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A REMARK ON THE KASPAROV GROUPS $\mathrm{Ext}^i(A, B)$

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Let A and B be C^*-algebras. We show that, under reasonable assumptions (A unital, nuclear and separable, B with a strictly positive element), the groups $\mathrm{Ext}^i(A, B)$ of Kasparov are isomorphic—up to a shift of dimension—to the K-theory groups of some commutant of A in the outer multiplier algebra of $B \otimes \mathcal{K}$. The main tool to prove this is Kasparov's "generalized theorem of Voiculescu". Following an idea of Paschke, we use our result to get a part of the "generalized Pimsner-Voiculescu exact sequence" for crossed products.

0. Introduction. The purpose of this note is to provide a K-theoretic interpretation for the Kasparov groups $\mathrm{Ext}^0(A, B)$ and $\mathrm{Ext}^1(A, B)$ (where A, B are C^*-algebras), by realizing them as the K-groups of some commutant of A in the outer multiplier algebra $Q(B \otimes \mathcal{K})$ of $B \otimes \mathcal{K}$ (the relevant definitions are given in §1).

The role of commutants can be explained very simply. Let $\tau\colon A \to Q(B \otimes \mathcal{K})$ be an extension of A, and p be a projection in the commutant $\tau(A)'$ of $\tau(A)$; the mapping

$$\alpha_\tau(p)\colon A \to Q(B \otimes \mathcal{K})$$

defined by

$$\alpha_\tau(p)(a) = p \cdot \tau(a)$$

is still an extension of A. Similarly, if u is a unitary in $\tau(A)'$, the mapping

$$\alpha_\tau(u)\colon C_0(\mathbf{R}) \otimes A \to Q(B \otimes \mathcal{K})$$

defined by

$$\alpha_\tau(u)(f \otimes a) = f(u) \cdot \tau(a)$$

is an extension of $C_0(\mathbf{R}) \otimes A$. It is shown in Lemma 1 that these mappings α_τ extend to well-defined homomorphisms $K_i(\tau(A)') \to \mathrm{Ext}^{i+1}(A, B)$ ($i \in \mathbf{Z}/2$). Moreover, we show in Proposition 3 that under mild assumptions on A and B, it is possible to find an extension τ such that α_τ is actually an isomorphism.

1. Definitions and notations. Let A and B be C^*-algebras. In the study of extensions of the form

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

(where $\mathcal{K}$ denotes the algebra of compact operators on the usual Hilbert space), Kasparov introduced in [**6**; §7] the classifying semigroup $\operatorname{Ext}(A, B)$ in the following way: let $M(B \otimes \mathcal{K})$ be the multiplier algebra of $B \otimes \mathcal{K}$, and let $Q(B \otimes \mathcal{K})$ be the outer multiplier algebra $M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$.

An *extension of* A (with respect to B) is a *-homomorphism τ: $A \to Q(B \otimes \mathcal{K})$; τ is *trivial* if there is a *-homomorphism σ: $A \to M(B \otimes \mathcal{K})$ such that $\tau = \pi \circ \sigma$, where π: $M(B \otimes \mathcal{K}) \to Q(B \otimes \mathcal{K})$ is the canonical projection. If τ_1, τ_2 are extensions, their sum is the *-homomorphism

$$\tau_1 \oplus \tau_2 : A \to Q(B \otimes \mathcal{K}) \oplus Q(B \otimes \mathcal{K}) \hookrightarrow M_2(Q(B \otimes \mathcal{K})) \simeq Q(B \otimes \mathcal{K}).$$

The extensions τ_1, τ_2 are *unitarily equivalent* if there is a unitary U in $M(B \otimes \mathcal{K})$ such that for any a in A:

$$\tau_2(a) = \pi(U)^* \tau_1(a) \pi(U).$$

Now $\operatorname{Ext}(A, B)$ is the set of equivalence classes of extensions, where τ_1, τ_2 are *equivalent* if there exist trivial extensions δ_1, δ_2 such that $\tau_1 \oplus \delta_1$ is unitarily equivalent to $\tau_2 \oplus \delta_2$. It is clear that $\operatorname{Ext}(A, B)$is a semi-group.

Kasparov has shown ([**6**; §7], Theorem 1) that $\operatorname{Ext}(A, B)$ is actually an abelian group if A is separable and nuclear and if B has a strictly positive element. Moreover, if we let A vary in the category of separable nuclear C^*-algebras, and B in the category of C^*-algebras with a strictly positive element, then we obtain a bi-functor which, in both variables, is exact ([**6**, §7], Lemmas 3 and 7), homotopically invariant ([**6**, §6], Theorem 1), stable ([**9**], 4.9), and satisfies Bott periodicity ([**6**, §7], Theorem 4). It is easy to check that $\operatorname{Ext}(A, \mathbf{C})$ is the usual Brown-Douglas-Fillmore group $\operatorname{Ext} A$, defined in [**1**, 3.1].

We fix some notations. Denote by $M_n(A)$ the algebra of $n \times n$-matrices with entries in A, and by 1_n (resp. 0_n) the unit (resp. zero) matrix in $M_n(\mathbf{C})$. For any sub-algebra C of $Q(B \otimes \mathcal{K})$, C' will be the commutant of C in $Q(B \otimes \mathcal{K})$. Finally, we set $\operatorname{Ext}^1(A, B) = \operatorname{Ext}(A, B)$ and $\operatorname{Ext}^0(A, B) = \operatorname{Ext}(A \otimes C_0(\mathbf{R}), B)$.

2. Main result. The basic idea to show that the Ext-groups are isomorphic, up to a shift of dimension, to the K-groups of certain commutants in $Q(B \otimes \mathcal{K})$ is provided by the following lemma:

LEMMA 1. *Let* τ *be an extension of* A. *For* $i \in \mathbf{Z}/2$, *there is a homomorphism*

$$\alpha_\tau : K_i(\tau(A)') \to \operatorname{Ext}^{i+1}(A, B)$$

which is natural in the two following senses:

(i) *If f is a *-homomorphism $A' \to A$, and if $\tilde{f}$ is the dual inclusion $\tau(A)' \to (\tau \circ f)(A')'$, then the diagram*

$$\begin{array}{ccc} K_i(\tau(A)') & \underset{\tilde{f}_*}{\rightarrow} & K_i(f^*\tau(A')') \\ \alpha_\tau \downarrow & & \downarrow \alpha_{f^*\tau} \\ \mathrm{Ext}^{i+1}(A, B) & \underset{f^*}{\rightarrow} & \mathrm{Ext}^{i+1}(A', B) \end{array}$$

commutes.

(ii) *Let g be a *-homomorphism $B \to B'$, and g_*: $Q(B \otimes \mathcal{K}) \to Q(B' \otimes \mathcal{K})$ the induced *-homomorphism. The diagram*

$$\begin{array}{ccc} K_i(\tau(A)') & \overset{g_*}{\rightarrow} & K_i\big((g_* \circ \tau(A))'\big) \\ \alpha_\tau \downarrow & & \downarrow \alpha_{g_* \circ \tau} \\ \mathrm{Ext}^{i+1}(A, B) & \overset{g_*}{\rightarrow} & \mathrm{Ext}^{i+1}(A, B') \end{array}$$

commutes.

Proof. Assume first that $i = 0$. For any projection p in $M_n(\tau(A)') = (1_n \otimes \tau(A))'$, the map $a \to p \cdot (1_n \otimes \tau(a))$ is an extension of A, and so defines an element $\alpha_\tau[p]$ of $\mathrm{Ext}(A, B)$. To check that $\alpha_\tau[p]$ only depends on the class of p in $K_0(\tau(A)')$, one simply notices that replacing p by $p \oplus 0_k$ is just adding a trivial extension to the above extension, and if p and q are projections in the same connected component in $M_n(\tau(A)')$, then $\alpha_\tau[p] = \alpha_\tau[q]$ because of the homotopy invariance of Ext. Assume now that $i = 1$. We regard $C_0(\mathbf{R})$ as a maximal ideal in $C(S^1)$. For any unitary u in $M_n(\tau(A)')$, the mapping

$$C_0(\mathbf{R}) \otimes A \to M_n(Q(B \otimes \mathcal{K}))$$

which is defined by

$$f \otimes a \to f(u)(1_n \otimes \tau(a))$$

defines an element $\alpha_\tau[u]$ of $\mathrm{Ext}^0(A, B)$. One checks as in the case $i = 0$ that $\alpha_\tau[u]$ only depends on the class of u in $K_1(\tau(A)')$. The naturalities (i) and (ii) are clear from the definitions.

In general, α_τ is neither injective nor surjective. However, in some cases, α_τ is an isomorphism. This is a consquence of the

GENERALIZED VOICULESCU'S THEOREM (*Kasparov*, [7], *Theorem* 6). *Let A be a unital, separable, nuclear C^*-algebra, and let B be a C^*-algebra with a strictly positive element. Let $\mathcal{H}$ be the usual Hilbert space, and let σ be a unital, faithful representation of A on $\mathcal{H}$, such that $\sigma(A) \cap \mathcal{K} = \{0\}$. We regard $\mathcal{L}(\mathcal{H})$ in a natural way as a C^*-sub-algebra of $M(B \otimes \mathcal{K})$ (by identifying $T \in \mathcal{L}(\mathcal{H})$ with $1 \otimes T \in M(B \otimes \mathcal{K})$). Set $\tau = \pi \circ \sigma$. If φ is a unital $*$-homomorphism $A \to M(B \otimes \mathcal{K})$, then the extensions $\pi \circ \varphi \oplus \tau$ and τ are unitarily equivalent.*

This theorem extends Theorem 1.3 of [**11**]. As a particular case, we see that if σ' is another unital faithful representation of A on $\mathcal{H}$ such that $\sigma'(A) \cap \mathcal{K} = \{0\}$ and if $\tau' = \pi \circ \sigma'$, then τ and τ' are unitarily equivalent (if B has a unit, this is actually Theorem 1.3 of [**11**]).

LEMMA 2. *Let A, B and σ be as in the preceding theorem. If φ: $A \to M(B \otimes \mathcal{K})$ is a $*$-homomorphism, then there is an isometry V in $M_3(M(B \otimes \mathcal{K}))$ such that for any a in A the element*

$$V^*(\varphi(a) \oplus 1_2 \otimes \sigma(a))V - 1_3 \otimes \sigma(a)$$

belongs to $M_3(B \otimes \mathcal{K})$ and

$$VV^* = \varphi(1) \oplus 1_2.$$

Proof. According to Theorem 2 of [7] or Lemma 3.6 of [3], one finds an isometry $W \in M_2(M(B \otimes \mathcal{K}))$ such that $WW^* = \varphi(1) \oplus 1$. By the Generalized Voiculescu's Theorem, there is a unitary U in $M_3(M(B \otimes \mathcal{K}))$ such that for any a in A the element $U^*[W^*(\varphi(a) \oplus \sigma(a))W \oplus \sigma(a)]U - 1_3 \otimes \sigma(a)$ belongs to $M_3(B \otimes \mathcal{K})$. So it suffices to take $V = (W \oplus 1) \cdot U$ to get the result.

This lemma will be used extensively in the proof of our main result.

PROPOSITION 3. *Let A, B, σ and τ be as in Voiculescu's theorem. Then α_τ: $K_i(\tau(A)') \to \operatorname{Ext}^{i+1}(A, B)$ is an isomorphism.*

Proof. (1) $i = 0$. If p is a projection in $M_k(\tau(A)')$ such that $\alpha_\tau[p] = 0$, we find, via Lemma 2, an n in $\mathbf{N}$ and an isometry $v \in M_n(Q(B \otimes \mathcal{K}))$ such that for any a in A:

$$v^*(p(1_k \otimes \tau(a)) \oplus 1_{n-k} \otimes \tau(a))v = 1_n \otimes \tau(a)$$

and $vv^* = p \oplus 1_{n-k}$. So v belongs to $M_n(\tau(A)')$, and consequently $[p] = [1_k]$ in $K_0(\tau(A)')$. Since in particular $[0_k] = [1_k]$, we have $[p] = 0$. This shows α_τ is one-to-one. To show it is onto, let ρ be any extension of A, and let ρ' be an extension such that $\rho \oplus \rho'$ is trivial (one can find such a ρ' since $\mathrm{Ext}(A, B)$ is a group). Let φ be a lifting of $\rho \oplus \rho'$, and let P' be the projection on the first factor. Now P' commutes with φ modulo $M_2(B \otimes \mathcal{K})$, a fortiori $P' \oplus 0_2$ commutes with $\varphi \oplus (1_2 \otimes \sigma)$ modulo $M_4(B \otimes \mathcal{K})$, so that if V is the isometry from Lemma 2, then $p = \pi(V^*(P' \oplus 0_2)V)$ is a projection in $M_4(\tau(A)')$, and $p(1_4 \otimes \tau)$ is obviously equivalent to ρ. This concludes the proof in the case $i = 0$.

(2) $i = 1$. By the remarks preceding Lemma 2, any two representations of A on $\mathcal{H}$ satisfying our assumptions on σ are unitarily equivalent modulo $B \otimes \mathcal{K}$. So we may begin by taking a unital, faithful representation $\tilde{\sigma}$ of $C(S^1) \otimes A$ on $\mathcal{H}$, such that $\tilde{\sigma}(C(S^1) \otimes A) \cap \mathcal{K} = \{0\}$, and set $\sigma = \tilde{\sigma}|_{j(A)}$, where $j: A \to C(S^1) \otimes A: a \to 1 \otimes a$ is the canonical inclusion. Now, it suffices to show that the sequence

$$0 \to K_1(\tau(A)') \underset{\alpha_\tau}{\to} \mathrm{Ext}(C(S^1) \otimes A, B) \underset{j^*}{\to} \mathrm{Ext}(A, B) \to 0$$

is exact. Indeed, by exactness of the functor Ext, we have

$$\mathrm{Ker}\, j^* = \mathrm{Ext}(C_0(\mathbf{R}) \otimes A, B).$$

Now, obviously j^* is onto, and $j^* \circ \alpha_\tau = 0$. If ρ is an extension of $C(S^1) \otimes A$ such that $j^*\rho = \rho|_{j(A)}$ is trivial, and if v is an isometry in $M_3(Q(B \otimes \mathcal{K}))$ transforming $j^*\rho \oplus (1_2 \otimes \tau)$ in $1_3 \otimes \tau$ as in Lemma 2, we set, for any f in $C(S^1) \otimes A$:

$$\psi(f) = v^*(\rho(f) \oplus 1_2 \otimes \tau(e(f)))v$$

where $e: C(S^1) \otimes A \to A$ is "evaluation at 1". So ψ is an extension of $C(S^1) \otimes A$ which is equivalent to ρ. Let u be the unitary in $C(S^1)$ defined by the identity of S^1, and set $w = \psi(u \otimes 1)$. Since $\psi(j(a)) = 1_3 \otimes \tau(a)$, we have $\alpha_\tau[w] = \psi$, i.e. $\mathrm{Ker}\, j^* \subset \mathrm{Im}\, \alpha_\tau$. The proof that α_τ is one-to-one is essentially a re-writing of this part of the proof for $i = 0$, and we feel allowed to skip it.

3. Concluding remarks. (i) Proposition 3 provides a way to express $\mathrm{Ext}(A, B)$ as a K-group, but unfortunately does not relate it to the K-groups of A and B. Such a link is given by the universal coefficient theorem of Rosenberg and Schochet ([**10**], 4.2). Note that this theorem, proved for A a type I C^*-algebra, remains to be proved for A nuclear.

(ii) For $A = \mathbf{C}$, Proposition 3 gives

$$\mathrm{Ext}^i(\mathbf{C}, B) \simeq K_{i+1}(Q(B \otimes \mathcal{K})).$$

But it was proved by Cuntz ([**3**], 3.2) and independently by Rosenberg ([**9**], 4.6) that

$$K_i(M(B \otimes \mathcal{K})) = 0 \qquad (i = 0, 1).$$

So, by the K-theory six terms exact sequence:

$$K_{i+1}(Q(B \otimes \mathcal{K})) \simeq K_i(B).$$

This shows that Ext-theory also covers K-theory.

(iii) Assume A is non-unital, nuclear, separable. If i denotes the inclusion of A in $\tilde{A}$, the algebra obtained by adjoining a unit, then we get a short exact sequence

$$0 \to \mathrm{Ext}^i(\mathbf{C}, B) \to \mathrm{Ext}^i(\tilde{A}, B) \underset{i^*}{\to} \mathrm{Ext}^i(A, B) \to 0$$

and a commutative diagram

$$\begin{array}{ccc} K_i(\tau(A)') & \underset{\alpha_\tau}{\to} & \mathrm{Ext}^{i+1}(\tilde{A}, B) \\ \| & & \downarrow i^* \\ K_i(\tau(A)') & \underset{\alpha_\tau}{\to} & \mathrm{Ext}^{i+1}(A, B) \end{array}$$

So we have

COROLLARY 4. *If A is non-unital, nuclear, separable, if B and τ are as in Proposition 3, then there is a short exact sequence*

$$0 \to K_i(B) \to K_{i+1}(\tau(A)') \overset{\alpha_\tau}{\to} \mathrm{Ext}^i(A, B) \to 0.$$

(iv) Proposition 3 was proved by Paschke [**8**] in the case $B = \mathbf{C}$ (up to a minor point: Paschke has to assume the homotopy-invariance property for $\mathrm{Ext}^0 A$. We do not have to make this assumption, because Kasparov has proved the homotopy-invariance of Ext in full generality for A separable, B with a strictly positive element). Paschke uses his result to construct a part of the Pimsner-Voiculescu exact sequence for the Ext-groups of the crossed product of a C^*-algebra by an automorphism ([**8**], Theorem 7). Paschke's reasoning can be generalized with little effort:

PROPOSITION 5. *Let A, B be as in Voiculescu's theorem. Let θ be an automorphism of A, let $A \times \theta$ be the crossed product of A by θ, and let i: $A \to A \times \theta$ be the canonical inclusion. There is a short exact sequence*:

$$\begin{array}{ccccc}
\mathrm{Ext}^0(A \times \theta, B) & \overset{i^*}{\to} & \mathrm{Ext}^0(A, B) & \overset{\theta^{*-1}-1}{\to} & \mathrm{Ext}^0(A, B) \\
 & & & & \downarrow \delta \\
\mathrm{Ext}^1(A, B) & \underset{\theta^{*-1}-1}{\leftarrow} & \mathrm{Ext}^1(A, B) & \underset{i^*}{\leftarrow} & \mathrm{Ext}^1(A \times \theta, B)
\end{array}$$

Proof. It is easy to see that this sequence is exact at the second and fifth terms. We concentrate on the central terms. To this end, let τ be an extension of $A \times \theta$ satisfying the conditions of Voiculescu's theorem, and let u_θ be the unitary element in $A \times \theta$ such that for any a in A:

$$\theta(a) = u_\theta a u_\theta^*.$$

We identify A with $i(A)$. Let w be a unitary element in $M_n(\tau(A)')$. We define a mapping

$$\gamma: K_1(\tau(A)') \to K_1(\tau(A)')$$

by

$$\gamma(w) = w \oplus (1_n \otimes \tau)(u_\theta^*)w^*(1_n \otimes \tau)(u_\theta)$$

and a mapping

$$\delta: K_1(\tau(A)') \to \mathrm{Ext}^1(A \times \theta, B)$$

by

$$\delta(w)(a) = (1_n \otimes \tau)(a) \quad \text{and} \quad \delta(w)(u_\theta) = (1_n \otimes \tau)(u_\theta) \cdot w.$$

It is clear that $\delta(w)$ provides an extension of $A \times \theta$, and that γ and δ extend to well-defined group homomorphisms. It is proved like in [**8**], Theorem 7, that the sequence

$$K_1(\tau(A)') \underset{\gamma}{\to} K_1(\tau(A)') \underset{\delta}{\to} \mathrm{Ext}^1(A \times \theta, B) \underset{i^*}{\to} \mathrm{Ext}^1(A, B)$$

is exact. Moreover, the diagram

$$\begin{array}{ccc}
K_1(\tau(A)') & \underset{\gamma}{\to} & K_1(\tau(A)') \\
\downarrow \alpha_\tau & & \downarrow \alpha_{\theta^{*-1}\tau} \\
\mathrm{Ext}^0(A, B) & \underset{\theta^{*-1}-1}{\to} & \mathrm{Ext}^0(A, B)
\end{array}$$

commutes, since

$$\begin{aligned}
&\alpha_{\theta^{*-1}\tau}(\gamma(w))(f\otimes a)\\
&\quad= \big(f(w)\oplus(1_n\otimes\tau)(u_\theta^*)f(w^*)(1_n\otimes\tau)(u_\theta)\big)\cdot(1_{2n}\otimes\tau)(\theta^{-1}a)\\
&\quad= \theta^{*-1}(\alpha_\tau(w))(f\otimes a)\oplus(1_n\otimes\tau)(u_\theta^*)\cdot\alpha_\tau(w^*)(f\otimes a)\cdot(1_n\otimes\tau)(u_\theta)
\end{aligned}$$

so that

$$\alpha_{\theta^{*-1}\tau}[\gamma(w)] = \theta^{*-1}[\alpha_\tau(w)] + [\alpha_\tau(w^*)] = (\theta^{*-1}-1)[\alpha_\tau(w)].$$

The exactness at the central terms of the sequence follows now from the fact that the vertical arrows in the preceding diagram are isomorphisms (by Proposition 3).

However, we have not been able to construct by our method the connecting map $\mathrm{Ext}^1(A,B)\to\mathrm{Ext}^0(A\times\theta,B)$ to get the generalized Pimsner-Voiculescu exact sequence of [**5**], Theorem 3(b).

(v) Kasparov also defined groups $KK^i(A,B)$, actually isomorphic to $\mathrm{Ext}^i(A,B)$ ([**6**, §7], Theorem 1). The interest of these groups is the existence of multiplicative structures (see [**6**, §4], Theorem 4). These were used in a crucial manner by Connes and Skandalis in their proof of the index theorem [**2**]. Perhaps we should note that, in the same paper, Connes and Skandalis prove isomorphisms,

$$\begin{aligned}
&\mathrm{Ext}^i(C(V),C(X)) \simeq K^{i+j}(V\times X),\\
&\mathrm{Ext}^i(C(X),C(V)) \simeq \mathrm{Ext}^{i+j}(V\times X) \qquad (j\equiv\dim V\ (\mathrm{mod}\ 2))
\end{aligned}$$

where X is a compact metrisable space and V is a compact C^∞-manifold endowed with a Spin^c-structure.

(vi) As pointed out by the referee, it is possible to develop an interesting analogy with a topological situation. Let X be a finite simplicial complex. By embedding X in a sphere S^n and taking a strong deformation retract D_nX fo $S^n\setminus X$, one obtains a Spanier-Whitehead dual of X. Then one can define the K-homology of X by

$$K_*(X) = K^*(D_nX)$$

(actually, $K_*(X)$ is isomorphic to Ext^*X. For the link between this definition of K-homology and Ext, see [**4**], Chapter 5). Now, let A be a unital, nuclear and separable C^*-algebra, and let $\tau\colon A\to Q(\mathcal{K})$ (the usual Calkin algebra) be an extension satisfying the conditions of Voiculescu's theorem. By Proposition 3:

$$\mathrm{Ext}^*A = K_*(\tau(A)').$$

Moreover, by Voiculescu's double commutant theorem ([**11**], 1.9)

$$\tau(A)'' = \tau(A)$$

so that $\tau(A)'$ seems to play the role of a (non-commutative) Spanier-Whitehead dual for A. Proposition 3 seems to indicate that such considerations still hold when $Q(\mathcal{K})$ is replaced by the generalized Calkin algebra $Q(B \otimes \mathcal{K})$

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