

COMPARISON THEOREMS FOR SECOND-ORDER OPERATOR-VALUED LINEAR DIFFERENTIAL EQUATIONS

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Let B be a Banach lattice with order continuous norm, $\mathcal{L}(B)$ the algebra of bounded linear operators. Let B_+ denote the positive cone induced by the lattice structure of B , and $\mathcal{L}_+(B)$ the corresponding positive cone in $\mathcal{L}(B)$. We consider second-order operator-valued differential equations of the form $Y'' + Q(x)Y = 0$, where $Q: [a, +\infty) \rightarrow \mathcal{L}(B)$ is continuous in the uniform topology and is such that $\int_a^\infty Q(t) dt \in \mathcal{L}_+(B)$ for all $x \geq a$. Comparison theorems of Hille-Wintner type are obtained.

1. Introduction. Consider the second-order linear differential equation

$$(1.1) \quad Y'' + Q(t)Y = 0$$

where $Q: [a, +\infty) \rightarrow \mathcal{L}(B)$ is a continuous operator-valued function and $\mathcal{L}(B)$ represents the Banach algebra of bounded linear operators $T: B \rightarrow B$, where B denotes a Banach space. By a solution Y of (1.1) we understand a function $Y: [a, \infty) \rightarrow \mathcal{L}(B)$ which is twice continuously differentiable in the uniform operator topology and satisfying (1.1) for all $t \in [a, \infty)$. We refer to the text of Hille [12] for a discussion of the concepts of differentiation and integration of functions from $[a, \infty)$ to a Banach algebra and to [15] for basic results concerning differential equations in Banach spaces. We shall be interested in comparing solutions of (1.1) with solutions of a second equation

$$(1.2) \quad Y'' + Q_1(t)Y = 0$$

with $Q_1: [a, \infty) \rightarrow \mathcal{L}(B)$ continuous. A solution $Y = Y(t)$ of (1.1) (or (1.2)) is said to be *non-singular* at a point $t_0 \in [a, \infty)$ if it has a bounded inverse $Y^{-1}(t_0) \in \mathcal{L}(B)$. If $Y(t)$ is non-singular for all $t \in [t_0, +\infty)$, some $t_0 \geq a$, then $Y = Y(t)$ is said to be a *non-oscillatory* solution of (1.1) on $[t_0, +\infty)$. Otherwise, $Y = Y(t)$ is said to be *oscillatory* on $[a, \infty)$. (Note that the inverse $Y^{-1}(t)$ of a non-oscillatory solution $Y(t)$ of (1.1) is continuously differentiable.)

Equations (1.1) and (1.2) have been the subject of numerous investigators ([4]–[11], [19], and the references therein) for the case that B is a

Hilbert space in which case $\mathcal{L}(B)$ is a B^* -algebra. Comparison and oscillation theorems have been obtained in many of the above references which are generalizations of the known theorems in the classical scalar case (e.g., the Sturm comparison theorem and the Hille-Wintner comparison theorem). In these results, the notion of comparison of operators in $\mathcal{L}(B)$ is used which is induced by the inner product; that is, if $T_1, T_2 \in \mathcal{L}(B)$ then $T_1 \geq T_2$ means $\langle (T_1 - T_2)x, x \rangle \geq 0$ for all x in the Hilbert space B . In this paper we shall be interested in employing an alternate notion of positivity in $\mathcal{L}(B)$ which is induced by a vector lattice structure on B . This has the advantage in that the class of positive operators is not only a positive cone but is also closed under multiplication, a fact which is not true in the B^* -algebra case.

A further advantage of the Banach lattice context is that it enables one to obtain certain results for nonselfadjoint equations. Ahmad and Lazer ([1], [2]) have extended the Sturm comparison theorem to nonselfadjoint equations, and Ahmad and Travis [3] and Keener and Travis [14] have obtained oscillation criteria for such systems. The emphasis of the papers [1], [2] and [14] is on conjugate point and focal point properties.

Henceforth, B will denote a Banach lattice. We recall that a Banach lattice B is a Banach space with a vector lattice structure such that $|x| \leq |y|$ implies $\|x\| \leq \|y\|$, $x, y \in B$, where $|x| \equiv x_v(-x)$ (see e.g. [17]). Let B_+ be defined by $B_+ \equiv \{x \in B: x \geq 0\}$. A subset $S \subset B$ is *order bounded* if there exists $z \in B_+$ with $|x| \leq z$ for all $x \in S$. The Banach lattice B is said to be *order complete* if for every non-empty majorized (with respect to the ordering) subset A of B , $\sup A$ exists in B . Also, B is said to have *order continuous norm* if each downward directed family A of B such that $\inf A = 0$ converges in norm to 0. A Banach lattice with order continuous norm is automatically order complete ([17], Theorem II.5.10, p. 89). Familiar examples of such Banach lattices are $l_p, \mathcal{L}_p, 1 \leq p < \infty, c_0$, and any reflexive Banach lattice. Let $\mathcal{L}_+(B)$ denote the set of positive bounded linear operators on B . That is, $T \in \mathcal{L}_+(B)$ iff $T(B_+) \subset B_+$. An operator $T \in \mathcal{L}(B)$ is said to be *regular* if $T = T_1 - T_2$ for some $T_1, T_2 \in \mathcal{L}_+(B)$. $\mathcal{L}'(B)$ denotes the vector space of regular operators on B . It can be shown (cf. [17], p. 229) that if B is order complete, then T is regular iff T maps order bounded sets onto order bounded sets iff $|T| \equiv T_v(-T)$ exists in the lattice structure induced on $\mathcal{L}(B)$. Furthermore, if B is order complete, then $\mathcal{L}'(B)$ is an order complete vector lattice ([17], p. 229). We refer to [16] and [17] for a more complete discussion of Banach lattices and their properties.

The most familiar example of lattice structure is with $B = \mathbf{R}^n$, $B_+ = \{x \in \mathbf{R}^n: x_i \geq 0, i = 1, 2, \dots, n\}$, $\mathcal{L}_+(B) = \{T: T \text{ is an } n \times n \text{ matrix } (t_{ij}) \text{ with } t_{ij} \geq 0; i, j = 1, 2, \dots, n\}$. Another example is $B = l_2$, $B_+ = \{x \in l_2: x_i \geq 0, i = 1, 2, \dots, \}$, where an important subset of $\mathcal{L}_+(B)$ is the set of positive Hilbert-Schmidt operators. These are compact operators T for which $Te_i \in B_+$, $i = 1, 2, \dots$, and $\sum_{i=1}^{\infty} \|Te_i\|^2 < \infty$, where $\{e_i\}_{i=1}^{\infty}$ is the usual basis for l_2 .

2. In this section it is our object to present in a more general setting some extensions of the classical Hille-Wintner comparison theorem for second-order scalar ordinary differential equations. The Hille-Wintner Theorem ([13], [20]) asserts that if $p(x) = \int_x^{\infty} q(t) dt$, $p_1(x) = \int_x^{\infty} q_1(t) dt$ exist (real-valued) and satisfy $|p_1(x)| \leq p(x)$ for all $x \in [a, \infty)$ then the existence of a non-oscillatory solution of

$$(2.1) \quad y''(x) + q(x)y = 0$$

on $[a, +\infty)$ implies the existence of a non-oscillatory solution of

$$(2.2) \quad y''(x) + q_1(x)y = 0$$

on $[a, +\infty)$. We shall consider the equations

$$(1.1) \quad Y'' + Q(t)Y = 0$$

and

$$(1.2) \quad Y'' + Q_1(t)Y = 0$$

and prove several generalizations of this result in the Banach lattice case. Etgen and Lewis [6] have extended the Hille-Wintner theorem in the B^* -algebra situation.

We begin with

THEOREM 2.1. *Let B be a Banach lattice with order continuous norm. Suppose the limits*

$$P(t) = \lim_{T \rightarrow \infty} \int_t^T Q(s) ds, \quad P_1(t) = \lim_{T \rightarrow \infty} \int_t^T Q_1(s) ds$$

exist (in the uniform operator topology of $\mathcal{L}(B)$) and

$$(i) \quad P(t), P_1(t), P(t) - P_1(t) \in \mathcal{L}_+(B), \quad t \in [a, +\infty).$$

If there exists a non-oscillatory solution $Y(t)$ of (1.1) such that

$$(ii) \quad Z(t) \equiv Y'(t)Y^{-1}(t) \in \mathcal{L}_+(B) \quad \text{for all } t \in [a, +\infty),$$

then (1.2) has a non-oscillatory solution on $[a, +\infty)$.

Proof. Throughout, inequality signs will refer to the appropriate positive cone.

The Riccati transformation $Z(t) = Y'(t)Y^{-1}(t)$ in (1.1) yields the equation

$$(2.3) \quad Z(t) = Z(T) + \int_t^T Q(s) ds + \int_t^T Z^2(s) ds, \quad a \leq t < T < \infty.$$

We assert that (2.3) implies that the set $\{\int_t^T Z^2(s) ds : a \leq T < \infty\}$ is an upward directed set in $\mathcal{L}^r(B)$ and is order bounded above by $Z(t)$. To see this, note first of all that the monotonicity of $\{\int_t^T Z^2(s) ds : a \leq T < \infty\}$ follows from the hypothesis that $Z(s)$ (and therefore $Z^2(s)$) is ≥ 0 for all $s \geq a$. Denote $\int_t^T Z^2(s) ds$ by $\phi(t; T)$. Then we can write (2.3) as

$$(2.4) \quad \phi(t; T) = Z(t) - Z(T) - P(t) + P(T).$$

Then if $a \leq T \leq \tau$ we have, by (2.4) $\phi(t; T) \leq \phi(t; \tau) \leq Z(t) + P(\tau)$, so

$$(2.5) \quad Z(t) + P(\tau) - \phi(t; T) \geq 0, \quad a \leq T \leq \tau.$$

For each $\tau \geq T$ the left-hand side of (2.5) is in $\mathcal{L}^r(B)$, which is a normed vector lattice and therefore has norm-closed positive cone $\mathcal{L}_+(B)$. Letting $\tau \rightarrow \infty$ in (2.5) we obtain $Z(t) - \phi(t; T) \geq 0$, which completes the proof of our assertion. Since $\mathcal{L}^r(B)$ is order complete, it follows that $\phi(t) \equiv \sup_T \phi(t; T)$ exists. Let $x \in B_+$. Then (2.3) implies

$$(2.6) \quad Z(T)x = Z(t)x - \left(\int_t^T Q(s) ds \right)x - \phi(t; T)x.$$

Since B has order continuous norm, $\phi(t; T)x \rightarrow \phi(t)x$ in norm, so $\phi(t)$ is the strong operator limit of $\phi(t; T)$ as $T \rightarrow \infty$. Thus the limit in norm as $T \rightarrow \infty$ exists for the right-hand side of (2.6). We conclude that the limit in norm of $Z(T)x$ as $T \rightarrow \infty$ exists (in B_+ by norm-closedness of B_+), call it $C(x)$. Thus

$$(2.7) \quad C(x) = Z(t)x - P(t)x - \phi(t)x.$$

It follows that $C(x)$ in fact defines a positive linear operator on B_+ and extends uniquely to a linear operator on B (cf. [17], p. 58). Thus, (2.7) holds for all x and we may write $C(x) = Cx$. Thus,

$$(2.8) \quad Z(t) = C + P(t) + \phi(t), \quad C \in \mathcal{L}_+(B), t \in [a, \infty).$$

We now define a sequence of operator-valued functions $\{W_n(t)\}$ as follows:

Let $W_0(t) \equiv C$, $t \geq a$, and inductively

$$(2.9)_n \quad W_{n+1}(t) \equiv C + P_1(t) + \text{s-lim}_{T \rightarrow \infty} \int_t^T W_n^2(s) ds, \\ n = 0, 1, 2, \dots, t \geq a,$$

where s-lim denotes the strong operator limit. To justify this, we make the inductive assumption

$$(2.10)_n \quad \begin{cases} 0 \leq W_n(t) \leq Z(t) \text{ and } W_n(t) \text{ is continuous in the} \\ \text{uniform operator topology of } \mathcal{L}(B), t \geq a. \end{cases}$$

Clearly $(2.10)_0$ is true. Assume $(2.10)_n$ is true. Then $0 \leq W_n(s) \leq Z(s)$ implies $0 \leq W_n^2(s) \leq Z^2(s)$. This is a consequence of the cone $\mathcal{L}_+(B)$ being closed under multiplication. It does not, in general, hold in the B^* -algebra case. Thus $\{\int_t^T W_n^2(s) ds\}$ is monotone increasing in T and is order bounded by $Z(t)$ since $\int_t^T W_n^2(s) ds \leq \int_t^T Z_n^2(s) ds \leq Z(t)$, and hence $\text{s-lim}_{T \rightarrow \infty} \int_t^T W_n^2(s) ds$ exists. Thus $W_{n+1}(t)$ is well defined by $(2.9)_n$. It follows from the hypothesis $0 \leq P_1(t) \leq P(t)$ that $0 \leq W_{n+1}(t) \leq Z(t)$. We also have

$$(2.11) \quad W_{n+1}(t) - W_{n+1}(s) = P_1(t) - P_1(s) - \int_s^t W_n^2(\tau) d\tau,$$

so it follows that $W_{n+1}(t)$ is continuous, hence $(2.10)_{n+1}$ is true and thus $(2.10)_n$ is verified for all $n \geq 0$.

Let $\psi_n(t; T) = \int_t^T W_n^2(s) ds$ and denote $\text{s-lim}_{T \rightarrow \infty} \psi_n(t; T)$ by $\psi_n(t)$. Since for fixed t , $\{W_n(t)\}_{n=1}^\infty$ is a monotone increasing sequence in $\mathcal{L}_+(B)$, bounded above by $Z(t)$, it follows that $\text{s-lim}_{n \rightarrow \infty} W_n(t)$ exists. Let $W(t) \equiv \text{s-lim}_{n \rightarrow \infty} W_n(t)$. By $(2.9)_n$ we have

$$(2.12) \quad W(t) = C + P_1(t) + \psi(t),$$

where $\psi(t) = \text{s-lim}_{n \rightarrow \infty} \psi_n(t)$. Note that $W(t)$ is continuous since

$$(2.13) \quad W(t) - W(s) = P_1(t) - P_1(s) - \text{s-lim}_{n \rightarrow \infty} \int_s^t W_n^2(\tau) d\tau$$

and $0 \leq W_n(\tau) \leq Z(\tau)$ implies

$$0 \leq \left\| \int_s^t W_n^2(s) ds \right\| \leq \int_s^t \|Z^2(s)\| ds \rightarrow 0 \quad \text{as } t - s \rightarrow 0$$

by continuity of Z . For each τ and each $x \in B_+$, $W_n^2(\tau)x \rightarrow W^2(\tau)x$ in norm and $W^2(\tau)x$ is continuous in τ . Now it follows from Dini's Theorem [21] that this convergence is uniform on any compact subinterval $[s, t]$ of $[a, +\infty)$. Thus

$$\lim_{n \rightarrow \infty} \left(\int_s^t W_n^2(\tau) d\tau \right) x = \left(\int_s^t W^2(\tau) d\tau \right) x, \quad x \in B_+,$$

so that

$$s\text{-}\lim_{n \rightarrow \infty} \int_s^t W_n^2(\tau) d\tau = \int_s^t W^2(\tau) d\tau.$$

We may therefore write (2.13) as

$$(2.14) \quad W(t) - W(s) = P_1(t) - P_1(s) - \int_s^t W^2(\tau) d\tau.$$

Dividing (2.14) by $t - s$ ($t \neq s$) and taking limits in the uniform operator topology as $s \rightarrow t$, we have

$$(2.15) \quad W'(t) = -Q_1(t) - W^2(t).$$

Now define $\hat{Y}(t)$ to be the solution of

$$(2.16) \quad \hat{Y}'(t) = W(t)\hat{Y}(t), \quad \hat{Y}(a) = I$$

($I =$ the identity operator in $\mathcal{L}(B)$). Then \hat{Y} is a non-oscillatory solution of (1.2) and this proves the Theorem.

We note that Theorem 2.1 does allow for two operator-valued equations to be compared as distinct from the comparison theorems of [6] which compare an operator-valued equation with a scalar equation.

One would like to weaken hypothesis (i) in Theorem 2.1 to

$$(i)' \quad P_1(t) \in \mathcal{L}'(B), \quad P(t), \quad P(t) - |P_1(t)| \in \mathcal{L}_+(B), \quad t \in [a, +\infty),$$

and to dispense with hypothesis (ii) (which holds automatically in the scalar case for a non-oscillatory solution $Y(t)$ of (1.1) when $P(t) \geq 0$).

As far as (ii) is concerned, we may make the following remarks:

Assume $P(t) \in \mathcal{L}^+(B)$, $t \in [a, \infty)$. Define $P_0(t)$ to be $P(t)$ and $P_n(t)$ by $P_n(t) = P(t) + s\text{-}\lim_{T \rightarrow \infty} \int_t^T P_{n-1}^2(s) ds$. Then the existence of a non-oscillatory solution $Y(t)$ of (1.1) satisfying (ii) is equivalent to

$$(iii) \quad P_n(t) \text{ is defined for all } t \in [a, \infty), \quad n = 0, 1, 2, \dots, \text{ and} \\ s\text{-}\lim_{n \rightarrow \infty} P_n(t) = \hat{P}(t) \text{ exists.}$$

The equivalence of (ii) and (iii) is established by essentially the same arguments as were used in the proof of Theorem 2.1 and is an extension of

a well-known result in the scalar case (see [18], Theorem 3.2, where the scalar result is given, though expressed in a slightly different form). One marked difference from the scalar case, however, is that there may be nonoscillatory solutions $Y(t)$ of (1.1) for which (ii) does not hold, as the following example illustrates. Let $B = \mathbf{R}^2$ and take

$$Q(t) = \frac{1}{288t^2} \begin{pmatrix} 24 & 5 \\ 5 & 48 \end{pmatrix}.$$

Then

$$Y(t) = \frac{1}{5} \begin{pmatrix} t^\mu & -25t^\nu \\ 5t^\mu & 5t^\nu \end{pmatrix}$$

is a solution of (1.1), where

$$\mu = \frac{1}{2} \left(1 - \sqrt{\frac{23}{72}} \right), \quad \nu = \frac{1}{2} \left(1 + \sqrt{\frac{49}{72}} \right).$$

It may be verified that

$$Z(t) = Y'(t)Y^{-1}(t) = \frac{1}{26t} \begin{pmatrix} \mu + 5\nu & -5(\nu - \mu) \\ -5(\nu - \mu) & \nu + 5\mu \end{pmatrix},$$

so $Z(t) \notin \mathcal{L}_+(B)$. In this case however, (1.1) does have a (different) nonoscillatory solution $Y(t)$ for which (ii) holds, namely

$$Y(t) = \frac{1}{5} \begin{pmatrix} t^\alpha & -25t^\beta \\ 5t^\alpha & 5t^\beta \end{pmatrix}, \quad \text{where } \alpha = \frac{1}{2} \left(1 + \sqrt{\frac{23}{72}} \right), \quad \beta = \frac{1}{2} \left(1 - \sqrt{\frac{49}{72}} \right).$$

We do not know whether $P(t) \in \mathcal{L}_+(B)$, $t \in [a, \infty)$, and the existence of a nonoscillatory solution of (1.1) together do imply that (ii) holds for some nonoscillatory solution $Y(t)$ (and therefore that (iii) holds).

In the case that B is finite dimensional, we can replace (i) by (i)'.

THEOREM 2.2. *Let $B = \mathbf{R}^n$ and suppose*

$$P(t) = \lim_{T \rightarrow \infty} \int_t^T Q(s) ds, \quad P_1(t) = \lim_{T \rightarrow \infty} \int_t^T Q_1(s) ds$$

exist such that (i)' holds. If there exists a non-oscillatory solution $Y(t)$ of (1.1) such that (ii) holds, then (1.2) has a non-oscillatory solution on $[a, \infty)$.

Proof. As in the proof of Theorem 2.1, we put $Z(t) = Y'(t)Y^{-1}(t)$ in (1.1) and obtain

$$(2.3) \quad Z(t) = Z(T) + \int_t^T Q(s) ds + \int_t^T Z^2(s) ds, \quad a \leq t \leq T < \infty,$$

which leads to

$$(2.8) \quad Z(t) = C + P(t) + \phi(t), \quad t \in [a, \infty),$$

where

$$C \in \mathcal{L}_+(B), \quad \phi(t) = \int_t^\infty Z^2(s) ds \in \mathcal{L}_+(B), \quad t \in [a, \infty).$$

Let \mathcal{W} be the set of continuous functions $W: [a, \infty) \rightarrow \mathcal{L}(R^n)$ such that $|W(t)| \leq Z(t)$ for all $t \in [a, \infty)$.

Then \mathcal{W} is a closed, convex subset of \mathcal{N} , the Fréchet space of continuous operator-valued functions on $[a, \infty)$ with the compact-open topology.

Now if $W \in \mathcal{W}$, then for each $t \in [a, \infty)$, $W(t)$ may be represented by an $n \times n$ matrix, say $W(t) = [(W(t))_{ij}]$. Similarly we may represent $Z(t)$ by the $n \times n$ matrix $[(Z(t))_{ij}]$, and we have

$$|(W(t))_{ij}| \leq (Z(t))_{ij}, \quad i, j = 1, \dots, n.$$

It follows that $|W^2(t)| \leq Z^2(t)$. Furthermore, we have $\|W^2(t)\| \leq \|Z^2(t)\|$, where $\|\cdot\|$ denotes the uniform operator norm on $\mathcal{L}(B)$. Since $Z^2(t)$ is integrable on $[a, \infty)$, it follows that $W^2(t)$ is integrable on $[a, \infty)$, and

$$(2.17) \quad \left| \int_t^\infty W^2(s) ds \right| \leq \int_t^\infty Z^2(s) ds.$$

It follows from (2.9), (2.17) and (i)' that the map F on \mathcal{W} defined by

$$(2.18) \quad (FW)(t) = C + P_1(t) + \int_t^\infty W^2(s) ds$$

is a map from \mathcal{W} into itself.

Let $\{W_n\}_{n=1}^\infty$, $W \in \mathcal{W}$ with $\lim_{n \rightarrow \infty} W_n = W$. For any $T \geq t > a$, we have

$$(2.19) \quad \begin{aligned} \|(FW_n)(t) - (FW)(t)\| &= \left\| \int_t^\infty W_n^2(s) ds - \int_t^\infty W^2(s) ds \right\| \\ &\leq \int_t^T \|W_n^2(s) - W^2(s)\| ds + \left\| \int_T^\infty W_n^2(s) ds \right\| \\ &\quad + \left\| \int_T^\infty W^2(s) ds \right\|. \end{aligned}$$

Since

$$\left\| \int_T^\infty W_n^2(s) ds \right\| \leq \left\| \int_T^\infty Z^2(s) ds \right\|$$

and

$$\left\| \int_T^\infty W^2(s) ds \right\| \leq \left\| \int_T^\infty Z^2(s) ds \right\|,$$

we may, for any given $\varepsilon > 0$, choose $T = T(\varepsilon)$ so that

$$\left\| \int_T^\infty W_n^2(s) ds \right\| < \frac{\varepsilon}{3}, \quad n = 1, 2, \dots,$$

and

$$\left\| \int_T^\infty W^2(s) ds \right\| < \frac{\varepsilon}{3}.$$

By the uniform convergence of W_n to W on $[a, T]$ we have

$$\int_t^T \|W_n^2(s) - W^2(s)\| ds < \frac{\varepsilon}{3} \quad \text{for all } n \geq N(\varepsilon),$$

say, and for all $t \in [a, T]$. Thus (2.19) gives

$$\|(FW_n)(t) - (FW)(t)\| < \varepsilon \quad \text{for all } n \geq N(\varepsilon), \quad t \in [a, \infty).$$

It follows that $\lim_{n \rightarrow \infty} FW_n = FW$, that is, $F: \mathcal{W} \rightarrow \mathcal{W}$ is continuous.

Now for $s, t \geq a$, we have

$$\|FW(t) - FW(s)\| \leq \int_s^t \|Q_1(\tau)\| d\tau + \int_s^t \|Z^2(\tau)\| d\tau.$$

It follows that $F(\mathcal{W})$ consists of equicontinuous, uniformly bounded operator-valued functions. Thus $F(\mathcal{W})$ is precompact.

By Tychonov's Theorem ([10], p. 405), F has a fixed point in \mathcal{W} , that is, there exists $W(t)$ such that

$$(2.20) \quad W(t) = C + P_1(t) + \int_t^\infty W^2(s) ds.$$

It follows that (1.2) has a non-oscillatory solution.

Our next result is of a somewhat different nature. Here we compare the operator equation (1.2) in $\mathcal{L}(R^n)$ with an appropriate scalar equation

$$(2.21) \quad y''(t) + q(t)y(t) = 0.$$

THEOREM 2.3. *Let $Q_1: [a, \infty) \rightarrow \mathcal{L}(R^n)$ be continuous and let $[(Q_1(t))_{ij}]$ be the usual matrix representation of $Q_1(t)$. Assume*

$$(P_1(t))_{ij} = \int_t^\infty (Q_1(s))_{ij} ds$$

exists for all i, j . Let $q: [a, \infty) \rightarrow R$ be continuous and suppose $p(t) = \int_t^\infty q(s) ds$ exists, with

$$(iii) \quad p(\sqrt{n}t) \geq \sqrt{n} \max_{i,j} (P_1(t))_{ij}, \quad t \in [a, \infty).$$

If (2.21) has a non-oscillatory solution on $[a, \infty)$, then so does (1.2).

Proof. Let $y(t)$ be a non-oscillatory solution of (2.21) and put $u(t) = y'(t)y^{-1}(t)$ to obtain the Riccati equation

$$u'(t) = -q(t) - u^2(t).$$

Put $v(t) = u(\sqrt{n}t)/\sqrt{n}$ to obtain

$$(2.22) \quad v'(t) = -q(\sqrt{n}t) - nv^2(t).$$

Integrate (2.22) to obtain

$$(2.23) \quad v(t) = v(T) + \int_t^T q(\sqrt{n}s) ds + n \int_t^T v^2(s) ds.$$

Letting $T \rightarrow \infty$ in (2.23), it follows that $\lim_{T \rightarrow \infty} v(T) = b$, where $-\infty \leq b < \infty$. In fact, it is easy to show that $0 \leq b < \infty$, and we have

$$(2.24) \quad v(t) = b + \frac{1}{\sqrt{n}} p(\sqrt{n}t) + n \int_t^\infty v^2(s) ds.$$

Let \mathcal{W}_0 be the set of continuous functions $W(t) = [(W(t))_{ij}]$ from $[a, \infty)$ to $\mathcal{L}(R^n)$ such that $|(W(t))_{ij}| \leq v(t)$, $t \in [a, \infty)$, for each i, j .

Define the map $F_0: \mathcal{W}_0$ to \mathcal{W}_0 by

$$(2.25) \quad ((F_0W)(t))_{ij} = (P_1(t))_{ij} + \int_t^\infty \left(\sum_{k=1}^n (W(s))_{ik} (W(s))_{kj} \right) ds, \\ t \in [a, \infty), i, j = 1, \dots, n.$$

It follows from hypothesis (iii) and (2.24) that F_0 is well defined. \mathcal{W}_0 is a closed, convex subset of \mathcal{W} , the Fréchet space of continuous operator-valued functions on $[a, \infty)$ with the compact-open topology. Arguments similar to those used in the proof of Theorem 2.2 show that F_0 is continuous on \mathcal{W}_0 and that $F_0(\mathcal{W}_0)$ is a precompact subset of \mathcal{W}_0 . Hence Tychonov's theorem yields the existence of a fixed point for F_0 , that is, there exists $W(t) = [(W(t))_{ij}]$ such that

$$(2.26) \quad (W(t))_{ij} = (P_1(t))_{ij} + \int_t^\infty \left(\sum_{k=1}^n (W(s))_{ik} (W(s))_{kj} \right) ds \\ t \in [a, \infty); i, j = 1, \dots, n.$$

By examining the matrix Riccati equation for (1.2), it is apparent that (2.26) enables us to define a non-oscillatory solution of (1.2).

3. We now present two examples which illustrate the results obtained.

EXAMPLE 3.1. Let B be a Banach lattice with order continuous norm and let B' be its dual space. (B' is also a Banach lattice, under the dual norm, which is order complete.) Let $c \in B_+$ and $\phi \in B'_+$. Let $q(t)$ be a real-valued positive, twice continuously differentiable function on $[a, +\infty)$ such that $q'(t) > 0$ and $q''(t) < 0$ for all $t \in [a, +\infty)$. Define $Q: [a, \infty) \rightarrow \mathcal{L}(B)$ by

$$Q(t)x = \frac{-q''(t)\phi(x)c}{1 + q(t)\phi(c)}, \quad x \in B.$$

Then

$$Q(t) \in \mathcal{L}_+(B) \quad \text{for all } t \in [a, \infty) \quad \left(\|Q(t)\| = \frac{|q''(t)| \cdot \|\phi\| \cdot \|c\|}{1 + q(t)\|\phi(c)\|} \right).$$

Note that

$$\left\| \int_t^\infty Q(s) ds \right\| = \left(\int_t^\infty \frac{-q''(s) ds}{1 + q(s)\phi(c)} \right) \|\phi\| \cdot \|c\| \leq q'(t) \|\phi\| \cdot \|c\|.$$

It may be verified that $Y'' + Q(t)Y = 0$ has a solution $Y(t)$ defined by

$$Y(t)x = x + q(t)\phi(x)c, \quad x \in B.$$

Since

$$Y^{-1}(t)x = x - \frac{q(t)\phi(x)c}{1 + q(t)\phi(c)}, \quad x \in B,$$

it follows that $Y(t)$ is a non-oscillatory solution and, furthermore, the Riccati variable $Z(t) = Y'(t)Y^{-1}(t)$ is given by

$$Z(t)x = \frac{q'(t)\phi(x)c}{1 + q(t)\phi(c)}, \quad x \in B,$$

so $Z(t) \in \mathcal{L}_+(B)$ on $[a, \infty)$. Therefore, if $c_1 \in B_+$, $c_1 \leq c$, $\phi_1 \in B'_+$, $\phi_1 \leq \phi$, and if $q_1(t)$ is a real-valued function such that

$$0 \leq \int_t^\infty \frac{-q_1''(s)}{1 + q_1(s)\phi_1(c_1)} ds \leq \int_t^\infty \frac{-q''(s)}{1 + q(s)\phi(c)} ds, \quad t \in [a, +\infty),$$

then the equation $Y'' + Q_1(t)Y = 0$ has a non-oscillatory solution by Theorem 2.1, where $Q_1(t)$ is given by

$$Q_1(t)x = \frac{-q_1''(t)\phi_1(x)c_1}{1 + q_1(t)\phi_1(c)}, \quad x \in B.$$

Note that $Q_1(t) \notin \mathcal{L}_+(B)$ in general.

EXAMPLE 3.2. As an application of Theorem 2.2, consider the case $n = 2$. Let

$$Q(t) = t^{-2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{where } a, b, c, d \geq 0, \quad t \in [1, \infty).$$

Invoking condition (iii) (see the remarks following Theorem 2.1), there is a nonoscillatory solution $Y(t)$ of (1.1) satisfying (ii) iff the sequence $P_n(t)$ converges as $n \rightarrow \infty$, where

$$\begin{aligned} P_0(t) &= P(t) = t^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ P_n(t) &= P(t) + \int_t^\infty P_{n-1}^2(s) ds, \quad n = 1, 2, \dots \end{aligned}$$

If

$$P_n(t) = t^{-1} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

we obtain the relations

$$(2.27) \quad \begin{aligned} a_{n+1} &= a + a_n^2 + b_n c_n, \\ b_{n+1} &= b + b_n(a_n + d_n), \\ c_{n+1} &= c + c_n(a_n + d_n), \\ d_{n+1} &= d + d_n^2 + b_n c_n, \quad n = 0, 1, \dots \end{aligned}$$

Since $a, b, c, d \geq 0$, the sequences a_n, b_n, c_n, d_n are increasing with n and have finite limits $\alpha, \beta, \gamma, \delta$ iff the system of equations

$$(2.28) \quad \begin{aligned} \alpha &= a + \alpha^2 + \beta\gamma, \\ \beta &= b + \beta(\alpha + \delta), \\ \gamma &= c + \gamma(\alpha + \delta), \\ \delta &= d + \delta^2 + \beta\gamma \end{aligned}$$

has a solution.

After some manipulation of these equations, we find that solvability of (2.28) is equivalent to the existence of a positive solution θ of the equation

$$(2.29) \quad \theta^2\{1 - 2(d + a) - \theta^2\} = (d - a)^2 + 4bc.$$

$\alpha, \beta, \gamma, \delta$ are then given by

$$\alpha = \frac{1}{2}\left(1 - \theta - \frac{d - a}{\theta}\right), \quad \beta = \frac{b}{\theta}, \quad \gamma = \frac{c}{\theta}, \quad \delta = \frac{1}{2}\left(1 - \theta + \frac{d - a}{\theta}\right).$$

This, in turn, is easily seen to be equivalent to the condition

$$(2.30) \quad \begin{cases} a < \frac{1}{4}, & \text{if } d = a \text{ and } bc = 0 \\ d + a + \sqrt{(d - a)^2 + 4bc} \leq \frac{1}{2}, & \text{if } d \neq a \text{ or } bc \neq 0. \end{cases}$$

Let a, b, c, d be any nonnegative constants satisfying (2.30) with equality. Now let $Q_1(t) = [q_{ij}(t)]$ be chosen so that

$$(2.31) \quad \begin{aligned} \overline{\lim}_{t \rightarrow \infty} t \int_t^\infty q_{11}(s) ds &< a, & \overline{\lim}_{t \rightarrow \infty} t \int_t^\infty q_{12}(s) ds &< b, \\ \overline{\lim}_{t \rightarrow \infty} t \int_t^\infty q_{21}(s) ds &< c, \\ \overline{\lim}_{t \rightarrow \infty} t \int_t^\infty q_{22}(s) ds &< d. \end{aligned}$$

Then by Theorem 2.2, (1.2) has a non-oscillatory solution.

This example may be thought of as an extension to the 2×2 matrix case of Hille's non-oscillation criterion [13]

$$(2.32) \quad \overline{\lim}_{t \rightarrow \infty} tq_1(t) < 1/4$$

for the scalar equation

$$(2.2) \quad y'' + q_1(t)y = 0.$$

In case $b = c = d = 0$, we have $a = 1/4$ and (2.31) reduces to (2.32).

Other examples can be readily found utilizing condition (iii).

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