PACIFIC JOURNAL OF MATHEMATICS
Vol. 110, No. 1, 1984

ON GAMELIN CONSTANTS

MASARU HARA

The purpose of this paper is to show that the corona theorem with bounds is valid for any finite bordered Riemann surface. As an application of it we construct an example of Riemann surfaces of infinite genus for which the corona theorem holds. The example can be chosen either from or not from the class of surfaces of Parreau-Widom type.

1. Introduction. Let R be a Riemann surface and $H^\infty(R)$ be the algebra of bounded analytic functions on R. Given a Riemann surface R, a natural number n and a positive number δ, we denote by $C_R(n,\delta)$ the infimum among constants C having the following property: For any $f_1,\ldots,f_n \in H^\infty(R)$ with $1 \geq \max_j |f_j| \geq \delta$ on R, there exist $g_1,\ldots,g_n \in H^\infty(R)$ such that $\sum_j f_j g_j = 1$ on R and $|g_j| \leq C$ on R $(j = 1,\ldots,n)$. If there exist no such constants, then we define $C_R(n,\delta) = \infty$. We call $C_K(n,\delta)$ the *Gamelin constant* for the triple (R, n, δ). If $C_R(n,\delta) < \infty$ for every n and $\delta > 0$, then we say that *the Gamelin constant of R is finite*.

Gamelin [3] proved that the Gamelin constant of any finitely connected planar domain R is finite in such a way that $C_R(n,\delta)$ is dominated by a constant $C_m(n,\delta)$ depending only on n, δ and the number m of boundary components of R. The primary purpose of this paper is to prove the following.

THEOREM 1. *The Gamelin constant of any Riemann surface which is the interior of any finite bordered Riemann surface is finite.*

We raise the question of whether the constants can be chosen to depend only on the genus or rather on the Euler characteristic of the surface.

We denote the maximal ideal space of $H^\infty(R)$ by $\mathfrak{M}(R)$. We set $\tau(R) = \{$the homomorphisms "evaluation at p": $p \in R\}$. If $H^\infty(R)$ separates the points of R, we identify $\tau(R)$ with R. When $\tau(R)$ is dense in $\mathfrak{M}(R)$, we say that the *corona theorem* holds for R. The set $\tau(R)$ is dense in $\mathfrak{M}(R)$ if and only if the following property holds: For each n and $\delta > 0$, given $f_1,\ldots,f_n \in H^\infty(R)$ such that $\max|f_j| \geq \delta$ on R, there exist $g_1,\ldots,g_n \in H^\infty(R)$ such that $\sum_j f_j g_j = 1$ on R. Therefore if the Gamelin constant of R is finite, then the corona theorem holds for R. It is well known (e.g. Gamelin [3]) that the corona theorem holds for any finite bordered Riemann surface.

Behrens [1] and Gamelin [3] proved that the corona theorem holds for some infinitely connected planar domains. Cole (cf. Gamelin [4]) gave an example of a Riemann surface for which the corona theorem is not valid. Nakai [6] gave an example of a Riemann surface of Parreau-Widom type for which the corona theorem is invalid. As the second purpose of this paper, we will give an example of a Riemann surface of infinite genus for which the corona theorem holds (Theorem 2). It is obtained from the Behrens example [1]. We will also show that example in Theorem 2 can be chosen from or not from the class of surfaces of Parreau-Widom type (Theorem 3).

2. The proof of Theorem 1. Let R be any finite bordered Riemann surface with genus g and m boundary components. If $g = 0$, then Theorem 1 is reduced to the Gamelin theorem. We assume that $g > 0$. Let $\gamma_1,\ldots,\gamma_g$ be simple closed curves on R such that $\gamma_1,\ldots,\gamma_g$ are mutually disjoint and $R - \bigcup_i \gamma_i$ is a plane domain. Let U_i be an annulus containing γ_i $(i = 1,\ldots,g)$ such that $\overline{U}_1,\ldots,\overline{U}_g$ are mutually disjoint. Let ρ be a smooth function on R such that $0 \leq \rho \leq 1$ on R, $\rho = 1$ on a neighbourhood of $\bigcup_i \gamma_i$ and the support of ρ is contained in $\bigcup_i U_i$.

Let $f_1,\ldots,f_n \in H^\infty(R)$ satisfy $1 \geq \max_j |f_j| \geq \delta$ on R. Since $R - \bigcup_i \gamma_i$ is a plane domain of connectivity $2g + m$, by the Gamelin theorem, there exist $p_1,\ldots,p_n \in H^\infty(R - \bigcup \gamma_i)$ such that $\sum f_j p_j = 1$ on $R - \bigcup \gamma_i$ and $\max_j |p_j| \leq C_{2g+m}(n, \delta)$. Also since each U_i is a plane domain of connectivity 2, there exist $q_1,\ldots,q_n \in H^\infty(\bigcup U_i)$ such that $\sum f_j q_j = 1$ on $\bigcup U_i$ and $\max_j |q_j| \leq C_2(n, \delta)$. Set

$$h_j = (1-\rho)p_j + \rho q_j \qquad (j = 1,\ldots,n).$$

Then h_j is smooth on R and $\sum f_j g_j = 1$ on R and $(\partial/\partial\bar{z})h_k = (q_k - p_k)(\partial/\partial\bar{z})\rho$. Set

$$w_{jk} = \frac{1}{\pi}\iint_R C(\zeta,\cdot)h_j\frac{\partial}{\partial\bar{z}}h_k\,d\xi\,d\eta,$$

where $C(\zeta,\cdot)$ is a Cauchy kernel on R which is regular on ∂R. Then $(\partial/\partial\bar{z})w_{jk} = h_j(\partial/\partial\bar{z})h_k$. If we set

$$g_j = h_j + \sum_{k=1}^{n}(w_{jk} - w_{kj})f_k \qquad (j = 1,\ldots,n),$$

then g_j is analytic on R and $\Sigma f_j g_j = 1$. Since

$$h_j \frac{\partial}{\partial \bar{z}} h_k - h_k \frac{\partial}{\partial \bar{z}} h_j = (p_j q_k - p_k q_j) \frac{\partial}{\partial \bar{z}} \rho,$$

if we set $C_1 = C_{2g+m}(n, \delta)$ and $C_2 = C_2(n, \delta)$, then

$$|g_j| \leq C_1 + C_2 + 2nC_1C_2 \frac{1}{\pi} \iint \left| C(\zeta, \cdot) \frac{\partial}{\partial \bar{z}} \rho \right| d\xi \, d\eta.$$

Therefore the Gamelin constant of R is finite.

3. An example. Given a domain V in the complex plane C, a sequence $\{\Delta_n\}_{n \geq 1}$ of open disks Δ_n is called a *Behrens sequence* in V if the following properties hold:

(1) for each Δ_n, there exists an open disk D_n such that $\bar{\Delta}_n \subset D_n \subset \bar{D}_n \subset V$ and Δ_n and D_n have the common center α_n;

(2) $d(\alpha_n, \partial V) \to 0$ as $n \to \infty$, where d is the Riemann spehre metric;

(3) the disks $\bar{D}_n$ in $\{D_n\}$ are mutually disjoint;

(4) $\Sigma(\text{rad } \Delta_n)/(\text{rad } D_n) < \infty$, where rad Δ is the radius of Δ;

(5) $(\text{rad } D_n)/d(\alpha, \partial V) \to 0$ as $n \to \infty$.

Behrens [1] proved that if the corona theorem holds for V and $\{\Delta_n\}$is a Behrens sequence of disks in V, then the corona theorem holds for the domain $U = V - \bigcup_n \bar{\Delta}_n$, which is called the region V with a Behrens sequence $\{\Delta_n\}$ in V *removed.*

let $\{\Delta_n\}$ be a Behrens sequence of disks in V. We denote by $\tilde{\Delta}_n$ a copy of Δ_n for each n. We introduce into each Δ_n a finite number (≥ 2) of mutually disjoint slits. Each slit is considered to have two banks: an N-bank and an S-bank. By joining every S- (resp. N-) bank of slits on Δ_n to an N- (resp. S-) bank of the corresponding slits on $\tilde{\Delta}_n$, we can construct a two sheeted covering Riemann surface of Δ_n which will be denoted by $\Delta_n + \tilde{\Delta}_n$. We assume that any two members in $\{\Delta_n + \tilde{\Delta}_n\}$ are mutually conformally equivalent. By welding the two sheeted disk $\Delta_n + \tilde{\Delta}_n$ to the Behrens domain $U = V - \bigcup_n \Delta_n$ along the boundary $\partial\Delta_n$ of $\Delta_n + \tilde{\Delta}_n$ and the boundary of U where $\bar{\Delta}_n$ is removed, we obtain a Riemann surface $R = V + \bigcup_n \tilde{\Delta}_n = U + \bigcup_n(\Delta_n + \tilde{\Delta}_n)$, which is called the Riemann surface V with a Behrens sequence $\{\Delta_n\}$ in V *attached.*

We are ready to state the following.

THEOREM 2. *If the corona theorem holds for a domain V in the complex plane C, then the corona theorem holds for the Riemann surface V with a Behrens sequence $\{\Delta_n\}$ in V attached.*

Let $\hat{C} = C \cup \{\infty\}$. We consider projections $P_n f$ of each function f in $H^\infty(R)$ to $H^\infty(\hat{C} + \tilde{\Delta}_n)$ by the following: First let

$$P_n f(z) = \frac{-1}{2\pi i} \int_{\partial \Delta_n} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for z in $\hat{C} - \bar{\Delta}_n$. Observe

$$P_n f(z) = f(z) - \frac{1}{2\pi i} \int_{\partial D_n} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for z in $D_n - \bar{\Delta}_n$. Since the right hand side of the above may be considered as a holomorphic function on $D_n + \tilde{\Delta}_n$, we can consider that $P_n f \in H^\infty(\hat{C} + \tilde{\Delta}_n)$. By Lemma 2.1 of Behrens [**1**], $\sum P_n f$ converges normally to a bounded analytic function, and therefore $f - \sum_{n=1}^{\infty} P_n f \in H^\infty(V)$. We have thus established the following decomposition:

$$H^\infty(R) = \sum_n H^\infty(\hat{C} + \tilde{\Delta}_n) + H^\infty(V).$$

Moreover, by Theorem 1, the Gamelin constant of $\Delta_n + \tilde{\Delta}_n$ is finite. Since any two members in $\{\Delta_n + \tilde{\Delta}_n\}$ are mutually conformally equivalent, the Gamelin constants of $\Delta_n + \tilde{\Delta}_n$ are all the same. The Behrens result [**1**] corresponding to Theorem 2 was proved based upon a decomposition corresponding to the above decomposition of $H^\infty(R)$ and the fact that Gamelin constants of removing disks are all the same finite constant. Since we have all the corresponding necessary machinery, we can repeat *almost* the same argument used by Behrens [**1**] to complete the proof of Theorem 2. We omit the details.

Next we will prove the following

THEOREM 3. *The Riemann surface $D + \bigcup_n \tilde{\Delta}_n$ can be made either of Parreau-Widom type or not by the choice of the Behrens sequence $\{\Delta_n\}$ in the unit disk D.*

Set $\log(1/r_m) = 2^{-m}$ $(m = 1, 2, \ldots)$. We give a Behrens sequence $\{\Delta_n\}$ in D as follows. The first p_1 number of D_n's have centers α_n on $\{|z| = r_1\}$, the next p_2 D_n's have centers α_n on $\{|z| = r_3\}$, etc., and all D_n's are disjoint from the circles $\{|z| = r_{2m}\}$, $1 \le m < \infty$. Let g be the Green's function of $D + \tilde{\Delta}_n$ with its pole at $z = 0$. Let π be the projection of $\Delta_n + \tilde{\Delta}_n$ onto $\{|z - \alpha_n| < \text{rad}\,\Delta_n\}$. We denote by $u(z)$ the harmonic function on $\Delta_n + \tilde{\Delta}_n$ which is equal to $\log(1/|z|)$ on $\partial\Delta_n$ and 0 on $\partial\tilde{\Delta}_n$. Then on the disk $\{|z - \alpha_n| < \text{rad}\,\Delta_n\}$, $u(z_1) + u(z_2) = \log(1/|z|)$ where

$\pi^{-1}(z) = \{z_1, z_2\}$. The function which is equal to $\log(1/|z|)$ on $D - \bar{\Delta}_n$ and $u(z)$ on $\Delta_n + \tilde{\Delta}_n$ is superharmonic on $D + \tilde{\Delta}_n$. Hence $\log(1/|z|) \geq g$ on $D - \bar{\Delta}_n$. If rad Δ_n is sufficiently small, then we have $\log(1/|z|) \geq g \geq \frac{1}{4}\log(1/|z|)$ on $D - \bar{D}_n$. We denote by G the Green's function of $D + \bigcup_n \tilde{\Delta}_n$ with its pole at $z = 0$. By the above argument, if each term of $\{\text{rad } \Delta_n\}$ is sufficiently small, then we have $\log(1/|z|) > G > \frac{1}{4}\log(1/|z|)$ on $D - \bigcup_{n\geq 1} \bar{D}_n$. The open set $(D + \bigcup_n \tilde{\Delta}_n) - \{|z| = r_{2m}\}$ consists of two components, one of which containing the center of D will be denoted by R_m. Then $\{|z| < r_{2m-2}\} - \bigcup_{n\geq 1} \bar{D}_n \subset \{G > 4^{-m}\} \subset R_m$. By the maximum principle, the complement of $\{G > 4^{-m}\}$ does not contain any compact component. Therefore if a cycle in $\{G > 4^{-m}\}$ is homologous to zero in R_m, then it is homologous to zero in $\{G > 4^{-m}\}$. Hence we have

$$p_1 + \cdots + p_{m-1} \leq B(0, 4^{-m}) \leq b(p_1 + \cdots + p_m),$$

where b (resp. $B(0, \alpha)$) is the first Betti number of $\Delta_n + \tilde{\Delta}_n$ (resp. $\{G > \alpha\}$). Since any two members in $\{\Delta_n + \tilde{\Delta}_n\}$ are mutually conformally equivalent, b does not depend on n. Hence

$$\int_0^\infty B(0, \alpha)\, d\alpha < \infty \quad \text{if and only if} \quad \sum_{m\geq 1} 4^{-m}(p_1 + \cdots + p_m) < \infty.$$

By the Widom theorem (cf. Widom [7], [8]), $D + \bigcup \tilde{\Delta}_n$ is of Parreau-Widom type if and only if $\Sigma_{n\geq 1} 4^{-m}(p_1 + \cdots + p_m) < \infty$.

References

[1] M. Behrens; *The maximal ideal space of algebras of bounded analytic functions on infinitely connected domains*, Trans. Amer. Math. Soc., **161** (1971), 359–380.

[2] L. Carleson, *Interpllation by bounded analytic functions and the corona problem*, Ann. of Math., **76** (1962), 547–559.

[3] T. Gamelin, *Localization of the corona problem*, Pacific J. Math., **34** (1970), 73–81.

[4] ______, *Uniform algebras and Jensen measures*, London Math. Soc. Lecture Note Series 32, Cambridge Univ. Press, 1978.

[5] ______, *Wolff's proof of the corona theorem*, Israel J. Math., **37** (1980), 113–119.

[6] M. Nakai, *Corona problem for Riemann surfaces of Parreau-Widom type*, Pacific J. Math., **103** (1982), 103–109.

[7] H. Widom, *The maximum principle for multiple-valued analytic functions*, Acta Math., **126** (1971), 63–82.

[8] ______, *H_p sections of vector bundles over Riemann surfaces*, Ann. of Math., **94** (1971), 305–324.

Received August 26, 1981 and in revised form November 19, 1982.

MEIJO UNIVERSITY
NAGOYA 468, JAPAN