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WEAK COMPACTNESS IN SPACES OF BOCHNER INTEGRABLE FUNCTIONS AND THE RADON-NIKODYM PROPERTY

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We characterize Banach spaces E such that E and E^* have the Radon-Nikodym property in terms of relatively weakly compact sets of $L^1[\lambda, E]$.

Introduction. It is well known [**1**] that if $(\Omega, \Sigma, \lambda)$ is a finite measure space and E is a Banach space, then a relatively weakly compact subset K of $L^1[\lambda, E]$ is *bounded*, *uniformly integrable* and for every $B \in \Sigma$, the set $\{\int_B f\,d\lambda, f \in K\}$ is *relatively weakly compact* in E. Moreover, it was shown in [**1**] that if the Banach space E and its dual E^* have the Radon-Nikodym property, then relatively weakly compact subsets of $L^1[\lambda, E]$ are completely characterized by the above three conditions. A question that arises naturally is the following: Are the conditions on E and E^* to have the Radon-Nikodym property necessary in order that relatively weakly compact subsets of $L^1[\lambda, E]$ be exactly those bounded, uniformly integrable subsets K such that for any $B \in \Sigma$, the set $\{\int_B f\,d\lambda,\ f \in K\}$ is relatively weakly compact in E? In [**1**], it was shown that the condition on E to have the Radon-Nikodym property is indeed necessary. The object of this paper is to show that the condition on E^* to have the Radon-Nikodym property is also necessary. This gives a new characterization of Banach spaces E such that E and E^* have the Radon-Nikodym property. We also study bounded linear operators T between Banach spaces such that T and its adjoint T^* are strong Radon-Nikodym operators.

Definitions and Preliminaries.

DEFINITION 1. A closed bounded convex subset C of a Banach space E is a *Radon-Nikodym* (R.N.P) *set* if for every finite measure space $(\Omega, \Sigma, \lambda)$ and any vector measure $G: \Sigma \to E$ such that the set $\{G(B)/\lambda(B), B \in \Sigma, \lambda(B) > 0\}$ is contained in C, there exists a Bochner integrable Radon-Nikodym derivative $f: \Omega \to C$ such that $G(B) = \int_B f\,d\lambda$, for every $B \in \Sigma$.

For more on (R.N.P) sets see [**3**] and [**4**].

DEFINITION 2. A bounded linear operator T from a Banach E into a Banach space F is called a *strong Radon-Nikodym operator* if the closure of $\{Tx, x \in E, \|x\| \leq 1\}$ is an (R.N.P) set in E.

Accordingly, a Banach space E has the Radon-Nikodym property (R.N.P) iff its closed unit ball is an (R.N.P) set or equivalently if the identity operator on E is a strong Radon-Nikodym operator.

If T: $E \to F$ is a strong Radon-Nikodym operator then T is an (R.N.P) operator see [**2**] i.e., for every vector measure G: $\Sigma \to E$ with $\|G(B)\| \leq \lambda(B)$ for all $B \in \Sigma$, there exists a Bochner integrable function f: $\Omega \to F$ such that $TG(B) = \int_B f\, d\lambda$ for all $B \in \Sigma$. The converse is not true as any quotient map Q from l^1 onto c_0 is an (R.N.P) operator but is not a strong Radon-Nikodym operator. But it follows from [**4**] if T: $E \to F$ is a bounded linear operator, then its adjoint T^* is a strong Radon-Nikodym operator if and only if T^* is an (R.N.P) operator.

Finally, given a finite measure space $(\Omega, \Sigma, \lambda)$ E and F two Banach spaces and T: $E \to F$ a bounded linear operator, we shall denote by $\tilde{T}$ the natural extension of T to a bounded linear operator from $L^1[\lambda, E]$ to $L^1[\lambda, F]$.

For all undefined statements and notations we refer the reader to [**1**].

The following theorem extends the result of [**1**, p. 101] to operators T: $E \to F$ such that T and T^* are strong Radon-Nikodym operators.

THEOREM 1. *Let E and F be two Banach spaces and let T: $E \to F$ be a bounded linear operator such that T and T^* are strong Radon-Nikodym operators. Then for any finite measure space $(\Omega, \Sigma, \lambda)$, the operator $\tilde{T}$: $L^1[\lambda, E] \to L^1[\lambda, F]$ sends into relatively weakly compact subsets of $L^1[\lambda, F]$ any bounded, uniformly integrable subsets K of $L^1[\lambda, E]$ such that for every $B \in \Sigma$ the set $\{\int_B f\, d\lambda, f \in K\}$ is relatively weakly compact in E.*

Proof. Let T: $E \to F$ be a bounded linear operator such that T and T^* are strong Radon-Nikdoym operators. Let $(\Omega, \Sigma, \lambda)$ be a finite measure space and let $K \subseteq L^1[\lambda, E]$ be a bounded and uniformly integrable subset of $L^1[\lambda, E]$ such that for any $B \in \Sigma$ the set $\{\int_B f\, d\lambda, f \in K\}$ is relatively weakly compact in E. Let $(f_n)_n$ be a sequence in K. Proceed now as in [**1**, p. 101] to get a countably generated σ-field Σ_1, such that each f_n is measurable with respect to Σ_1, find a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and define a countably additive vector measure $G: \Sigma_1 \to E$ of bounded variation by

$$G(B) = \underset{k}{\text{weak limit}} \int_B f_{n_k}\, d\lambda, \quad \text{for every } B \in \Sigma_1.$$

Since T^* is a Radon-Nikodym operator, it follows from **[4]** that there exist a Banach space Z, such that Z^* has RNP, and bounded linear operators T_1: $E \to Z$ and T_2: $Z \to F$ such that the following diagram commutes

$$\begin{array}{ccc} E & \overset{T}{\to} & F \\ T_1 \searrow & & \nearrow T_2 \\ & Z & \end{array}$$

Case 1. Assume that for some $\alpha > 0$ $\|G(B)\| \leq \alpha\lambda(B)$, for all $B \in \Sigma_1$. It follows that the set $\{T_1G(B)/\lambda(B),\ \lambda(B) > 0,\ B \in \Sigma_1\}$ is contained in the closure C in Z of the set $\{T_1x,\ x \in E,\ \|x\| \leq \alpha\}$. But a glance at the construction of **[4]** reveals that the set C is affinely homeomorphic to the closure in F of the set $\{Tx,\ x \in E,\ \|x\| \leq \alpha\}$, and one can show that the set C is an R.N.P. set. Therefore there exists a Bochner integrable function h: $\Omega \to C$ such that

$$T_1G(B) = \int_B h\, d\lambda, \quad \text{for all } B \in \Sigma_1.$$

Moreover since Z^* has R.N.P and since $(\int_B T_1 f_{n_k}\, d\lambda)_k$ converges weakly to $\int_B h\, d\lambda$ in Z for every $B \in \Sigma_1$, it follows that the sequence $(\tilde{T}_1 f_{n_k})_k$ converges weakly to h in $L^1[\Sigma_1, \lambda, Z]$, thus $(\tilde{T}f_{n_k})_k$ converges weakly to $\tilde{T}_2 h$ in $L^1[\Sigma_1, \lambda, F]$, and hence in $L^1[\lambda, F]$. An appeal to Eberlein's theorem shows that $\{\tilde{T}f,\ f \in K\}$ is relatively weakly compact in $L^1[\lambda, F]$ and completes the proof of Case 1.

General case. Let $(\Omega_m)_m$ be a partition of Ω of elements of Σ_1 and such that

$$\|G(B)\| \leq m\lambda(B)$$

for all elements B of Σ_1 contained in Ω_m. By restricting the sequence $(f_{n_k})_k$ to each of the sets Ω_m, by Case 1, and by an appropriate diagonal process, one can produce a subsequence $(h_j)_j$ of $(f_{n_k})_k$ and a sequence $(g_m)_m$ of Bochner integrable functions g_m: $\Omega_m \to F$ such that:

(i) the sequence $(\tilde{T}h_{j|\Omega_m})_m$ converges weakly to g_m in $L^1[\Omega_m, \lambda, F]$,

(ii) $TG(B \cap \Omega_m) = \int_{B\cap\Omega_m} g_m\, d\lambda$, for $B \in \Sigma_1$.

Let g: $\Omega \to F$ be defined as follows:

$$g(w) = g_m(w) \quad \text{if } w \in \Omega_m.$$

It is clear that $g \in L^1[\lambda, F]$ for g is obviously measurable and it follows from (ii) that

$$\int_\Omega \|g(w)\| d\lambda \leq \sum_m |TG|(\Omega_m) = |TG|(\Omega) < \infty.$$

The proof will be complete when we show that the sequence $(\tilde{T}h_j)_j$ converges weakly to g in $L^1[\lambda, F]$. For this let $L \in (L^1[\lambda, F])^*$. For each $m \geq 1$ let L_m be the restriction of L to $L^1[\Omega_m, \lambda, F]$. For every $m \geq 1$ we have

$$\left|L(\tilde{T}h_j - g)\right| \leq \left|\sum_{i=1}^m L_i(\tilde{T}h_{j|\Omega_i} - g_i)\right| + \|L\| \int_{U\Omega_i;\, i>m} \|\tilde{T}h_i - g\| d\lambda.$$

Since the sequence $(h_j)_j$ is uniformly integrable, there exists $m \geq 1$ such that $\int_{U\Omega_i; i\geq m} \|\tilde{T}h_j - g\| d\lambda$ is arbitrary small for all $j \geq 1$. Since $\tilde{T}h_{j|\Omega_i}$ converges weakly to g_i, it follows that $|\sum_{i=1}^m L(\tilde{T}h_{j|\Omega_i} - g_i)|$ is arbitrary small as $j \to \infty$. Hence $L(\tilde{T}h_j - g) \to 0$ as $j \to \infty$. This completes the proof.

The following proposition establishes the fact that if T^* fails to be a strong Radon-Nikodym operator, then the conclusion of Theorem 1 is no more valid.

PROPOSITION. *If T is a bounded linear operator from a Banach space E into a Banach space F such that T^* fails to be a strong Radon-Nikodym operator, then there exists a finite measure space $(\Omega, \Sigma, \lambda)$, a bounded uniformly integrable subset K of $L^1[\lambda, E]$ such that the set $\{\int_B f\, d\lambda, f \in K\}$ is relatively weakly compact in E for any $B \in \Sigma$, but the set $\{\tilde{T}f, f \in K\}$ is not relatively weakly compact in $L^1[\lambda, F]$.*

Proof. Suppose that T^* fails to be a strong Radon-Nikodym operator. Let $\Delta = \{-1, 1\}^{\mathbf{N}}$ denote the Cantor group with Haar measure m and let $\{\Delta_{n,i}, 1 \leq i \leq 2^n\}$ denote the standard nth partition of Δ with $\Delta_{01} = \Delta$, $\Delta_{n,i} = \Delta_{n+1,2i-1} \cup \Delta_{n+1,2i}$, $\Delta_{n,i}$ is clopen, and $m(\Delta_{n,i}) = 1/2^n$. It follows from the dichotomy theorem of Stegall [4] that the operator T must factor the Haar operator H: $l^1 \to L_\infty(m)$ which takes the basis of l^1 into the usual Haar basis of $C(\Delta)$ considered as a subspace of $L_\infty(m)$. Indeed the Haar operator is defined as follows:

$$\text{if } h_{ni} = \chi_{\Delta_{n+1,2i-1}} - \chi_{\Delta_{n+1,2i}}, \qquad n \geq 0,\ 1 \leq i \leq 2^n \quad \text{then} \quad He_{ni} = h_{ni},$$

here $\{e_{ni}, n \geq 0, 1 \leq i \leq 2^n\}$ is an enumeration of the usual l^1 basis. Let U: $l^1 \to E$ and V: $F \to L_\infty(m)$ be bounded linear operators such that $H = V \circ T \circ U$ as illustrated in the following diagram.

$$\begin{array}{ccc} E & \overset{T}{\rightarrow} & F \\ U\uparrow & & \downarrow V \\ l^1 & \overset{H}{\rightarrow} & L_\infty(m) \end{array}$$

Consider the following sequence $(f_n)_n$ in $L^1[m, l^1]$ with

$$f_n(t) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{2^j} h_{ji}(t) e_{ji}, \quad \text{for } t \in \Delta.$$

The sequence $(f_n)_n$ is easily seen [2] to have the following properties

(i) $\sup_n \| f_n(t) \| = 1$ m.a.e.

(ii) $\sup_{\|x^*\| \leq 1} \int | x^* \circ f_n | \, dm$ approaches zero as $n \to \infty$.

It follows that for every Borel B set in Δ the sequence $(\| \int_B f_n \, dm \|)_n$ approaches zero as $n \to \infty$. The sequence $(f_n)_n$ is bounded and uniformly integrable in $L^1[m, l^1]$ and $(\int_B fn \, dm)_n$ is a null sequence in l^1. We claim that the sequence $(\tilde{H}f_n)$ is not relatively weakly compact in $L^1[m, L_\infty(m)]$. For this note that for each $n \geq 1$

$$\tilde{H}f_n(t) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{2^j} h_{ji}(t) h_{ji}, \qquad t \in \Delta;$$

therefore $\tilde{H}f_n(t)$ takes its values in $C(\Delta)$, to prove the claim all we need to show is that $(\tilde{H}f_n)_n$ is not relatively weakly compact in $L^1[m, C(\Delta)]$. To this end note that since for every Borel set B the sequence $(\int_B \tilde{H}f_n \, dm)_n$ converges to zero in $C(\Delta)$, it follows that every weakly convergent subsequence of $(\tilde{H}f_n)_n$ in $L^1[m, C(\Delta)]$ must converge to zero. Let $L \in (L^1[m, C(\Delta)])^*$ be defined as follows: for $\psi \in L^1[m, C(\Delta)]$

$$L(\psi) = \int_\Delta \psi(t)(t) \, dm$$

then

$$L(\tilde{H}f_n) = \frac{1}{n} \int_\Delta \sum_{j=1}^{n} \sum_{i=1}^{2^j} h_{ji}(t) h_{ji}(t) \, dm = 1.$$

This shows that the sequence $(\tilde{H}f_n)_n$ has no weakly convergent subsequence in $L^1[m, C(\Delta)]$. The sequence $(\tilde{U}f_n)_n$ is bounded and uniformly integrable in $L^1[m, E]$ and the set $\{\int_B \tilde{U}f_n \, d\lambda, \, n \geq 1\}$ is relatively weakly compact in E for all Borel sets B of Δ, yet since T factors the Haar operator H, the sequence $(\tilde{T}\tilde{U}f_n)_n$ cannot have a weakly convergent subsequence in $L^1[m, F]$. This completes the proof.

COROLLARY 3. *A Banach space E and its dual E^* have* (R.N.P) *if and only if for every finite measure space* $(\Omega, \Sigma, \lambda)$, *any bounded and uniformly integrable subset K of $L^1[\lambda, E]$ is relatively weakly compact whenever for every $B \in \Sigma$, the set $\{\int_B f\, d\lambda, f \in K\}$ is relatively weakly compact in E.*

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