0 \in \operatorname{dom} f$. Again the proof is by contradiction. So suppose that the claim of the Lemma is not true. Then there is a $y \in (\mathring{K}_Y^+)$ such that for every $U \in \mathfrak{F}(x_0)$ there is an $x \in U$ for which $f(x) + y - f(x_0) \notin (\mathring{K}_Y^+)$. So we form a sequence $\{x_n\}_{n\in N}$ such that $x_n \to x_0$. By hypothesis then we have that

$$\begin{aligned} f(x_0) &\leq \underline{\lim}_{n\to\infty} f(x_n) \\ &\to f(x_0) < \underline{\lim}_{n\to\infty} f(x_n) + y \quad \text{for } y \in (\mathring{K}_Y^+) \\ &\to \underline{\lim}_{n\to\infty} f(x_n) + y - f(x_0) \in (\mathring{K}_Y^+). \end{aligned}$$

Call $y_n = f(x_n) + y - f(x_0)$. So we have $y_n \notin (\mathring{K}_Y^+)$ $\forall n \in N$; but $\underline{\lim}_{n\to\infty} y_n \in (\mathring{K}_Y^+)$.

If $\underline{\lim}_{n\to\infty} y_n = +\infty$ it is easy to see, using H_∞, that we get a contradiction. If $\underline{\lim}_{n\to\infty} y_n < +\infty$ then let $\beta_n = \bigwedge_{n'\geq n} y_n \leq y_n$. We see that $\beta_n \uparrow \underline{\lim}_{n\to\infty} y_n$ which by the diagonal property of Y implies that $\beta_n \xrightarrow{ru} \underline{\lim}_{n\to\infty} y_n$ and since Y is normal, $\beta_n \xrightarrow{\tau_Y} \underline{\lim}_{n\to\infty} y_n$. So for $n \geq n_0$, we have that $\beta_n \in (\mathring{K}_Y^+) \to y_n \in (\mathring{K}_Y^+)$ for $n \geq n_0$, a contradiction.

So f is l.s.c. at x_0. □

Now we will introduce some notions that we will be using in the sequence.

DEFINITION 5.2. We call the set UL$y = \{x: f(x) - y \in (0, +\infty]\}$ the *upper level set of f at y*.

Similarly, we call the set L$y = \{x: f(x) \leq y\}$ the *lower level set of f at y*.

DEFINITION 5.3. We call the set hg$f = \{[x, y] \in X \times \overline{Y} f(x) - y \in (0, \infty]\}$ the *hypograph of f*.

Similarly, we call the set epi $f = \{[x, y] \in X \times Y$: $f(x) - y\}$ the *epigraph of f*.

Then we have the following complete characterization of the lower semicontinuous mappings.

THEOREM 5.1. *The following are equivalent*:

(1) $f: X \to \overline{Y}$ *is l.s.c.*

(2) ULy *is open* $\forall y \in Y$.

If in addition Y satisfies the assumptions of Lemma 5.2 *or of Lemma* 5.2′, *then* (1) *and* (2) *are equivalent to* (3) *below and they all iimply* (4) *and* (5).

(3) hg$f$ *is open in* $X \times \overline{Y}$.

(4) Ly *is closed in X for every* $y \in Y$.

(5) epi f *is closed in* $X \times Y$.

Proof. (1) → (2). Let UL$\hat{y} = \{x: f(x) - \hat{y} \in (0, \infty]\}$ and take $x_0 \in$ UL$\hat{y}$. If $f(x_0) = +\infty$ then for every $y \in (\mathring{K}_Y^+)$, there is a neighborhood U of x_0 such that, for $x \in U$, $f(z) > y$. Let y be such that $y \geq \hat{y}$. This is possible since every interior point of K_Y^+ is an order unit. Using that y, we are done with the case where $f(x_0) = +\infty$.

Now suppose that $f(x_0) < +\infty$. Then we have $f(x_0) - \hat{y} \in (\mathring{K}_Y^+)$. By the lower semicontinuity of f at x_0 for every $y \in (\mathring{K}_Y^+)$ there is a neighborhood U of x_0 such that, for all $z \in U$, $f(x_0) - y < f(z)$. Let $y = f(x_0) - \hat{y} \in (\mathring{K}_Y^+)$. Let U' be the neighborhood of x_0 corresponding to this particular choice of y. Then for $z \in U'$, we have $f(x_0) - (f(x_0) - \hat{y}) < f(z) \to \hat{y} < f(z)$ for all $z \in U' \to U' \subseteq \mathrm{UL}\hat{y} \to \mathrm{UL}\hat{y}$ is open in X.

(2) $\to$ (1). Let $y \in (\mathring{K}_Y^+)$. If $f(x_0) = +\infty$, then (2) is just the second part of Definition 5.1. If $x_0 \in \operatorname{dom} f$ then $f(x_0) > f(x_0) - y$. By (2) there is a neighborhood U of x_0 such that for all $z \in U$ $f(z) > f(x_0) - y$ which is just our definition of lower semicontinuity.

Now consider Y with the assumption either of Lemma 5.2 or Lemma 5.2′.

(1) $\leftrightarrow$ (3). Consider the mapping F: $X \times Y \to \overline{Y}$ defined by

$$F(x, y) = f(x) - y.$$

Under those assumptions F is l.s.c. if and only if f is. So (1) $\leftrightarrow$ (3) follows from (1) $\leftrightarrow$ (2).

(1) $\leftrightarrow$ (4). Let $x_n \in \mathrm{L}y$ and $x_n \to x$. Then since $f(x_n) \le y \to \underline{\lim}_{n\to\infty} f(x_n) \le y$. But $f(x) \le \underline{\lim}_{n\to\infty} f(x_n) \le y$ (by Lemma 5.1) $\to x \in \mathrm{L}y \to \mathrm{L}y$ is closed in X.

(1) $\to$ (5). Again by considering the mapping $F(x, y) = f(x) - y$, this follows from (1) $\to$ (4) above. $\square$

Next we will present a Weierstrass type theorem for our class of lower semicontinuous mappings.

For that, let Y be a Frechet o.t.v.s. which is a vector lattice and which satisfies the assumptions of Lemma 5.2 or of Lemma 5.2′.

THEOREM 5.2. *If f: $K \to \overline{Y}$ where $K \subseteq X$ is compact and $\overline{\operatorname{Im} f}$ is o-complete then f attains its infimum on K.*

Proof. Let $y_0 = \inf_{x\in K} f(x) \in \overline{\operatorname{Im} f}$ by its o-completeness. So there are $x_n \in K$ such that $f(x_n) \to y_0$.

Since, by hypothesis, K is compact, there is a subsequence of x_n (which for simplicity in the notation we will denote again by x_n) such that $x_n \to x \in K$.

By the lower semicontinuity of f, we deduce that

$$\varliminf_{n\to\infty} f(x_n) \geq f(x).$$

Also $f(x_n) \to y_0 \to f(x_n) \overset{*-ru}{\to} y_0$ (see Peressini **[24]**, p. 162) $f(x_{n_{k_l}}) \overset{o-ru}{\to} y_0$ (just from the definition of $*$-convergence). But $\varliminf_{l\to\infty} f(x_{n_{k_l}}) \geq f(x)$. So $f(x) = y_0$ or otherwise we will contradict the choice of y_0. □

DEFINITION 5.4. A mapping f: $X \to \overline{Y}$ is said to be u.s.c. *at* x_0 if and only if $-f$ is lower semicontinuous at x_0.

Hence using Definition 5.1 we have the following topological description of upper semicontinuity.

"f: $X \to \overline{Y}$ which is finite at x_0 is u.s.c. there if and only if for every $y \in (\mathring{K}_Y^+)$ there is a neighborhood U of x_0 such that $f(x_0) + y - f(z) \in (\mathring{K}_Y^+)$ for all $z \in U$. If $f(x_0) = -\infty$, then f is u.s.c. at x_0 if and only if for every $y \in (-\mathring{K}_Y^+)$ there is a neighborhood U of x_0 such that $f(z) < y$ $\forall z \in U$".

The next result shows that our definition of semicontinuity is the appropriate one.

THEOREM 5.3. *If f: $X \to \overline{Y}$ is finite at x_0 and both u.s.c. and l.s.c. there then f is continuous at x_0.*

Proof. Let $V \in \mathcal{F}(0)$ where $\mathcal{F}(0)$ is the filter of neighborhoods of the origin in Y.

Take $y \in (\mathring{K}_Y^+)$ such that $[-y, y] \subseteq V$. This is possible by the normality assumption, since in that case order bounded sets are bounded.

Now form the lower semicontinuity of f at x_0 there is a neighborhood U_1 of x_0 such that

(1) $$f(x_0) < f(z) + y \quad \forall z \in U_1.$$

Also from the upper semicontinuity of f at x_0 there is a neighborhood U_2 of x_0 such that

(2) $$f(z) < f(x_0) + y \quad \forall z \in U_2.$$

Let $U = U_1 \cap U_2$.

Combining **(1)** and **(2)** above we get that

$$\begin{aligned} &-y < f(z) - f(x_0) < y \quad \forall z \in U \\ &\to f(z) - f(x_0) \in [-y, y] \subseteq V \quad \forall z \in U \\ &\to f(U) \subseteq V + f(x_0) \\ &\to f \text{ is continuous at } x_0. \end{aligned}$$

□

A converse of this result can be obtained if we note that the first part of hypothesis H holds, since $\mathring{K}_Y^+ \neq \varnothing$.

THEOREM 5.4. *If f: $X \to Y$ is finite and continuous at x_0 then f is both u.s.c. and l.s.c. at x_0.*

Proof. By the continuity of f at x_0 for every $V \in \mathfrak{F}(0)$ which is symmetric, there is a neighborhood U of x_0 such that

$$\pm\left(f(x_0) - f(x)\right) \in V \quad \text{for all } z \in U$$

Let $V = (-y, y)$ for some $y \in (\mathring{K}_Y^+)$ (this is possible by H). Then $f(x_0) < f(z) + y \to f$ is l.s.c. at x_0 and also $f(z) < f(x_0) + y \to f$ is u.s.c. at x_0. □

An important class of mappings that will have a key role in the second paper is the class of o-Lipschitz mappings. By that we mean those functions for which for U, a bounded, open subset of X, and for all $x, z \in U$, there is a $y \in K_Y^+$ such that $|f(x) - f(z)| \leq y\|x - z\|$.

So we see that X must be a Banach space. Let Y be an order complete Banach lattice. Recall that since Y is a Banach lattice, then K_Y^+ is automatically closed (see [35]).

PROPOSITION 5.1. *If $f \in \mathrm{L}_{\mathrm{ip}}^{\mathrm{oc}}(X, Y)$ then f is l.s.c.*

Proof. Fix any $x \in X$. Let U be an open bounded subset of X containing x such that for some $y \in K_Y^+$ we have

$$|f(x) - f(z)| \leq y\|x - z\|.$$

Let $d = \operatorname{diam} U < \infty$. Then $y\|x - z\| < dy$.

Consider $y \in (\mathring{K}_Y^+)$. Then being an interior point of K_Y^+ is an order unit. So there is $\lambda \in R^+$ such that $dy \leq \lambda\hat{y} \to dy/\lambda \leq \hat{y}$. So if $z \in B = \{z'\colon \|x - z'\| < d/\lambda\}$ then $f(x) < f(z) + \hat{y} \to f$ is l.s.c. at x. Since x was arbitrary in X, then f is l.s.c. □

In the sequence we will get some results that relate the notions of the subdifferential and of the o-directional derivative of a convex mapping with that of lower semicontinuity.

PROPOSITION 5.2. *If* (1) *Y is complete* (i.e. *is Frechet*) *with the diagonal property and K_Y^+ is closed.*

(2) *f: $X \to \overline{Y}$ is a convex mapping with $\partial f(x_0) \neq \varnothing$ then f is l.s.c. at x_0.*

Proof. By hypothesis (1) above, we deduce that $\tau_Y = \tau_0$ (see **[24]**) and furthermore that topological convergence in Y is equivalent to relative uniform convergence which obviously implies order convergence. Since by hypothesis $\partial f(x_0) \neq \varnothing$, there is $A \in \mathfrak{L}(X, Y)$ such that

$$A(z - x_0) < f(z) - f(x_0) \quad \forall z \in \operatorname{dom} f.$$

Let $z_n \to x_0$. Then by the continuity of A we have that

$$A(z_n - x_0) \to 0$$

$$\to A(z_n - x_0) \overset{o}{\to} 0 \quad \text{(by the remark in the beginning of the proof)}.$$

Then we have that

$$\varliminf_{n\to\infty} A(z_n - x_0) \leq \varliminf_{n\to\infty} f(z_n) - f(x_0)$$

$$\to 0 \leq \varliminf_{n\to\infty} f(z_n) - f(x_0)$$

$$\to f(x_0) \leq \varliminf_{n\to\infty} f(z_n)$$

$\to f$ is l.s.c. at x_0 by Lemma 5.2′. □

Note. In the above Proposition, we did not need assumption H since from the fact that $\partial f(x_0) \neq \varnothing$, we have $x_0 \in \operatorname{dom} f$ and so as it is easy to see from the proof of Lemma 5.2′, its result holds.

PROPOSITION 5.3. *If $f\colon X \to \overline{Y}$ is convex, $x_0 \in (\operatorname{dom} f)^{\alpha i}$ and $f'(x_0, \cdot)$ is l.s.c. at* 0 *then f is l.s.c. at x_0.*

Proof. Directly from the definition of the o-directional derivative we know that for $h \in X$

$$\textbf{(1)} \qquad f'(x_0; h) \leq f(x_0 + h) - f(x_0).$$

Also from the lower semicontinuity of $f'(x_0; \cdot)$ at the origin, we know that for any $y \in (\mathring{K}_Y^+)$ there is U_0, a neighborhood of the origin in X, such that

$$0 = f'(x_0; 0) < f'(x_0; h) + y \quad \forall h \in U_0.$$

Using **(1)** we deduce that

$$0 < f(x_0 + h) - f(x_0) + y \quad \forall h \in U_0$$

$$\to f(x_0) < f(x_0 + h) + y \quad \forall h \in U_0$$

$\to f$ is l.s.c. at x_0. □

The next result is a very interesting one because it reveals another case where the topological and algebraic subdifferential coincide (see also Valadier's Theorem 6 in [**38**] and Lemma 3.2 of this paper).

PROPOSITION 5.4. *If $f\colon X \to Y$ is convex and u.s.c. at $x \in (\operatorname{dom} f)^{\alpha i}$ such that $f(x) \leq 0$ and $f'(x; x) \leq f(x)$ then $\partial^\alpha f(x) = \partial f(x)$. If in addition $f(0) \geq 0$ and $f(x) = 0$ then $\partial^\alpha f(z) = \partial f(z)$ $\forall z \in X$.*

Proof. Let V' be any neighborhood of $f(x)$ in Y. Consider $V = V' - f(x)$. This is a neighborhood of the origin in Y. Since by hypothesis Y is normal, there is $y \in (\mathring{K}_Y^+)$ such that $[-y, y] \subseteq V$. Then

$$[-y, y] + f(x) \subseteq V + f(x) = V'$$
$$\to [-y + f(x), y + f(x)] \subseteq V'.$$

Now let $T \in \partial^\alpha f(x)$. As we have already pointed out in a previous section, such an operator exists by the Hahn-Banach-Kantorovich theorem.

By definition we know that

$$T(z - x) \leq f(z) - f(x) \quad \forall z \in \operatorname{dom} f.$$

Now by the upper semicontinuity of f at x we know that for every $y' \in (\mathring{K}_Y^+)$ there is U' a symmetric neighborhood of x such that

$$f(z) < f(x) + y' \quad \forall z \in U'$$
$$\to T(z - x) \leq f(z) - f(x) < y' \quad \forall z \in U'$$
$$\to T(z) < y' + T(x).$$

On the other hand

$$-T(z) = T(-z) > -y' - T(x) \quad \forall z \in U'$$

and since U' is a symmetric neighborhood of x we get that

$$T(z) > -y' - T(x).$$

Hence we have

$$-y' - T(x) < T(z) < y' + T(x) \quad \forall z \in U'.$$

Since $T(x) \leq f'(x; x)$ we get that

$$-y' - f'(x; x) < T(z) < y' + f'(x; x). \tag{1}$$

Now specify y' to be $y' = -f'(x; x) + y + f(x)$. From our hypothesis we know that $y' \in (\mathring{K}_Y^+)$.

Substituting this choice of y' in relation (**1**) above, we get that

$$\textbf{(2)}\qquad \begin{aligned} -f'(x; x) + f'(x; x) - y - f(x) &< T(z) \\ &< -f'(x; x) + y + f'(x; x) + f(x) \\ &\to -y - f(x) < T(z) < y + f(x)\ \forall z \in U! \end{aligned}$$

Since $f(x) \leq 0$, we have that $-f(x) \geq f(x) \to -y - f(x) \geq -y + f(x)$. Hence from (**2**) above we deduce that

$$T(U') \subseteq [-y - f(x), y + f(x)] \subseteq [-y + f(x), y + f(x)] \subseteq V'.$$

But then this implies that $T \in \mathfrak{L}(X, Y)$. Since in general $\partial^{\alpha} f(x) \supseteq \partial f(x)$, we conclude that $\partial^{\alpha} f(x) = \partial f(x)$.

For the second part of the Proposition, we proceed as follows.

Let w be any point in X and let $A \in \partial^{\alpha} f(w)$. Let V be a symmetric and full neighborhood of the origin in Y. Let V_1 be such that $V_1 + V_1 \subseteq V$. For some $\lambda > 0$, we have that $\lambda(A(w) - f(w)) \in V_1 \to A(w) - f(w) \in V_1/\lambda$. Let $y \in (\mathring{K}_Y^+)$ such that $[-y, y] \subseteq V_1/\lambda$. By the upper semicontinuity of f at x, we have that there is a neighborhood $U' = x + U$ of x ($U =$ neighborhood of the origin in X) such that $f(z) < y\ \forall z \in U'$. Assume that U' is symmetric.

Now $-f(z) > -y$. But since $0 \leq f(0) = f(\frac{1}{2}z + \frac{1}{2}(-z)) \leq \frac{1}{2}f(z) + \frac{1}{2}f(-z) \to f(-z) \geq -f(z)$, we get that $f(-z) > -y\ \forall z \in U'$. Since U' is symmetric again we conclude that $f(U') \subseteq [-y, y] \subseteq V_1/\lambda$.

Now $z \in U'$ we have

$$\begin{aligned} A(z) &= A(z + w - w) = A(z - w) + A(w) \leq f(z) - f(w) + A(w) \\ &= A(w) - f(w) + f(z). \end{aligned}$$

But

$$\begin{aligned} &A(w) - f(w) \in V_1/\lambda \text{ and } f(z) \in V_1/\lambda \\ &\qquad \to A(U') \subseteq V_1/\lambda + V_1/\lambda \subseteq V/\lambda \\ &\qquad \to A(\lambda U') \subseteq V \\ &\qquad \to A \in \mathfrak{L}(X, Y) \to A \in \partial f(w). \end{aligned}$$

Hence we conclude that $\partial^{\alpha} f(z) = \partial f(z)\ \forall z \in X$. □

The next two results will show that the supremum of certain affine continuous mappings are lower semicontinuous ones.

So let Y satisfy the hypotheses of Lemma 5.2 or those of Lemma 5.2′.

Let $f\colon X \to \overline{Y}$ be an l.s.c. mapping which is minorized by y_0.

Let $A \in \mathfrak{L}(X, Y)$, consider the affine continuous mapping

$$T_x(z) = A(z - x) + f(x).$$

Then the following result holds.

PROPOSITION 5.5. *If* $\phi(z) = \sup_{x \in X} A(z - x) + f(x)$ *then* ϕ *is convex and l.s.c.*

Proof. Clearly ϕ is convex.

Next we will show its lower semicontinuity. Let $z_n \to z$. Then by its definition

$$\phi(z_n) \geq A(z_n - x) + f(x).$$

Also

$$A(z_n - x) \overset{\tau_Y = \tau_0}{\to} A(z - x) \to A(z_n - x) \overset{ru}{\to} A(z - x).$$

So

$$\underline{\lim}_{n \to \infty} \phi(z_n) \geq \underline{\lim}_{n \to \infty} A(z_n - x) + f(x) = A(z - x) + f(x)$$

$$\to \underline{\lim}_{n \to \infty} \phi(z_n) \geq \sup_{x \in X} A(z - x) + f(x) = \phi(x).$$

Hence from Lemma 5.2 or 5.2′, we get that ϕ is l.s.c. □

Before stating the final result in this direction, we need the following auxiliary Lemma.

Let Y be any order complete vector lattice.

LEMMA 5.3. *If* $\{\alpha_n\}_{n \in N}$ *and* $\{\beta_n\}_{n \in N} \subseteq Y$ *then*

$$\underline{\lim}_{n \to \infty} (\alpha_n + \beta_n) \geq \underline{\lim}_{n \to \infty} \alpha_n + \underline{\lim}_{n \to \infty} \beta_n.$$

Proof. First we have that

$$\left.\begin{array}{l} p_m = \bigwedge_{n \geq m} \alpha_n \leq \alpha_n \quad \forall n \geq m \\ q_m = \bigwedge_{n \geq m} \beta_n \leq \beta_n \quad \forall n \geq m \end{array}\right\} \to \bigwedge_{n \geq m} \alpha_n + \bigwedge_{n \geq m} \beta_n \leq \alpha_n + \nu_n \quad \forall n \geq m$$

$$\to \bigwedge_{n \geq m} \alpha_n + \bigwedge_{n \geq m} \beta_n \leq \bigwedge_{n \geq m} (\alpha_n + \beta_n) = r_m$$

$$\to p_m + q_m \leq r_m \quad \forall m \in N.$$

But

$$p_m \uparrow \varliminf_{n\to\infty} \alpha_n, \qquad q_m \uparrow \varliminf_{n\to\infty} \beta_n \quad \text{and} \quad r_m \uparrow \varliminf_{n\to\infty} (\alpha_n + \beta_n).$$

Then

$$p_m + q_m \uparrow \varliminf_{n\to\infty} \alpha_n + \varliminf_{n\to\infty} \beta_n \le o\text{-}\lim r_m = \varliminf_{n\to\infty} (\alpha_n + \beta_n). \qquad \square$$

Now let everything be as in Proposition 5.5. Then we have

PROPOSITION 5.6. *If $\phi(x) = \sup_{z\in X} T_x(z)$ and $f\colon X \to Y$ is convex and l.s.c. then ϕ is convex and l.s.c.*

Proof. Clearly ϕ is convex.

Let $x_n \to x$. Then by the lower semicontinuity of f we have that $\varliminf_{n\to\infty} f(x_n) \ge f(x)$. Also

$$A(z - x_n) \overset{\tau_Y}{\to} A(z - x)$$
$$\to A(z - x_n) \overset{ru}{\to} A(z - x).$$

Since $\phi(x_n) \ge A(z - x_n) - f(x_n)\ \forall n \in N$

$$\varliminf_{n\to\infty} \phi(x_n) \ge \varliminf_{n\to\infty} (A(z - x_n) + f(x_n)),$$

$$\varliminf_{n\to\infty} \phi(x_n) \ge \varliminf_{n\to\infty} A(z - x_n) + \varliminf_{n\to\infty} f(x_n) \quad \text{(by Lemma 5.3)},$$

$$\varliminf_{n\to\infty} \phi(x_n) \ge A(z - x) + f(x) \quad \text{for all } z \in X,$$

$$\varliminf_{n\to\infty} \phi(x_n) \ge \phi(x).$$

So by Lemma 5.2 or 5.2′, is l.s.c. $\square$

Now we pass to the duality theory of convex mappings.

DEFINITION 5.5. Let $f\colon X \to \overline{Y}$ be a mapping.

Then we define the *Fenchel Transform of f* to be the mapping $f^*\colon \mathfrak{L}_s(X, Y) \to \overline{Y}$ given by

$$f^*(A) = \sup_{x\in \operatorname{dom} f} \{A(x) - f(x)\}.$$

Miming the notation existing for real valued functions, we will denote the collection of l.s.c. convex mappings going from X into $\overline{Y}$ by $\Gamma(X, Y)$.

PROPOSITION 5.7. *If Y has the diagonal property and K_Y^+ is complete then $f^* \in \Gamma(\mathcal{L}_s(X, Y), Y)$.*

Proof. Let $A_n \to^s A$. this means that for every $x \in X$, we have that $A_n(x) \xrightarrow{\tau_Y} A(x)$. But from the assumptions on Y, we have that $\tau_Y = \tau_0$ where, as always, $\tau_0 =$ the order topology on Y (see [24]). So $A_n(x) \xrightarrow{\tau_0} A(x)$ $\to A_n(x) \xrightarrow{ru} A(x)$ (again by [24], p. 160) $\to A_n(x) \xrightarrow{o} A(x)$.

Now from the definition of the Fenchel transform of f, we have

$$A_n(x) - f(x) \leq f^*(A_n) \quad \forall x \in \operatorname{dom} f$$

$$\to \varliminf_{n\to\infty} A_n(x) - f(x) \leq \varliminf_{n\to\infty} f^*(A_n)$$

$$\to A(x) - f(x) \leq \varliminf_{n\to\infty} f^*(A_n) \quad \forall x \in \operatorname{dom} f$$

$$\to \sup_{x\in\operatorname{dom} f} A(x) - f(x) \leq \varliminf_{n\to\infty} f^*(A_n)$$

$$\to f^*(A) \leq \varliminf_{n\to\infty} f^*(A_n)$$

$\to f$ is l.s.c. using Lemma 5.2′.

Next, let us show the convexity of $f^*(\cdot)$.

Let $A^\lambda = \lambda A_1 + (1-\lambda)A_2$ for $\lambda \in [0, 1]$ and $A_1, A_2 \in \operatorname{dom} f^*$. Then

$$\begin{aligned}
f^*(A^\lambda) &= f^*(\lambda A_1 + (1-\lambda)A_2) \\
&= \sup_{x\in\operatorname{dom} f} [(\lambda A_1 + (1-\lambda)A_2)(x) - f(x)] \\
&= \sup_{x\in\operatorname{dom} f} [\lambda(A_1(x) - f(x)) + (1-\lambda)(A_2(x) - f(x))] \\
&\leq \sup_{x\in\operatorname{dom} f} \lambda[A_1(x) - f(x)] + \sup_{x\in\operatorname{dom} f} (1-\lambda)[A_2(x) - f(x)] \\
&= \lambda f^*(A_1) + (1-\lambda)f^*(A_2).
\end{aligned}$$

So f^* is convex. Therefore $f^* \in \Gamma(\mathcal{L}_s(X, Y), Y)$. □

We can continue this process and define

$$f^{**}\colon (\mathcal{L}_s(X, Y), Y) \to \overline{Y}$$

by

$$f^{**}(T) = \sup_{A \in \operatorname{dom} f^*} (T(A) - f^*(A)).$$

Recall that X can be identified with a subspace of $\mathfrak{L}(\mathfrak{L}_s(X, Y), Y)$ via the identification $i: X \to \mathfrak{L}(\mathfrak{L}_s(X, Y), Y)$ where $[i(x)](A) = A(x)$.

So it is natural to consider f^{**} restricted on $i(x)$, which, for simplicity, we will denote again by X and then compare it with f.

A straightforward result in this direction is the following.

LEMMA 5.4. *In general we have that* $f^{**}|_x \leq f$.

Proof. Observe that from the definition of f^* we have

$$\begin{aligned} A(x) - f^*(A) &\leq A(x) - [A(x) - f(x)] \\ &\to A(x) - f^*(A) \leq f(x) \quad \forall A \in \operatorname{dom} f^* \\ &\to f^{**}(x) \leq f(x) \\ &\to f^{**}|_x \leq f. \end{aligned}$$

□

Also, as we did for f^*, similarly we can show that f^{**} is l.s.c. and convex, i.e. $f^{**} \in \Gamma(\mathfrak{L}(\mathfrak{L}_s(X, Y)), Y)$. (Note that we can derive this also from Proposition 5.6.)

The next proposition is a step further in understanding the relation between f and $f^{**}|_x$.

PROPOSITION 5.8. *If* $\partial f(x_0) \neq \varnothing$ *then* $f(x_0) = f^{**}(x_0)$. *Hence, if* Y *satisfies the assumptions of Proposition* 5.7 *then* f *is l.s.c. at* x_0.

Proof. Let $A \in \partial f(x_0)$.

From the definition of the subdifferential we have that

$$\begin{aligned} A(z - x_0) &\leq f(z) - f(x_0) \quad \forall z \in \operatorname{dom} f \\ &\to A(z) - A(x_0) + f(x_0) \leq f(z). \end{aligned}$$

Call $y_0 = A(x_0) - f(x_0)$.

Then $A(z) - y_0 \leq f(z)$ $\forall z \in \operatorname{dom} f$. So $A(z) - y_0$ is a continuous affine minorant of f.

Let any $y < f(x_0)$ and take $z = x_0$. In that case, $A(x_0) - y_0 = f(x_0)$ just from the definition of y_0. So we have

$$A(z) - y_0 \leq f(z) \quad \forall z \in \operatorname{dom} f \tag{1}$$

and for $y < f(x_0)$

$$(2) \qquad y < A(x_0) - y_0.$$

From (1) we get that $\forall z \in \operatorname{dom} f,\ A(z) - f(z) \leq y_0 \to f^*(A) \leq y_0$. From (2) and since $A \in \partial f(x_0)$, we have that $y < A(x_0) - f^*(A) \to y < f^*(x_0)$.

Let $y = y_n = f(x_0) - w/n$ for $w \in (\mathring{K}_Y^+)$

$$(3) \qquad \to f(x_0) - \frac{1}{n} w < f^{**}|_x (x_0) \quad \forall n \in N$$

$$\to f(x_0) \leq f^{**}|_x (x_0).$$

Since by Lemma 5.4, we know that in general $f^{**}|_x \leq f \to f(x_0) = f^{**}(x_0)$.

The last statement of the Proposition follows from the fact that f^{**} is convex and l.s.c. □

Note. Observe that the last part of the above Proposition reproduces with a different approach the result of Proposition 5.2.

Next we will formulate a theorem that is the analog of the well-known Moreau Theorem for real valued convex functions.

Assume that Y has an order bounded neighborhood of the origin. Then we have

PROPOSITION 5.9. *If f is u.s.c. at $x_0 \in (\operatorname{d\mathring{o}m} f)$ and $\partial^\alpha f(x) \neq \varnothing$ $\forall x \in \operatorname{dom} f$ then $f = f^{**}|_x$. Hence if the hypotheses of Lemmas 5.2 and 5.2′ hold then f is l.s.c. mapping.*

Proof. For any $y \in (\mathring{K}_Y^+)$, there is U, a neighborhood of x_0 such that $y + f(x_0) > f(z)$ for all $z \in U$. So f is majorized in a neighborhood of x_0. By Theorem 3.1, we conclude that f is continuous at int dom f. Hence by Lemma 3.2 $\partial^\alpha f(x) = \partial f(x)$ $\forall x \in X$. Since for $x \in \operatorname{dom} f$, we have by hypothesis that $\partial^\alpha f(x) \neq \varnothing$, then $\partial f(x) \neq \varnothing$ for $x \in \operatorname{dom} f$. Applying Proposition 5.8, we conclude that $f(x) = f^{**}(x)$, $x \in \operatorname{dom} f$.

Clearly if $f(x) = +\infty$, then $f^{**}(x) = +\infty$, too.

So $f = f^{**}|_x$ on all X.

Finally the last part of the Proposition follows again from the fact that f^{**} is l.s.c. □

We will conclude this section with a result that is suggested from the proof of Proposition 5.8.

PROPOSITION 5.10. *If for every* $x \in \operatorname{dom} f$, *given* $y < f(x)$, *there is* A, *a* $(\tau_X o)$*-bounded linear operator from* X *to* Y *such that* $y < A(x) - y_0$ *and*

$$A(z) - y_0 \leq f(z) \quad \forall z \in \operatorname{dom} f$$

then f *is an l.s.c. mapping.*

Proof. Consider the upper level set $\mathrm{UL}y = \{x \colon y < f(x)\}$.

We will show that this is open in X.

Let $x_0 \in \mathrm{UL}y$. Then $y < f(x_0)$. By hypothesis, we can find $A \colon X \to Y$ which is $(\tau_X o)$-bounded and linear such that

$$y < A(x_0) - y_0$$

and

$$A(z) - y_0 < f(z) \quad \forall z \in \operatorname{dom} f.$$

Let $\hat{y} = A(x_0) - y_0 - y > 0$.

By the $(\tau_X o)$-boundedness of A, we deduce that for a neighborhood U of x_0,

$$\begin{aligned}
&|A(x) - A(x_0)| < \hat{y} \quad \forall x \in U \\
&\to -\hat{y} + A(x_0) < A(x) < A(x_0) + \hat{y} \quad \forall x \in U \\
&\to y < A(x) - y_0 \leq f(x) \\
&\to x \in \mathrm{UL}y \quad \text{for all } x \in U \\
&\to U \subset \mathrm{UL}y.
\end{aligned}$$

So $\mathrm{UL}y$ is open and so by Theorem 3.1 f is l.s.c. □

Note. In a normal space a $(\tau_X o)$-bounded operator is bounded. So the operator A in Proposition 5.10, since it is linear, is continuous i.e. $A \in \mathfrak{L}(X, Y)$.

But in general an operator $B \in \mathfrak{L}(X, Y)$ is not necessarily $\tau_X o$-bounded.

This concludes our study of the duality theory of convex operators.

In the next, last, section of this first paper, we will study some useful generalizations of convex operators and of the notion of the subdifferential.

6. Quasidifferentiability, quasiconvexity and closedness. The notion of quasidifferentiability, introduced first by Pshenichnyi, was the first attempt to generalize the subdifferential calculus to nonconvex functions.

Here we undertake this task for vector valued mappings.

DEFINITION 6.1. Let

$$f'(x; h) = o\text{-}\lim_{\lambda \downarrow 0} \frac{f(x + \lambda h) - f(x)}{\lambda}$$

and we call the limit when it exists "*the o-directional derivative of f at x in the direction h*".

REMARK. Clearly if $f'(x; \cdot)$ exists for all $h \in X$, then $x \in (\text{dom } f)^{\alpha i}$ i.e. $\text{dom } f - x$ is absorbing.

We know that for f convex and $x \in (\text{dom } f)^{\alpha i}$ then $f'(x; h)$ exists for all $h \in X$.

What we mean by Definition 6.1 is that for every sequence $\lambda_n > 0$ $\lambda_n \downarrow 0$, the limits exist. We need to check that the limit is unique, i.e. independent of the sequence λ_n chosen.

Suppose that for $\lambda_n^1 > 0$, we have

$$\frac{f(x + \lambda_n^1 h) - f(x)}{\lambda_n^1} \xrightarrow{o} f_1'(x; h)$$

and for $\lambda_n^2 > 0$

$$\frac{f(x + \lambda_n^2 h) - f(x)}{\lambda_n^2} \xrightarrow{o} f_2'(x; h).$$

Now consider the combined sequence

$$\lambda_n = \begin{cases} \lambda_n^1, & n = \text{even}, \\ \lambda_n^2, & n = \text{odd}, \end{cases}$$

Then

$$\frac{f(x + \lambda_n h) - f(x)}{\lambda_n} \xrightarrow{o} f'(x; h).$$

But every subsequence of the above sequence also converges to the same limit $f'(x; h)$.

So by taking the subsequence $\{\lambda_n\}_{n=\text{even}}$ and $\{\lambda_n\}_{n=\text{odd}}$, we conclude that

$$f_1'(x; h) = f_2'(x; h) = f'(x; h).$$

Hence, indeed, the limit is unique and so the notion of the o-directional derivative is well defined.

Note. The definition of Valadier [38] given for f convex is just a particular case of our general definition 6.1.

DEFINITION 6.2. If $f'(x; h) = \max_{A \in m_1(x)} A(h)$, where $M_1(x)$ is a nonempty convex equicontinuous subset of $\mathfrak{L}(X, Y)$ then f is said to be *quasidifferentiable*.

In the sequence, we will investigate in detail this new class of mappings and relate it with notions and results obtained in the previous sections.

PROPOSITION 6.1. *If* $f_1, f_2 \colon X \to Y$ *are quasidifferentiable mappings then so is* $f_1 + f_2$.

Proof. We have

$$\begin{aligned}
&\underset{\lambda \downarrow 0}{o\text{-lim}}\left[\frac{(f_1+f_2)(x+\lambda h)-(f_1+f_2)(x)}{\lambda}\right]\\
&\quad\to \underset{\lambda \downarrow 0}{o\text{-lim}}\left[\frac{f_1(x+\lambda h)-f_1(x)+f_2(x+\lambda h)-f_2(x)}{\lambda}\right]\\
&\quad= \underset{\lambda \downarrow 0}{o\text{-lim}}\left[\frac{f_1(x+\lambda h)-f_1(x)}{\lambda}\right] + \underset{\lambda \downarrow 0}{o\text{-lim}}\left[\frac{f_2(x+\lambda h)-f_2(x)}{\lambda}\right]\\
&\quad= f_1'(x; h) + f_2'(x; h)\\
&\quad= \max_{T \in M_1(x)} T(h) + \max_{R \in M_2(x)} R(h) \quad (\text{since } f_1, f_2 \text{ are quasidifferentiable})\\
&\quad= \max_{Q \in [M_1(x)+M_2(x)]} Q(h).
\end{aligned}$$

We have to check that $M_1(x) + M_2(x)$ is convex and equicontinuous subset of $\mathfrak{L}(X, Y)$. Clearly it is convex since $M_1(x)$ and $M_2(x)$ are.

Now to show the equicontinuity of $M_1(x) + M_2(x)$, we proceed as follows. Let V be a neighborhood of the origin in Y. Then there is W such that $W + W \subseteq V$.

Let $U_1 \in \mathfrak{F}(0_x)$ such that $T(U_1) \subseteq W$ for all $T \in M_1(x)$ and $U_2 \in \mathfrak{F}(0_x)$ such that $R(U_2) \subseteq W$ for all $R \in M_2(x)$ (such neighborhoods exist from the equicontinuity of M_1 and M_2). Define $U = U_1 \cap U_2$. Then $(T + R)(U) = Q(U) = T(U) + R(U) \subseteq W + W \subseteq V$. Hence $M_1(x) + M_2(x)$ is equicontinuous and so we conclude that $f_1 + f_2$ is indeed quasidifferentiable. □

PROPOSITION 6.2. *If* f *is quasidifferentiable and* $\mu \in R^+$ *then* μf *is quasidifferentiable also.*

Proof. Obvious from the definition.

From those two Propositions, we get immediately the following Corollary.

COROLLARY. *Positive linear combinations of quasidifferentiable mappings give a quasidifferentiable mapping.*

Next we will examine the quasidifferentiability of the supremum of two quasidifferentiable mappings.

Recall that in a vector lattice Y $a + \beta = (a \vee \beta) + (a \wedge \beta)$. Also $a, \beta \in Y$ are said to be orthogonal if $|a| \wedge |\beta| = 0$. We denote that by writing $a \perp \beta$. In that case, it is easy to see that $|a| \vee |\beta| = |a| + |\beta|$.

So now we can proceed to the next proposition.

PROPOSITION 6.3. *If $f_1, f_2 \geq 0$, $f_1 \perp f_2$ and both are quasidifferentiable then $f = f_1 \vee f_2$ is quasidifferentiable too.*

Proof. By definition

$$
\begin{aligned}
f'(x; h) &= o\text{-}\lim_{\lambda \downarrow 0} \frac{f(x + \lambda h) - f(x)}{\lambda} \\
&= o\text{-}\lim_{\lambda \downarrow 0} \frac{(f_1 \vee f_2)(x + \lambda h) - (f_1 \vee f_2)()}{\lambda} \\
&= o\text{-}\lim_{\lambda \downarrow 0} \frac{f_1(x + \lambda h) - f_1(x)}{\lambda} + o\text{-}\lim_{\lambda \downarrow 0} \frac{f_2(x + \lambda h) - f_2(x)}{\lambda}
\end{aligned}
$$

(from the fact that $f_1 \perp f_2$ and the previous remarks)

$$= f_1'(x; h) + f_2'(x; h).$$

The rest are as in the proof of Proposition 6.1. □

Now assume that Y is an algebra lattice.

DEFINITION 6.3. *If f: $X \to Y$ is* o-continuous along lines then f is called *o-hemicontinuous* i.e.

$$o\text{-}\lim_{\lambda \downarrow 0} f(x + \lambda h) = f(x).$$

PROPOSITION 6.4. *If f_1: $X \to Y$ is quasidifferentiable at x and $f_1(x + \lambda h) \leq \lambda f_1(x)$ for $\lambda > 0$ f_2: $X \to Y$ is o-hemicontinuous and nonnegative at x then $f_1 \cdot f_2$ is quasidifferentiable at x.*

Proof. We have that

$$\begin{aligned}(f_1 \cdot f_2)'(x; h) &= \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{(f_1 \cdot f_2)(x + \lambda h) - (f_1 \cdot f_2)(x)}{\lambda} \\ &= \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{f_1(x + \lambda h)\cdot f_2(x + \lambda h) - f_1(x)\cdot f_2(x)}{\lambda} \\ &= \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{f_1(x + \lambda h)\cdot f_2(x + \lambda h) - f_1(x + h)\cdot f_2(x)}{\lambda} \\ &\quad + \frac{f_1(x + \lambda h)\cdot f_2(x) - f_1(x)\cdot f_2(x)}{\lambda} \\ &= \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{f_1(x + \lambda h)[f_2(x + \lambda h) - f_2(x)]}{\lambda} \\ &\quad + \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{f_2(x)[f_1(x + \lambda h)\cdot f_1(x)]}{\lambda}.\end{aligned}$$

By the o-hemicontinuity of f_2 and since $f_1(x + \lambda h) \leq \lambda f_1(x)$, the first summand goes to zero, while the second is

$$\begin{aligned}f_2(x) f_1'(x; h) &= f_2(x) \underset{T \in M_1(x)}{\operatorname{Max}} T(h) \\ &= \underset{T \in M_1(x)}{\operatorname{Max}} \rightarrow f_2(x) T(h) \quad (\text{since } f_2(x) \geq 0).\end{aligned}$$

So indeed $f_1 \cdot f_2$ is quasidifferentiable at x. □

The next result will give us some sufficient conditions for quasidifferentiability.

PROPOSITION 6.5. *If $f'(x; \cdot)$ exists and is convex and continuous then f is quasidifferentiable at x. Furthermore, if Y has the order intervals $[x, y]$ weakly relatively compact and the cone K_Y^+ is closed, then the set $M(x)$ is also compact in $\mathfrak{L}_s(X, Y_w)$.*

Proof. Let $\phi(h) = f'(x; h)$ for all $h \in X$.

By hypothesis ϕ is convex and continuous. So it has a directional derivative at every point. Hence we have

$$(1) \qquad \phi'(0; h) = \underset{\lambda \downarrow 0}{\operatorname{Inf}} \frac{\phi(\lambda h) - \phi(0)}{\lambda} = \underset{\lambda \downarrow 0}{o\text{-lim}} \frac{\phi(\lambda h) - \phi(0)}{\lambda}.$$

But $\phi(0) = f'(x; 0) = 0$ since $f'(x; \cdot)$ is positively homogeneous and

$$\phi(\lambda h) = f'(x; \lambda h) = \lambda f'(x; h) = \lambda \phi(h).$$

So going back into (**1**) we get that

$$f'(x;\, h) = \phi'(0;\, h) = \phi(h).$$

But ϕ is convex and continuous. So by Valadier's Theorem 6 (see [**38**]) we know that

$$\phi'(0;\, h) = \operatorname*{Max}_{T \in \partial\phi(0)} T(h) \to f'(x;\, h) \operatorname*{Max}_{T \in \partial\phi(0)} T(h).$$

Since $M(x) = \partial\phi(x)$, it is convex and equicontinuous and so f is quasidifferentiable at x.

The final part of the Proposition follows from Valadier's Corollary 7. $\square$

Now assume that X is a normed space and Y an order complete Banach lattice with $(\mathring{K}_Y^+) \neq \varnothing$.

Let $f\colon X \to Y$ be a quasidifferentiable mapping i.e.

$$f'(x;\, h) = \max_{T \in M(x)} T(h)$$

where $M(x)$ is convex and equicontinuous subset of $\mathfrak{L}(X, Y)$.

Suppose that $M\colon X \to 2^{\mathfrak{L}(X,Y)}$ is an u.s.c. multioperator from X with its norm topology to the power set of $\mathfrak{L}(X, Y)$ with the strong operator topology.

PROPOSITION 6.6. *If Y has a strong order unit, whose linear hull is order dense and $f'(z;\, h)$ is finite $\forall h$, $x \in X$ then $f'(\cdot\,;\, h)$ is u.s.c. $\forall h \in X$.*

Proof. Fix $h \in X$. Let $0 < \varepsilon \leq 1$ and consider the neighborhood in the strong operator topology, given by

$$B_\varepsilon(0, h) = \{P \in \mathfrak{L}(X, Y)\colon \|P(h)\| \leq \varepsilon\}.$$

From the upper semicontinuity of the multioperator M, we deduce that there is a neighborhood U of x such that for $z \in U$

$$M(z) \subseteq M(x) + B_\varepsilon(0;\, h).$$

Then

$$\max_{T \in M(z)} T(h) \leq \sup_{F \in (M(x) + B_\varepsilon(0;h))} F(h)$$

$$\to \max_{T \in M(z)} T(h) \leq \sup_{(R+P) \in M(x) + B_\varepsilon(0;h)} (R + P)(h)$$

$$\to \max_{T \in M(z)} T(h) \leq \sup_{R \in M(x)} R(h) + \sup_{P \in B_\varepsilon(0;h)} P(h).$$

But by the quasidifferentiability of f, we have that

$$\sup_{R\in M(x)} R(h) = \max_{R\in M(x)} R(h) = f'(x;h).$$

Also $\|P(h)\| \leq \varepsilon \leq 1 \to P(h) \leq \varepsilon e$, where e is the strong order unit.

Hence $f'(z;h) \leq f'(x;h) + \varepsilon e$.

For $\varepsilon' > \varepsilon$, we have $f'(z;h) < f'(x;h) + \varepsilon' e$ for all $z \in U$.

Since Re is order dense in Y, then for every $y \in (\mathring{K}_Y^+)$ there is an ε'' such that $\varepsilon'' e \leq y$. In that case

$$f'(z;h) < f'(x;h) + y \quad \text{for all } z \in U$$
$$\to f'(\cdot\,;h) \text{ is u.s.c.} \qquad \square$$

We will continue by examining some useful generalizations of convex mappings.

Given a convex mapping $f\colon X \to Y$, we know that the level sets $\mathrm{L}y = \{x\colon f(x) \leq y\}$ are convex for every $y \in Y$. It is natural then to ask "What about the converse?" This problem, for $Y = R$, was studied in detail by Fenchel. We will briefly examine its vector valued counterpart.

We start with a definition.

DEFINITION 6.6. A mapping $f\colon X \to Y$ is called *quasiconvex* if and only if for every $y \in Y$ the level sets $\mathrm{L}y = \{x\colon f(x) \leq y\}$ are convex.

In the following Propositions, we will study further this generalization of convexity of mappings, which in the real valued case was successfully used in optimization theory (see [**4**]).

PROPOSITION 6.7. *If $f\colon X \to Y$ is quasiconvex then for $\lambda \in [0,1]$ $f(\lambda x + (1-\lambda)z) \leq f(x) \vee f(z)$. So if $f(z) \leq f(x)$ then*

$$f(\lambda x + (1-\lambda)z) \leq f(x) \quad \textit{for } \lambda \in [0,1].$$

Proof. Clearly $f(x) \leq f(x) \vee f(z)$ and $f(z) \leq f(x) \vee f(z)$.

So $x, z \in L_{f(x)\vee f(z)}$ which is by quasiconvexity a convex set. Therefore $f(\lambda x + (1-\lambda)z) \leq f(x) \vee f(z)$ for $\lambda \in [0,1]$. $\square$

PROPOSITION 6.8. *If $f\colon X \to Y$ is quasiconvex and $f(x) \geq f(0)$ then $f(\lambda x) \leq f(x)$ for $\lambda \in [0,1]$. Also $\phi(\lambda) = f(\lambda x)$ is increasing for $\lambda \geq 0$.*

Proof. Consider the level set $L_{f(x)} = \{z\colon f(z) \leq f(x)\}$. This is convex since f is assumed to be quasiconvex.

By hypothesis $0 \in L_{f(x)} \to (1-\lambda)0 + \lambda x \in L_{f(x)}$ for $\lambda \in [0,1] \to f(\lambda x) \leq f(x)$.

Now consider the mapping $\phi(\lambda) = f(\lambda x)$ for $\lambda \geq 0$.

Let $\lambda_2 \geq \lambda_1$. Call $\lambda_2 x = y$. then $\lambda_1 x = (\lambda_1/\lambda_2)y$ and $(\lambda_1/\lambda_2) \in [0,1]$. From the first part of the proposition we have that

$$f((\lambda_1/\lambda_2)y) \leq f(y) \to f(\lambda_1 x) \leq f(\lambda_2 x) \to \phi(\lambda_1) \leq \phi(\lambda_2). \qquad \square$$

PROPOSITION 6.9. *If f: $X \to Y$ is quasiconvex and g: $Y \to Y$ is an increasing lattice homomorphism then $g \circ f$ is quasiconvex too.*

Proof. Consider the level set $\mathrm{L}y = \{x\colon (g \circ f)(x) \leq y\}$.

Let $x, z \in \mathrm{L}y$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} f(\lambda x + (1-\lambda)z) &\leq f(x) \vee f(z) \\ &\to g(f(\lambda x + (1-\lambda)z)) \leq g(f(x) \vee f(z)) \\ &= g(f(x)) \vee g(f(z)) \leq y. \end{aligned}$$

So we get that $\lambda x + (1-\lambda)z \in \mathrm{L}y$ for $\lambda \in [0,1] \to \mathrm{L}y$ is convex $\to g \circ f$ is a quasiconvex mapping. $\square$

PROPOSITION 6.10. *If f_α: $X \to Y$, $\alpha \in A$, are quasiconvex mappings then $\sup_{\alpha \in A} f_\alpha$: $X \to \overline{Y}$ is quasiconvex too.*

Proof. Let $\mathrm{L}y = \{x\colon \sup_{\alpha\in A} f_\alpha(x) \leq y\}$ and let $x, z \in \mathrm{L}y$.

Then $f_\alpha(z) \leq y$ and $f_\alpha(x) \leq y$ $\forall \alpha \in A$. Since the f_α's are quasiconvex, we have that for

$$\begin{aligned} &\forall \lambda \in [0,1]\, f_\alpha(\lambda x + (1-\lambda)z) \leq y \quad \forall \alpha \in A \\ &\to \sup_{\alpha\in A} f_\alpha(\lambda x + (1-\lambda)z) \leq y \\ &\to \lambda x + (1-\lambda)z \in \mathrm{L}y \to \sup_{\alpha \in A} f_\alpha \end{aligned}$$

is quasiconvex. $\square$

We conclude with the following useful property of the *o*-directional derivative of quasiconvex mappings.

PROPOSITION 6.11. *If f is quasiconvex and $f'(x; h)$ exists for all $h \in X$ then $f(z) \leq f(x)$ implies that $f'(x; z-x) \leq 0$.*

Proof. By definition

$$f'(x; z-x) = o\text{-}\lim_{\lambda \downarrow 0} \frac{f(x+\lambda(z-x)) - f(x)}{\lambda}.$$

Let $\lambda \in [0, 1]$. Then by the quasiconvexity of f and by Proposition 6.7 we have that

$$f(x+\lambda(z-x)) \leq f(x) \to f'(x; z-x) \leq 0. \qquad \square$$

Finally we will study a class of mappings which under some appropriate assumptions on the nature of the space Y constitutes a generalization of our class of lower semicontinuous mappings (see §5). So let X be a first countable t.v.s. and Y a first countable o.t.v.s. which is order complete and a lattice.

DEFINITION 6.7. $f: X \to \overline{Y}$ is said to be *closed* if its level sets $\mathrm{L}y = \{x: f(x) \leq y\}$ $\forall y \in Y$ are closed.

Note. By Theorem 5.1 of this work (in particular (1) $\to$ (4)) if Y satisfies the assumptions of Lemma 5.2 or Lemma 5.2′, then an l.s.c. mapping is closed.

LEMMA 6.1. *f is closed if and only if epif is closed.*

Proof. Define $F(x, y) = f(x) - y$. Clearly F is closed if and only if f is closed. Next note that *epif* can be considered as a level set of F. $\square$

PROPOSITION 6.12. *If X is a Banach space, Y a Banach lattice and $f \in \mathrm{L}^{\mathrm{op}}_{\mathrm{ip}}(X, Y)$ then f is closed.*

Proof. Let $y \in Y$ and consider the level set $\mathrm{L}y = \{x: f(x) \leq y\}$. Let $\{x_n\}_{n \in N} \subseteq \mathrm{L}y$ such that $x_n \to x$. By the o-Lipschitz property of f for n large enough $|f(x) - f(x_n)| \leq y'\|x - x_n\|$ and $\|x_n - x\| \to 0$. So $f(x_n) \xrightarrow{o} f(x)$. Since $f(x_n) \leq y \to f(x) \leq y \to x \in \mathrm{L}y \to \mathrm{L}y$ is closed and so f is closed. $\square$

PROPOSITION 6.13. *If K_Y^+ is closed and f is continuous then f is closed.*

Proof. Again let $\mathrm{L}y = \{x: f(x) \leq y\}$ and $\{x_n\}_{n \in N} \subseteq \mathrm{L}y$ such that $x_n \to x$. By the continuity of f, $f(x_n) \xrightarrow{\tau_Y} f(x)$. Since $f(x_n) \leq y$ $\forall n \in N$ and K_Y^+ is closed, then by Lemma 2.1, we have that $f(x) \leq y \to x \in \mathrm{L}y \to \mathrm{L}y$ is closed $\to f$ is a closed mapping. $\square$

Recall the Fenchel transform of f given by

$$f^*(A) = \sup_{x \in \operatorname{dom} f} \{A(x) - f(x)\}$$

where $A \in \mathcal{L}(X, Y)$ (see §5).

PROPOSITION 6.14. *If K_Y^+ is closed then $f^*(\cdot): \mathcal{L}(X, Y) \to \overline{Y}$ is closed in $\mathcal{L}_s(X, Y)$.*

Proof. Consider $\mathrm{L}y = \{A \in \mathcal{L}(X, Y): f^*(A) \leq y\}$. We will show that this is closed in $\mathcal{L}_s(X, Y)$.

Let $\{A_n\}_{n \in N} \subseteq \mathrm{L}y$ and

$$A_n \overset{\mathcal{L}_s(X,Y)}{\to} A.$$

Then for every $x \in X$

$$A_n(x) \to A(x) \to A_n(x) - f(x) \to A(x) - f(x) \quad \forall x \in X.$$

Since $A_n(x) - f(x) \leq y$ $\forall n \in N$ and K_Y^+ is closed, then $A(x) - f(x) \leq y$ for all $x \in X \to f^*(A) \leq y \to A \in \mathrm{L}y \to \mathrm{L}y$ is closed in $\mathcal{L}_s(X, Y)$ and then so is the mapping f^*. □

PROPOSITION 6.15. *If for every $y^* \in (K_Y^+)^*$, $(f(x), y^*)$ is an l.s.c. real valued function then f is closed.*

Proof. For $y \in Y$, consider the level set $\mathrm{L}y = \{x: f(x) \leq y\}$. Let $\{x_n\}_{n \in N} \subseteq \mathrm{L}y$, $x_n \to x$. Then since $(f(\cdot), y^*)$ is l.s.c. $\forall y^* \in (K_Y^+)^*$, we have that $\underline{\lim}_{n \to \infty}(f(x_n), y^*) \geq (f(x), y^*)$. But $(f(x_n), y^*) \leq (y, y^*)$ $\to \underline{\lim}(f(x_n), y^*) \leq (y, y^*) \to (f(x), y^*) \leq (y, y^*) \to \forall y^* \in (K_Y^+)^*$, $(f(x) - y, y^*) \leq 0 \to f(x) \leq y \to x \in \mathrm{L}y \to f$ is closed. □

Finally we conclude this section with an important generalization of a well known result for real valued l.s.c. functions.

THEOREM 6.1. *If $(\overset{\circ}{\operatorname{dom}} f) \neq \varnothing$ then a closed map is the supremum of the affine maps that it majorizes.*

Proof. Let $x \in (\overset{\circ}{\operatorname{dom}} f)$. Consider $y \leq f(x)$, $y \neq f(x)$.

Then since f is a closed mapping, we have that

$$Uy = \{x': f(x') \nleq y\}$$

is open.

We will find an affine mapping h such that

$$y \leq h(x) \leq f(x)$$

and

$$h(z) \leq f(z) \quad \forall z \in X.$$

For that purpose, let U be a neighborhood of x such that $U \subseteq (\overset{\circ}{\operatorname{dom}}\ f)$ and $Uy \supseteq U$. Then for all $z \in U f(z) \geq y$.

Define

$$g(z) = \begin{cases} -\infty & \text{if } z \notin U, \\ y & \text{if } z \in U. \end{cases}$$

Clearly $g\colon X \to \overline{Y}$ is concave.

Now $(\operatorname{dom} f)^{\alpha i} \cap (\operatorname{dom} g) = (\operatorname{dom} f)^{\alpha i} \cap U \neq \varnothing$. So we can apply Zowe's "Sandwich Theorem" (see **[42]**) to get the desired affine map. Then obviously this concludes the proof of the Theorem. □

This concludes the convex part of our work. In the second paper, we deal with nonconvex extensions of this material and in particular with Clarke's Theory, which was formulated during the last six years. This theory extended significantly most of the results of Convex Analysis and widened the spectrum of optimization problems that we can solve.

In the second paper (see **[22]**) we examine to what extent we can have such a powerful theory for vector valued mappings and we generalize several of the results obtained in this part of the work.

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