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FLAT HILBERT CUBE MANIFOLD PAIRS

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The purpose of this paper is to study embeddings of Q-manifolds into Q-manifolds. Mainly, we relate flat Q-manifold pairs with PL manifold pairs by using a relative version of the Chapman Splitting Theorem. The concepts of QPL embedding and QPL homeomorphism are introduced.

1. Introduction and definitions. For topological spaces (polyhedra) X and Y an embedding f: $X \to Y$ is said to be a (PL) locally flat embedding provided that every point of X has a neighborhood U and an open (PL) embedding h: $U \times \mathbf{R}^m \to Y$ such that $h(x, 0) = f(x)$, for all $x \in U$. If U can be taken to be all of X, then the embedding is said to be a (PL) flat embedding. Furthermore, the pair (Y, X) is said to be a flat pair if the inclusion $X \hookrightarrow Y$ is a (PL) flat embedding. Note that if (M, N) is a flat finite-dimensional manifold pair, then $N \cap \partial M = \partial N$ and $(\partial M, \partial N)$ is a flat manifold pair.

We use Q to denote the Hilbert cube and by a Q-manifold we mean a separable metric manifold modeled on Q.

The purpose of this paper is to relate flat Q-manifold pairs with flat PL manifold pairs by using a relative version of the Chapman Splitting Theorem [**6**]. The following is our first result in this direction.

THEOREM 1. *Let $(\mathfrak{M}, \mathfrak{N})$ be a flat compact Q-manifold pair. Then there exists a flat PL manifold pair (M, N) and a homeomorphism h: $(\mathfrak{M}, \mathfrak{N}) \to (M, N) \times Q$.*

Chapman [**4**] has proved that there exists a codimension 3 locally flat embedding $\mathfrak{N} \hookrightarrow \mathfrak{M}$ between Q-manifolds such that $\mathfrak{N}$ has no tubular neighborhood and, moreover, no stabilization $\mathfrak{N} \times \{0\} \hookrightarrow \mathfrak{M} \times \mathbf{R}^n$ has a tubular neighborhood. On the other hand, Milnor [**9**] and Kister [**8**] proved the stable existence of tubular neighborhoods for embeddings of finite dimensional manifolds. Consequently an analogue of Theorem 1 for locally flat Q-manifold pairs is not possible.

Let M and N be PL manifolds. An embedding (homeomorphism) f: $N \times Q \to M \times Q$ is said to be a QPL embedding (homeomorphism) if there exists a PL embedding (homeomorphism) g: $N \times I^n \to M \times I^m$

such that $f = g \times \mathrm{Id}$: $(N \times I^n) \times Q_{n+1} \to (M \times I^m) \times Q_{m+1}$, where $\mathrm{Id}(q_{n+1}, \ldots) = (q_{n+1}, \ldots) \in Q_{m+1}$. If, in addition, g is a PL flat embedding, then f is called a QPL-flat embedding. The next theorem relates flat embeddings of Q-manifolds with PL flat embeddings of their underlying spaces.

THEOREM 2. *Let N be a compact PL manifold and let M be a PL manifold. A flat embedding f: $N \times Q \to M \times Q$ is isotopic to a QPL-flat embedding if and only if f is homotopic to a QPL-flat embedding.*

We say that two maps f_0, f_1: $(X, X_0) \to (Y, Y_0)$ are homotopic by pairs if there exists a homotopy h_t: $(X, X - X_0, X_0) \to (Y, Y - Y_0, Y_0)$ such that $h_0 = f$ and $h_1 = f_1$. If, in addition, h_t is a homeomorphism for every $t \in I$, we say f_0 is isotopic by pairs to f_1.

The next theorem, which was virtually proved by Chapman in [**5**] for $M_0 = N_0 = \varnothing$, relates homeomorphisms of flat Q-manifold pairs with homeomorphisms of flat PL manifold pairs.

THEOREM 3. *Let (M^{m+k}, M_0^m) and (N^{n+k}, N_0^n) be flat compact PL manifold pairs. A homeomorphism h: $(N, N_0) \times Q \to (M, M_0) \times Q$ is isotopic by pairs to a QPL homeomorphism if and only if h is homotopic by pairs to a QPL homeomorphism.*

At the end we give an example showing that the condition

$$h_t((N - N_0) \times Q) \subset (M - M_0) \times Q,$$

in the homotopy of Theorem 3, is necessary.

We let $\mathbf{R}^n$ denote Euclidean n-space, I the closed unit interval $[0, 1]$ and for $r > 0$, $B_r^n = [-r, r]^n \subset \mathbf{R}^n$. As usual, ∂B_r^n denotes the boundary of B_r^n and $\mathring{B}_r^n$ denotes its interior. For any space X and $A \subset X$ we use $\mathrm{Int}_X A$ and $\mathrm{Bd}_X A$ to denote the topological interior and boundary of A in X. The subscript will be omitted when the meaning is clear.

We represent Q as $Q = I_1 \times I_2 \times \cdots$, where I_i is a copy of the closed unit interval $[0, 1]$. We also let $I^n = I_1 \times \cdots \times I_n$ and $Q_n = I_n \times I_{n+1} \times \cdots$, so that $Q = I^n \times Q_{n+1}$. We use 0 to represent $(0, 0, \ldots) \in Q_n$. In this paper it will be convenient to identify X with $X \times \{0\}$ in $X \times Q$ and, in general, $X \times I^n$ with $X \times I^n \times \{0\}$ in $X \times I^n \times Q_{n+1}$.

A compact subpolyhedron Y of a polyhedron X is said to be straight provided that $\mathrm{Bd}\, Y$ is PL collared in both Y and $X - \mathrm{Int}\, Y$. By a PL manifold we will mean a piecewise-linear manifold with or without boundary as in [**1**].

In general we use results and notation from [**2**] concerning Q-manifolds and from [**7**] concerning PL-topology.

2. A relative splitting theorem. Let $\mathfrak{M}$ be a compact connected Q-manifold and let (M, N) be a flat PL manifold pair. Let h: $\mathfrak{M} \times (B_1^m, \{0\}) \times \mathbf{R} \to (M, N) \times Q$ be an open embedding.

A splitting of h is a decomposition, $\mathfrak{M} \times B_1^m \times \mathbf{R} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, such that if $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \mathfrak{M}_2$, $\mathfrak{N}_1 = \mathfrak{M}_1 \cap (\mathfrak{M} \times \{0\} \times \mathbf{R})$, $\mathfrak{N}_2 = \mathfrak{M}_2 \cap (\mathfrak{M} \times \{0\} \times \mathbf{R})$, and $\mathfrak{N}_0 = \mathfrak{N}_1 \cap \mathfrak{N}_2$, then

(1) $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are non-compact Q-manifolds which are closed in $\mathfrak{M} \times B_1^m \times \mathbf{R}$,

(2) $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are non-compact Q-manifolds which are closed in $\mathfrak{M} \times \{0\} \times \mathbf{R}$,

(3) there is a polyhedron $A \subset M \times I^n$ such that if $B = A \cap (N \times I^n)$, then A is PL bicollared in $M \times I^n$, B is PL bicollared in $N \times I^n$, and $h(\mathfrak{M}_0, \mathfrak{N}_0) = (A \times Q_{n+1}, B \times Q_{n+1})$,

(4) $(\mathfrak{M}_0, \mathfrak{N}_0)$ is a compact Q-manifold pair, and

(5) there is an open PL embedding φ: $N \times I^n \times \mathbf{R}^m \to M \times I^n$ such that $\varphi = \mathrm{Id}$ on $N \times I^n \times \{0\}$ and $\varphi(N \times I^n \times \mathbf{R}^m) \cap A = \varphi(B \times \mathbf{R}^m)$.

The purpose of this section is to prove the following relative version of the Chapman Splitting Theorem [**6**].

THEOREM 2.1. *There exists a splitting of h*, $\mathfrak{M} \times B_1^m \times \mathbf{R} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, *such that the inclusions* $\mathfrak{M}_0 \hookrightarrow \mathfrak{M} \times B_1^m \times \mathbf{R}$, $\mathfrak{N}_0 \hookrightarrow \mathfrak{M} \times \{0\} \times \mathbf{R}$, *and* $\mathfrak{M}_0 - \mathfrak{N}_0 \hookrightarrow \mathfrak{M} \times (B_1^m - \{0\}) \times \mathbf{R}$ *are homotopy equivalences.*

LEMMA 2.1. *Splittings of h exist.*

Proof. Since $h(\mathfrak{M} \times B_1^m \times \{0\})$ is compact and $h(\mathfrak{M} \times B_1^m \times \mathbf{R})$ is open in $M \times Q$, it follows that there is a compact polyhedron $K \subset M \times I^n l$ and an open set $U \subset M \times I^n$ such that

$$h(\mathfrak{M} \times B_1^m \times \{0\}) \subset K \times Q_{n+1} \subset U \times Q_{n+1} \subset h(\mathfrak{M} \times B_1^m \times \mathbf{R}) \subset M \times Q.$$

Let $\tilde{K} = K \cap (N \times I^n)$. Since $(M \times I^n, N \times I^n)$ is a codimension m flat PL manifold pair, we may assume without loss of generality that $N \times I^n \times \mathbf{R}^m \subset M \times I^n$ and, for some $r > 0$, $K \cap (N \times I^n \times B_r^m) = \tilde{K} \times B_r^m$. Therefore, there exists a polyhedron R in U such that

(a) $K \subset \operatorname{Int} R \subset R \subset U$,

(b) $\operatorname{Bd} R$ is PL bicollared in $M \times I^n$,

(c) if $R_1 = R \cap (N \times I^n)$, then Bd R_1 is PL bicollared in $N \times I^n$, and

(d) for some $r_1 > 0$ Bd $R \cap (N \times I^n \times B^m_{r_1}) = \text{Bd } R_1 \times B^m_{r_1}$.

Since Int $R \times Q_{n+1}$ is a neighborhood of $h(\mathfrak{M} \times B^m_1 \times \{0\})$ in $h(\mathfrak{M} \times B^m_1 \times \mathbf{R})$, we can decompose Bd R as Bd $R = R' \cup R''$, where

$$R' \times Q_{n+1} \subset h(\mathfrak{M} \times B^m_1 \times (-\infty, 0))$$

and

$$R'' \times Q_{n+1} \subset h(\mathfrak{M} \times B^m_1 \times (0, \infty)).$$

Similarly Bd $R_1 = R'_1 \cup R''_1$, where

$$R'_1 \times Q_{n+1} \subset h(\mathfrak{M} \times \{0\} \times (-\infty, 0))$$

and

$$R''_1 \times Q_{n+1} \subset h(\mathfrak{M} \times \{0\} \times (0, \infty)).$$

Let

$$\mathfrak{M}_1 = (\mathfrak{M} \times B^m_1 \times (-\infty, 0)) - h^{-1}(\text{Int } R \times Q_{n+1})$$

and

$$\mathfrak{M}_2 = (\mathfrak{M} \times B^m_1 \times (0, \infty)) \cup h^{-1}(R \times Q_{n+1}).$$

Then we have $h(\mathfrak{M}_0) = R' \times Q_{n+1}$,

$$\mathfrak{N}_1 = (\mathfrak{M} \times \{0\} \times (-\infty, 0)) - h^{-1}(\text{Int } R_1 \times Q_{n+1}),$$

$$\mathfrak{N}_2 = (\mathfrak{M} \times \{0\} \times (0, \infty)) \cup (R_1 \times Q_{n+1}),$$

and $h(\mathfrak{N}_0) = R'_1 \times Q_{n+1}$, thus giving the desired splitting of h.

LEMMA 2.2. *Let $\mathfrak{M} \times B^m_1 \times \mathbf{R} = \mathfrak{M}_1 \cup \mathfrak{M}_2$ be a splitting of h. Then we may assume there is a compact polyhedron $K \subset N \times I^n$ containing B such that the inclusion $K \hookrightarrow h(\mathfrak{N}_1)$ is a homotopy equivalence.*

Proof. We will first prove there is a compact polyhedron K_1 containing B and an embedding f: $K_1 \to h(\mathfrak{N}_1)$ such that $f|B = \text{Id}$ and f is a homotopy equivalence.

Since $\mathfrak{M}$ is compact, then $_1$ has the homotopy type of a compact polyhedron K_2. Let g: $K_2 \to h(\mathfrak{N}_1)$ be a homotopy equivalence and let $\tilde{g}$: $h(\mathfrak{N}_1) \to K_2$ be a homotopy inverse of g. Let φ: $B \to K_2$ be a PL map homotopic to $\tilde{g}|B$ and let K_1 be the mapping cylinder of φ. Let ρ: $K_1 \to K_2$ be the mapping cylinder retraction onto the base. Note that K_1 is a compact polyhedron containing B and that the map $g\rho$: $K_1 \to h(\mathfrak{N}_1)$ is a homotopy equivalence with the property that $g\rho | B$: $B \to h(\mathcal{N}_1)$ is homotopic to the inclusion. Since $B \subset h(\mathcal{N}_1)$ is a Z-set, it follows that there is a Z-embedding f: $K_1 \to h(\mathcal{N}_1)$ such that $f | B = \mathrm{Id}$, f is homotopic to $g\rho$ and consequently, f is a homotopy equivalence.

The compact set $f(K_1) \cup (B \times Q_{n+1})$ is contained in

$$h(\mathcal{M} \times \{0\} \times \mathbf{R})$$

which is open in $N \times Q$. Therefore, there is an $l \geq 0$ and an open subset U of $N \times I^l$ such that $f(K_1) \cup (B \times Q_{n+1}) \subset U \times Q_{l+1} \subset h(\mathcal{M} \times \{0\} \times \mathbf{R})$. Choose $U = U_1 \cup U_2$, where $U_1 \times Q_{l+1} \subset h(\mathcal{N}_1)$ and $U_2 \times Q_{l+1} \subset h(\mathcal{N}_2)$. We may assume $f = (f_1, f_2)$: $K_1 \to U_1 \times Q_{l+1}$ is an embedding. Let f_1': $K_1 \to U_1$ be a PL map homotopic to f_1 such that $f_1' | B = \mathrm{Id}$. Let f_2': $K_1 \to I_{l+1} \times \cdots \times I_k$ be a PL map such that $f_2'(B) = \{0\}$ and $f_2' | K_1 - B$: $K_1 - B \to I_{l+1} \times \cdots \times I_k - \{0\}$ is one to one. It is easy to see that $f' = (f_1', f_2')$: $K_1 \to U_1 \times I_{l+1} \times \cdots \times I_k$ is a PL embedding which is homotopic to f in $h(\mathcal{N}_1)$. Furthermore, since $B \times I_{n+1} \times \cdots \times I_k$ is bicollared, we can push $f'(K_1 - B)$ off $B \times I_{n+1} \times \cdots \times I_k$. This means we may assume $f'(K_1) \cap (B \times I_{n+1} \times \cdots \times I_k) = B$. Consider the compact subpolyhedron K of $N \times I^k$ defined by

$$K = f'(K_1) \cup_B (B \times I_{n+1} \times \cdots \times I_k).$$

Then we have $B \times I_{n+1} \times \cdots \times I_k \subset K \subset N \times I^k$. Furthermore, the inclusion $K \hookrightarrow h(\mathcal{N}_1)$ is a homotopy equivalence. This completes the proof of Lemma 2.2.

LEMMA 2.3. *There exists a splitting of h such that the inclusion* $\mathcal{N}_0 \hookrightarrow \mathcal{M} \times \{0\} \times \mathbf{R}$ *is a homotopy equivalence.*

Proof. Assertion. Let $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1 \cup \mathcal{M}_2$ be a splitting of h. Then there is a splitting of h, $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1' \cup \mathcal{M}_2'$, such that $\mathcal{N}_1' \subset \operatorname{Int} \mathcal{N}_1$ and the inclusions $\mathcal{N}_0' \hookrightarrow \mathcal{N}_2' - \operatorname{Int} \mathcal{N}_2$ and $\mathcal{N}_0' \hookrightarrow \mathcal{N}_1'$ are homotopy equivalences.

Proof of Assertion. Let K be as in the Lemma 2.2. Without loss of generality we may assume there is an open set U of $M \times I^n$ containing A such that $U = U_1 \cup U_2, U_1 \cap U_2 = A$, $U_1 \times Q_{n+1} \subset h(\mathcal{M}_1), U_2 \times Q_{n+1} \subset h(\mathcal{M}_2)$, and for some $r > 0$, $\varphi(K \times B_r^m) \subset U_1$. Let D be a regular neighborhood of $(\varphi(K \times B_r^m) \times \{1\}) \cup (A \times I_{n+1})$ in $U_1 \times I_{n+1}$ satisfying the following properties:

(a) $D_1 = D \cap (N \times I^{n+1})$ is a regular neighborhood of $(K \times \{1\}) \cup (B \times I_{n+1})$ in $(U_1 \cap (N \times I^n)) \times I_{n+1}$,

(b) Bd $D \cap (N \times I^{n+1}) = \text{Bd } D_1$, and

(c) Bd $D \cap \varphi \times \text{Id}_{I_{n+1}}(N \times I^{n+1} \times B_{r_1}^m) = \varphi \times \text{Id}_{I_{n+1}}(\text{Bd } D_1 \times B_{r_1}^m)$ for some $r_1 > 0$.

Note that the inclusion $K \hookrightarrow D_1$ is a homotopy equivalence. Since $(K \times \{1\}) \cup (B \times I_{n+1}) \subset D_1$ is a Z-set, it follows that the inclusion Bd $D_1 \hookrightarrow D_1$ is a homotopy equivalence.

Let

$$\mathcal{M}_1' = \mathcal{M}_1 - h^{-1}(\text{Int } D \times Q_{n+2}) \quad \text{and} \quad \mathcal{M}_2' = \mathcal{M}_2 \cup h^{-1}(D \times Q_{n+2}).$$

Then we have

$$h(M_0') = \text{Bd } D \times Q_{n+2}, \qquad \mathcal{N}_1' = \mathcal{N}_1 - h^{-1}(\text{Int } D_1 \times Q_{n+2}),$$

$$\mathcal{N}_2' = \mathcal{N}_2 \cup h^{-1}(D_1 \times Q_{n+2}), \quad \text{and} \quad h(\mathcal{N}_0') = \text{Bd } D_1 \times Q_{n+2}.$$

Since Bd $D_1 \hookrightarrow D_1$ is a homotopy equivalence, the inclusion $\mathcal{N}_0' \hookrightarrow \mathcal{N}_2' -$ Int $\mathcal{N}_2$ is a homotopy equivalence, hence there is a strong deformation *retraction of $\mathcal{N}_2'$ − Int $\mathcal{N}_2$ onto $\mathcal{N}_0'$ (see* **[11**, *p. 31] for further details),* and consequently, the inclusion $\mathcal{N}_1' \hookrightarrow \mathcal{N}_1$ is a homotopy equivalence. On the other hand, since the inclusions $K \hookrightarrow D_1 \times Q_{n+2}$ and $K \hookrightarrow h(\mathcal{N}_1)$ are homotopy equivalences, the inclusion Bd $D_1 \times Q_{n+2} \hookrightarrow h(\mathcal{N}_1)$ is a homotopy equivalence but, hence, $\mathcal{N}_0' \hookrightarrow \mathcal{N}_1$ is a homotopy equivalence and, *consequently, $\mathcal{N}_0' \hookrightarrow \mathcal{N}_1'$ is a homotopy equivalence. This concludes the* proof of the Assertion.

Let us now return to the proof of the lemma. By Lemma 2.1 and the Assertion, there is a splitting of h, $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1 \cup \mathcal{M}_2$, such that the inclusion $\mathcal{N}_0 \hookrightarrow \mathcal{N}_2$ is a homotopy equivalence. Again, by the Assertion, there is a splitting of h, $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1' \cup \mathcal{M}_2'$, such that $\mathcal{N}_1' \subset \text{Int } \mathcal{N}_1$ and the inclusions $\mathcal{N}_0' \hookrightarrow \mathcal{N}_2' - \text{Int } \mathcal{N}_2$ and $\mathcal{N}_0' \hookrightarrow \mathcal{N}_1'$ are homotopy equivalences. Since there is a strong deformation retraction of $\mathcal{N}_2$ onto $\mathcal{N}_0$, the inclusion $\mathcal{N}_2' - \text{Int } \mathcal{N}_2 \hookrightarrow \mathcal{N}_2'$ is a homotopy equivalence, but hence, $\mathcal{N}_0' \hookrightarrow \mathcal{N}_2'$ is a homotopy equivalence and, consequently, the inclusion $\mathcal{N}_0' \hookrightarrow \mathcal{M} \times \{0\} \times \mathbf{R}$ is a homotopy equivalence. This completes the proof of Lemma 2.3.

LEMMA 2.4. *Let $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1 \cup \mathcal{M}_2$ be a splitting of h such that the inclusion $\mathcal{N}_0 \hookrightarrow \mathcal{M} \times \{0\} \times \mathbf{R}$ is a homotopy equivalence and for some $r > 0$, $\mathcal{M}_0 \cap (\mathcal{M} \times B_r^m \times \mathbf{R}) = \mathcal{N}_0 \times B_r^m$. Then there exists a splitting of h, $\mathcal{M} \times B_1^m \times \mathbf{R} = \mathcal{M}_1' \cup \mathcal{M}_2'$, such that the inclusions $\mathcal{N}_0' \hookrightarrow \mathcal{M} \times \{0\} \times \mathbf{R}$, $\mathcal{M}_0' \hookrightarrow \mathcal{M} \times B_1^m \times \mathbf{R}$, and $\mathcal{M}_0' - \mathcal{N}_0' \hookrightarrow \mathcal{M} \times (B_1^m - \{0\}) \times \mathbf{R}$ are homotopy equivalences.*

Proof. The proof is similar to the proof of Lemma 2.3, but using the following fact for ANR's. If $Z \subset S$, $S = S_1 \cup S_2$, $S_0 = S_1 \cap S_2$, and the inclusions $Z \cap S_1$, $Z \cap S_2 \hookrightarrow S_2$, and $Z \cap S_0 \hookrightarrow S_0$ are homotopy equivalences, then the inclusion $Z \hookrightarrow S$ is a homotopy equivalence.

Proof of Theorem 2.1. By Lemma 2.3, there is a splitting of h, $\mathfrak{M} \times B_1^m \times \mathbf{R} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, such that $\mathfrak{M}_0 \hookrightarrow \mathfrak{M} \times \{0\} \times \mathbf{R}$ is a homotopy equivalence. We will first show there exists a homeomorphism h_1: $\mathfrak{M} \times B_1^m \times \mathbf{R} \to \mathfrak{M} \times B_1^m \times \mathbf{R}$ such that $h_1 = \mathrm{Id}$ on $\mathfrak{M} \times \{0\} \times \mathbf{R}$ and $h_1(\mathfrak{M}_0) \cap (\mathfrak{M} \times B_{r_1}^m \times \mathbf{R}) = \mathfrak{M}_0 \times B_{r_1}^m$ for some $r_1 > 0$.

Let φ: $N \times I^n \times \mathbf{R}^m \to M \times I^n$ be an open PL embedding such that $\varphi = \mathrm{Id}$ on $N \times I^n \times \{0\}$ and $\varphi(N \times I^n \times \mathbf{R}^m) \cap A = \varphi(B \times \mathbf{R}^m)$. Therefore, $\varphi \times \mathrm{Id}_{Q_{n+1}}$: $N \times Q \times \mathbf{R}^m \to M \times Q$ is an open embedding such that $\varphi \times \mathrm{Id}_{Qn+1} = \mathrm{Id}$ on $N \times Q \times \{0\}$ and

$$\varphi \times \mathrm{Id}_{Q_{n+1}}(N \times Q \times \mathbf{R}^m) \cap h(\mathfrak{M}_0) = \varphi \times \mathrm{Id}_{Q_{n+1}}(B \times \mathbf{R}^m).$$

Hence, using a small continuous function λ: $\mathfrak{M} \times \mathbf{R} \to (0, 1)$ such that if $(m, x, t) \in \mathfrak{M} \times \mathbf{R}^m \times \mathbf{R}$ and $\|x\| \times \lambda(m, t)$ then

$$\varphi \times \mathrm{Id}_{Q_{n+1}}(h(m, 0, t), x) \subset h(\mathfrak{M} \times B_1^m \times \mathbf{R}),$$

we can construct an open embedding g: $\mathfrak{M} \times \mathbf{R}^m \times \mathbf{R} \to \mathfrak{M} \times B_1^m \times \mathbf{R}$ such that $g = \mathrm{Id}$ on $\mathfrak{M} \times \{0\} \times \mathbf{R}$ and $g(\mathfrak{M}_0 \times \mathbf{R}^m) = g(\mathfrak{M} \times \mathbf{R}^m \times \mathbf{R}) \cap \mathfrak{M}_0$. Using the proof of Lemma 4.3 of **[10]**, it is easy to find a homeomorphism f: $\mathfrak{M} \times B_1^m \times \mathbf{R} \to \mathfrak{M} \times B_1^m \times \mathbf{R}$ such that $f = \mathrm{Id}$ on $\mathfrak{M} \times \{0\} \times \mathbf{R}$ and $f = g$ on $\mathfrak{M} \times B_r^m \times B^1$ for some small $r > 0$. Hence, there exists $r_1 > 0$ such that if $h_1 = f^{-1}$, then $h_1(\mathfrak{M}_0) \cap (\mathfrak{M} \times B_{r_1}^m \times \mathbf{R}) = \mathfrak{M}_0 \times B_{r_1}^m$. This completes the construction of h_1.

By Lemma 2.4 there exists a splitting of hh_1^{-1}, $\mathfrak{M} \times B_1^m \times \mathbf{R} = \mathfrak{M}_1' \cap \mathfrak{M}_2'$, such that the inclusions $\mathfrak{M}_0' \hookrightarrow \mathfrak{M} \times B_1^m \times \mathbf{R}$, $\mathfrak{M}_0 \hookrightarrow \mathfrak{M} \times \{0\} \times \mathbf{R}$ and $\mathfrak{M}_0' - \mathfrak{M}_0' \hookrightarrow \mathfrak{M} \times (d\mathrm{B}_1^m - \{0\}) \times \mathbf{R}$ are homotopy equivalences. Therefore, $\mathfrak{M} \times B_1^m \times \mathbf{R} = h_1^{-1}(\mathfrak{M}_1') \cup h_1^{-1}(\mathfrak{M}_2')$ is the desired splitting of h. This completes the proof of Theorem 2.1.

3. QPL-flat embeddings. The purpose of this section is to prove Theorem 2, which can be restated as follows.

THEOREM 2. *Let N^n be a compact PL manifold and let (M^{n+m}, N^n) be a flat PL manifold pair. If h: $N \times Q \to M \times Q$ is a codimension m flat embedding homotopic to the inclusion, then there exists an $l \geq 0$ and a codimension m PL flat embedding g: $N \times I^l \to M \times I^l$ such that h is isotopic to the QPL embedding $g \times \mathrm{Id}_{Q_{l+1}}$.*

COROLLARY 3.1. *Let M^{n+m} be a PL manifold and let f: $Q \to M \times Q$ be a codimension m locally flat embedding. Then there exists an $l \geq n$ and a codimension m PL flat embedding g: $I^l \to M \times I^{l-n}$ such that* $g \times \mathrm{Id}$: $I^l \times Q_{l+1} \to (M \times I^{l-n}) \times Q_{l-n+1}$ *is isotopic to* f: $Q \to M \times Q$.

COROLLARY 3.2. *Any two codimension m ($m \neq 2$) locally flat embeddings f_0, f_1: $Q \to Q$ are ambient isotopic.*

The following lemma is the main ingredient in the proof of Theorem 2. Its proof is virtually identical to the proof of Theorem 2 of [**5**].

LEMMA 3.1. *Let M^{n+k+1} and N^n be PL manifolds and let $K \subset N$ be a compact set. Let α: $N \times B_1^k \to \partial M$ be a PL embedding and let h: $I \times N \times B_1^k \times Q \to M \times Q$ be an open embedding such that $h = \alpha \times \mathrm{Id}_Q$ on $\{0\} \times N \times B_1^k \times Q$. Then there exists an $l \geq 0$, a compact PL submanifold N_1 of N with $K \subset \mathrm{Int}\, N_1$, a PL embedding g: $I \times N_1 \times B_1^k \times I^l \to M \times I^l$ such that $g(I \times \mathrm{Int}\, N_1 \times B_1^k \times I^l)$ is open in $M \times I^l$, and an open embedding f: $I \times N \times B_1^k \times Q \to M \times Q$ such that*

(1) *$f = \alpha \times \mathrm{Id}_Q$ on $\{0\} \times N \times B_1^k \times Q$,*
(2) *$f = g \times \mathrm{Id}_{Q_{l+1}}$ on $I \times N_1 \times B_1^k \times Q$,*
(3) *$f = h$ outside a neighborhood U of $I \times N_1 \times B_1^k \times Q$, and*
(4) *f is isotopic to h relative to*

$$\big(\{0\} \times N \times B_1^k \times Q\big) \cup \big((I \times N \times B_1^k \times Q) - U\big).$$

Proof of Theorem 2. Let α: $N \times \mathbf{R}^m \to M$ be a PL embedding such that $\alpha = \mathrm{Id}$ on $N \times \{0\}$. Let $\tilde{h}$: $N \times \mathbf{R}^m \times Q \to M \times Q$ be an open embedding such that $\tilde{h} = h$ on $N \times \{0\} \times Q$. Since

$$\alpha \times \mathrm{Id}_Q\colon (N \times B_2^m) \times \{0\} \times Q_2 \to M \times I_1 \times Q_2$$

and

$$\tilde{h} | N \times B_2^m \times \{0\} \times Q_2\colon N \times B_2^m \times \{0\} \times Q_2 \to M \times I_1 \times Q_2$$

are homotopic Z-embeddings, we may assume without loss of generality that $\tilde{h}$: $I_1 \times (N \times \mathring{B}_2^m) \times Q_2 \to (I_1 \times M) \times Q_2$ is an open embedding such that $\tilde{h} = \alpha \times \mathrm{Id}_{Q_2}$ on $\{0\} \times (N \times \mathring{B}_2^m) \times Q_2$, where α: $N \times \mathring{B}_2^m \to \partial(I_1 \times M)$ is a PL embedding. By Lemma 3.1 there exists a PL embedding $\tilde{g}$: $N \times B_1^m \times I^l \to M \times I^l$ such that $\tilde{g}(N \times \mathring{B}_1^m \times I^l)$ is open in $M \times I^l$ and $\tilde{g} \times \mathrm{Id}_{Q_{l+1}}$: $N \times B_1^m \times Q \to M \times Q$ is isotopic to $\tilde{h} | N \times B_1^m \times Q$. Therefore, $g = \tilde{g} | N \times \{0\} \times I^l$ is a codimension m PL flat embedding and $g \times \mathrm{Id}_{Q_{l+1}}$ is isotopic to h. This concludes the proof of Theorem 2.

4. Triangulating flat Q-manifold pairs. Throughout this section, by a flat Q manifold pair $(\mathfrak{M}, \mathfrak{N})$, we mean a flat Q-manifold pair for which $\mathfrak{N}$ is compact and $\mathfrak{M}$ can be triangulated with a PL manifold. We say that the pair $(\mathfrak{M}, \mathfrak{N})$ can be triangulated if there exists a flat PL manifold pair (M, N) and a homeomorphism $h: (\mathfrak{M}, \mathfrak{N}) \to (M, N) \times Q$.

THEOREM 1. *Every flat Q-manifold pair can be triangulated.*

Proof. Let $(\mathfrak{M}, \mathfrak{N})$ be a flat Q-manifold pair. Let φ: Bd $I^n \to \mathfrak{N}$ be a continuous map and let $\tilde{\varphi}$: Bd $I^n \times Q \to \mathfrak{N}$ be the composition Bd $I^n \times Q \overset{\text{proj}}{\to}$ Bd $I^n \overset{\varphi}{\to} \mathfrak{N}$. Let ψ: Bd $i^n \times Q \to \mathfrak{N}$ be a Z-embedding homotopic to $\tilde{\varphi}$. Since $\psi(\text{Bd } I^n \times Q) \subset \mathfrak{N}$ is a Z-set, there is an open embedding f: Bd $I^n \times Q \times [0, 1) \to \mathfrak{N}$ such that $f = \psi$ on Bd $I^n \times Q \times \{0\}$. Furthermore, since the inclusion $\mathfrak{N} \hookrightarrow \mathfrak{M}$ is a flat embedding, there exists an open embedding $\tilde{f}$: (Bd $I^n \times Q \times [0, 1) \times B_1^m$, Bd $I^n \times Q \times [0, 1) \times \{0\}) \to (\mathfrak{M}, \mathfrak{N})$ such that $\tilde{f} = f$ on Bd $I^n \times Q \times [0,1) \times \{0\}$. Let $\tilde{\psi} = \tilde{f}|$Bd $I^n \times Q \times \{0\} \times B_1^m$. Put

$$\mathfrak{M}_1 = \mathfrak{M} \cup_{\tilde{\psi}} (I^n \times Q \times B_1^m) \quad \text{and} \quad \mathfrak{N}_1 = \mathfrak{N} \cup_{\psi} (I^n \times Q \times \{0\}).$$

Note that $(\mathfrak{M}_1, \mathfrak{N}_1)$ is a flat Q-manifold pair and $\mathfrak{N}_1$ has the homotopy type of $\mathfrak{N} \cup_\varphi I^n$. Here is the main step in the proof. After having established this, it will be easy to deduce Theorem 1.

Assertion. If $(\mathfrak{M}_1, \mathfrak{N}_1)$ can be triangulated, then so can $(\mathfrak{M}, \mathfrak{N})$.

Proof of Assertion. Let $(M^{\lambda+m}, N^\lambda)$ be a flat PL manifold pair and let h: $(\mathbf{R}^n \times Q \times B_1^m, \mathbf{R}^n \times Q \times \{0\}) \to (M \times Q, N \times Q)$ be an open embedding. Assume, without loss of generality, that $(\mathfrak{M}_1, \mathfrak{N}_1) = (M \times Q, N \times Q)$ and

$$(\mathfrak{M}, \mathfrak{N}) = \left(M \times Q - h\left(\mathring{B}_1^n \times Q \times B_1^m\right), N \times Q - h\left(\mathring{B}_1^n \times Q \times \{0\}\right)\right).$$

By Theorem 2.1, there exists a straight PL submanifold A of $M \times I^q$ such that if $B = A \cap (N \times I^q)$, then

(a) B is a straight PL submanifold of $N \times I^q$,

(b) $(M \times I^q - \text{Int } A, N \times I^q - \text{Int } B)$ is a flat PL manifold pair,

(c)

$$h\left(B_{1/2}^n \times Q \times B_1^m\right) \subset \text{Int } A \times Q_{q+1} \subset A \times Q_{q+1}$$
$$\subset h\left(\mathring{B}_2^n \times Q \times B_1^m\right),$$

(d)

$$h\left(B^n_{1/2} \times Q \times \{0\}\right) \subset \operatorname{Int} B \times Q_{q+1} \subset B \times Q_{q+1}$$
$$\subset h\left(\mathring{B}^n_2 \times Q \times \{0\}\right),$$

and

(e) the inclusions $\operatorname{Bd} A \times Q_{q+1} \hookrightarrow h((\mathring{B}^n_2 - B^n_{1/2}) \times Q \times B^m_1)$ and $\operatorname{Bd} B \times Q_{q=1} \hookrightarrow h((\mathring{B}^n_2 - B^n_{1/2}) \times Q \times \{0\})$ are homotopy equivalences.

By (a) thru (e), there is a homotopy equivalence between Q-manifolds

$$\tau\colon h\left(\left(B^n_2 - \mathring{B}^n_1\right) \times Q \times B^m_1\right) \to h\left(B^n_2 \times Q \times B^m_1\right) - \operatorname{Int} A \times Q_{q+1}$$

such that

$$\tau \mid h\left(\partial B^n_2 \times Q \times B^m_1\right)\colon h\left(\partial B^n_2 \times Q \times B^m_1\right)$$
$$\to h\left(B^n_2 \times Q \times B^m_1\right) - \operatorname{Int} A \times Q_{q+1}$$

is homotopic to the inclusion. Hence, there is a homeomorphism

$$\tilde{g}\colon M \times Q - h\left(\mathring{B}^n_1 \times Q \times B^m_1\right) \to \left(M \times I^n - \operatorname{Int} A\right) \times Q_{q+1}$$

such that $\tilde{g} = \operatorname{Id}$ on $M \times Q - h(\mathring{B}^n_2 \times Q \times B^m_1)$. Similarly there is a homeomorphism

$$g\colon N \times Q - h\left(\mathring{B}^n_1 \times Q \times \{0\}\right) \to \left(N \times I^n - \operatorname{Int} B\right) \times Q_{q+1}$$

such that $g = \operatorname{Id}$ on $N \times Q - h(\mathring{B}^n_2 \times Q \times \{0\})$.

Let

$$r_t\colon M \times Q - h\left(\mathring{B}^n_1 \times Q \times B^m_1\right) \to M \times Q - h\left(\mathring{B}^n_1 \times Q \times B^m_1\right)$$

be the strong deformation retraction of $M \times Q - h(\mathring{B}^n_1 \times Q \times B_1m)$ onto $M \times Q - h(\mathring{B}^n_2 \times Q \times B^n_1)$ along the rays of

$$h\left(\left(B^n_2 - \mathring{B}^n_1\right) \times Q \times B^m_1\right).$$

It is not difficult to prove, using r_t, that the following map is homotopic to the inclusion:

$$\left(N \times I^q - \operatorname{Int} B\right) \times Q_{q+1} \overset{g^{-1}}{\to} N \times Q - h\left(\mathring{B}^n_1 \times Q \times \{0\}\right)$$
$$\hookrightarrow M \times Q - h\left(\mathring{B}^n_1 \times Q \times B^m_1\right) \overset{\tilde{g}}{\to} \left(M \times Q^q - \operatorname{Int} A\right) \times Q_{q+1}.$$

Therefore, by Theorem 2, the pair

$$\left(M \times Q - h\left(\mathring{B}^n_1 \times Q \times B^m_1\right), N \times Q - h\left(\mathring{B}^n_1 \times Q \times \{0\}\right)\right) = (\mathfrak{M}, \mathfrak{N})$$

can be triangulated. This concludes the proof of the Assertion.

We now return to the proof of Theorem 1. Since every compact ANR can be transformed into a compact contractible ANR by attaching a finite number of cells, it follows that we can construct a sequence of flat Q-manifold pairs $(\mathfrak{M}, \mathfrak{N}) = (\mathfrak{M}_0, \mathfrak{N}_0), (\mathfrak{M}_1, \mathfrak{N}_1), \ldots, (\mathfrak{M}_p, \mathfrak{N}_p)$, where each pair $(\mathfrak{M}_{i+1}, \mathfrak{N}_{i+1})$ is obtained from $(\mathfrak{M}_i, \mathfrak{N}_i)$ by attaching a copy of $(I^n \times Q \times B_1^m, I^n \times Q \times \{0\})$ as above, and $\mathfrak{N}_p$ is homeomorphic to Q. By Corollary 3.1 and, of course, since locally flat embeddings of Hilbert cubes are flat [3], the pair $(\mathfrak{M}_p, \mathfrak{N}_p)$ can be triangulated and, therefore, by the Assertion, the pair $(\mathfrak{M}, \mathfrak{N})$ can be triangulated. This concludes the proof of Theorem 1.

REMARK. For flat compact Q-manifold pairs, there is a completely different proof of Theorem 1 which avoids the use of the Relative Splitting Theorem 2.1.

5. QPL homeomorphisms. Let M and N be compact PL manifolds. Chapman [5] proved that a homeomorphism $h: N \times Q \to M \times Q$ is isotopic to a QPL homeomorphism if and only if h is homotopic to a QPL homeomorphism. The purpose of this section is to obtain the same result at the level of flat Q-manifold pairs. We shall relay heavily on Chapman's paper [5].

Our first task is to prove the following theorem.

THEOREM 5.1. *Let (M^{n+m+1}, M_0^{n+1}) be a flat compact PL manifold pair, N^n a compact PL manifold and $\alpha: N \times (B_1^m, \{0\}) \to (\partial M, \partial M_0)$ a PL embedding. Let $h: I \times N \times (B_1^m, \{0\}) \times Q \to (M, M_0) \times Q$ be a homeomorphism such that $h = \alpha \times \mathrm{Id}_Q$ on $\{0\} \times N \times B_1^m \times Q$. Then there exists an $l \geq 0$ and a PL homeomorphism $g: I \times N \times (B_1^m, \{0\}) \times I^l \to (M, M_0) \times I^l$ such that $g = \alpha \times \mathrm{Id}_{I^l}$ on $\{0\} \times N \times B_1^m \times I^l$ and h is isotopic by pairs to $g \times \mathrm{Id}_{Q_{l+1}}$ relative to $\{0\} \times N \times B_1^m \times Q$.*

The proof of Theorem 5.1 requires some lemmas.

LEMMA 5.1. *Let $(M^{n+k+m+1}, M_0^{n+k+1})$ be a flat compact PL manifold pair, $\alpha: \mathbf{R}^n \times (B_1^{k+m}, B_1^k) \to (\partial M, \partial M_0)$ a PL embedding and $h: I \times \mathbf{R}^n \times (B_1^{k+m}, B_1^k) \times Q \to (M, M_0) \times Q$ an open embedding such that $h = \alpha \times \mathrm{Id}_Q$ on $\{0\} \times \mathbf{R}^n \times B_1^{k+m} \times Q$. Then there exists an $l \geq 0$ and a straight submanifold $A \subset M \times I^l$ such that:*

(1) *$B = A \cap (N \times I^l)$ is a straight submanifold of $N \times I^l$,*

(2) *$\mathrm{Bd}\, B = \mathrm{Bd}\, A \cap (N \times I^l)$ (where $\mathrm{Bd}\, B$ is computed in $N \times I^l$),*

(3)

$$h\left(I \times B^n_{1/2} \times B^{k+m}_1 \times Q\right) \subset \operatorname{Int} A \times Q_{l+1} \subset A \times Q_{l+1}$$
$$\subset h\left(I \times \mathring{B}^n_2 \times B^{k+m}_1 \times Q\right),$$

(4) *the following inclusions are homotopy equivalences*:

$$\operatorname{Bd} A \times Q_{l+1} \hookrightarrow h\left(I \times \left(\mathring{B}^n_2 - B^n_{1/2}\right) \times B^{k+m}_1 \times Q\right),$$
$$\operatorname{Bd} B \times Q_{l+1} \hookrightarrow h\left(I \times \left(\mathring{B}^n_2 - B^n_{1/2}\right) \times B^k_1 \times Q\right),$$

and

$$(\operatorname{Bd} A - \operatorname{Bd} B) \times Q_{l+1} \hookrightarrow h\left(I \times \left(\mathring{B}^n_2 - B^n_{1/2}\right) \times \left(B^{k+m}_1 - B^k_1\right) \times Q\right),$$

(5) *there exists an open PL embedding* φ: $N \times I^l \times \mathbf{R}^m \to M \times I^l$ *such that* $\varphi = \operatorname{Id}$ *on* $N \times I^l \times \{0\}$ *and*

$$\operatorname{Bd} A \cap \varphi\left(N \times I^l \times \mathbf{R}^m\right) = \varphi(\operatorname{Bd} B \times \mathbf{R}^m),$$

(6)

$$(A, B) \cap \left[\alpha\left(\mathbf{R}^n \times \left(B^{k+m}_1, B^k_1\right)\right) \times I^l\right] = \alpha\left(B^n_1 \times \left(B^{k+m}_1, B^k_1\right)\right) \times I^l,$$

and

(7)

$$(\operatorname{Bd} A, \operatorname{Bd} B) \cap \left[\alpha\left(\mathbf{R}^n \times \left(B^{k+m}_1, B^k_1\right)\right) \times I^l\right]$$
$$= \alpha\left(\partial B^n_1 \times \left(B^{k+m}_1, B^k_1\right)\right) \times I^l.$$

Proof. The proof is similar to the proof of Lemma 2.3 of [**5**], but using the Relative Splitting Theorem 2.1 and its proof instead of the Chapman Splitting Theorem [**6**].

If X is a compact space, we define Cone $X = X \times [0,1]/X \times \{1\}$ and we shall assume $X \times \{0\} \subset \operatorname{Cone} X$.

LEMMA 5.2. *Let* h: $I \times (B^m_1, \{0\}) \times Q \to I \times (B^m_1, \{0\}) \times Q$ *be a homeomorphism such that* $h = \operatorname{Id}$ *on* $\{0\} \times B^m_1 \times Q$. *Then* h *is ambient isotopic by pairs to* Id *relative to* $\{0\} \times B^m_1 \times Q$.

Proof. Since there is a homeomorphism δ: $I \times (B^m_1, \{0\}) \times Q \to \operatorname{Cone}((B^m_1, \{0\}) \times Q)$ such that $\delta = \operatorname{Id}$ on $\{0\} \times B^m_1 \times Q$ (for the construction of δ, see proof of Lemma 5.1, IV of [**3**]), the problem reduces to proving that if h': $\operatorname{Cone}((B^m_1, \{0\}) \times Q) \to \operatorname{Cone}((B^m_1, \{0\}) \times Q)$ is a homeomorphism such that $h' = \operatorname{Id}$ on $\{0\} \times B^m_1 \times Q$, then h' is ambient isotopic by pairs to Id relative to $\{0\} \times B^m_1 \times Q$. But this is just a version of the well-known Alexander trick.

LEMMA 5.3. *Let $\mathfrak{N}$ be a compact Q-manifold with $\pi_1(\mathfrak{N})$ free or free abelien and let f: $I \times \mathfrak{N} \times \mathbf{R}^m \to I \times \mathfrak{N} \times B_1^m$ be an open embedding such that*

(1) $f = \mathrm{Id}$ *on* $\{0\} \times \mathfrak{N} \times \{0\}$,

(2) $f(I \times \mathfrak{N} \times \mathbf{R}^m) \cap \{0\} \times \mathfrak{N} \times B_1^m = f(\{0\} \times \mathfrak{N} \times \mathbf{R}^m)$, *and*

(3) *the inclusion*

$$\{0\} \times \mathfrak{N} \times (B_1^m - \{0\}) \hookrightarrow I \times \mathfrak{N} \times B_1^m - f(I \times \mathfrak{N} \times \{0\})$$

is a homotopy equivalence.

Then there is a homeomorphism h: $I \times \mathfrak{N} \times B_1^m \to I \times \mathfrak{N} \times B_1^m$ such that $h = \mathrm{Id}$ on $\{0\} \times \mathfrak{N} \times B_1^m$ and $hf = \mathrm{Id}$ on $I \times \mathfrak{N} \times \{0\}$.

Proof. Using the construction of h_2 in the proof of Theorem 2 of **[3]** and the construction of u in the proof of Assertion 1, Lemma 4.3 of **[10]**, we may assume without loss of generality that $f = \mathrm{Id}$ on $\{0\} \times \mathfrak{N} \times B_r^m$ for some $r > 0$. Since the inclusion $\{0\} \times \mathfrak{N} \times (B_1^m - \{0\}) \hookrightarrow I \times \mathfrak{N} \times B_1^m - f(I \times \mathfrak{N} \times \{0\})$ is a homotopy equivalence, it follows that the inclusion $f(I \times \mathfrak{N} \times \partial B_r^m) \hookrightarrow I \times \mathfrak{N} \times B_1^m - f(I \times \mathfrak{N} \times \mathring{B}_r^m)$ is a homotopy equivalence and, consequently a simple homotopy equivalence. Hence, there exists a homeomorphism

$$\lambda: I \times \mathfrak{N} \times (B_1^m - \mathring{B}_r^m) \to I \times \mathfrak{N} \times B_1^m - f(I \times \mathfrak{N} \times \mathring{B}_r^m)$$

such that $\lambda = f$ on $I \times \mathfrak{N} \times \partial B_r^m$. Furthermore, we may choose λ in such a way that $\lambda = \mathrm{Id}$ on $\{0\} \times \mathfrak{N} \times (B_1^m - \mathring{B}_r^m)$. Then λ and $f \mid I \times \mathfrak{N} \times B_r^m$ piece together to give a homeomorphism whose inverse is our desired homeomorphism h.

LEMMA 5.4. *Let f: $B_3^n \times \{0\} \times Q \to B_3^n \times B_2^m \times Q$ be a locally flat embedding such that*

(1) $f = \mathrm{Id}$ *on* $B_1^n \times \{0\} \times Q$,

(2) $f(B_3^n \times \{0\} \times Q) \cap B_1^n \times B_2^m \times Q = B_1^n \times \{0\} \times Q$,

(3) *the inclusion*

$$\partial B_1^n \times (B_2^m - \{0\}) \times Q \hookrightarrow (B_3^n - \mathring{B}_1^n) \times B_2^m \times Q - f(B_3^n \times \{0\} \times Q)$$

is a homotopy equivalence, and

(4) *there exists an open embedding ϕ: $\mathring{B}_2^n \times \mathring{B}_1^m \times Q \to B_3^n \times B_2^m \times Q$ such that $\phi = f$ on $\mathring{B}_2^n \times \{0\} \times Q$ and $\phi = \mathrm{Id}$ on $B_1^n \times \mathring{B}_1^m \times Q$.*

Then there exists a homeomorphism h: $B_3^n \times B_2^m \times Q \to B_3^n \times B_2^m \times Q$ such that $h = \mathrm{Id}$ on $B_1^n \times B_2^m \times Q$ and $hf = \mathrm{Id}$.

Proof. By Theorem 1 of [**3**], f is a flat embedding. Moreover, by Lemma 4.3 of [**10**], there exists an open embedding $\varphi\colon B_3^n \times \mathring{B}_r^m \times Q \to B_3^n \times B_2^m \times Q$ such that $\varphi = f$ on $B_3^n \times \{0\} \times Q$, $\varphi = \text{Id}$ on $B_1^n \times \mathring{B}_r^m \times Q$ for some $r > 0$, and $\varphi(B_3^n \times \mathring{B}_r^m \times Q) \cap B_1^n \times B_2^m \times Q = B_1^n \times \mathring{B}_r^m \times Q$. Lemma 5.4 now follows from Lemma 5.3.

LEMMA 5.5. *Let*

$$f\colon I \times B_1^n \times (B_1^m, \{0\}) \times Q \to I \times \mathbf{R}^n \times (B_1^m, \{0\}) \times Q$$

be an embedding such that

(1) $$f = \text{Id} \quad \textit{on } \{0\} \times B_1^n \times B_1^m \times Q,$$

(2) $$f(I \times B_1^n \times B_1^m \times Q) \cap (\{0\} \times \mathbf{R}^n \times B_1^m \times Q) = \{0\} \times B_1^n \times B_1^m \times Q,$$

(3) $$\text{Bd}\, f(I \times B_1^n \times B_1^m \times Q) = f(I \times \partial B_1^n \times B_1^m \times q)$$

is bicollared in $I \times \mathbf{R}^n \times B_1^m \times Q$ *and* $f(I \times \partial B_1^n \times \{0\} \times Q)$ *is bicollared in* $I \times \mathbf{R}^n \times \{0\} \times Q$,

(4) *the inclusions*

$$f(I \times \partial B_1^n \times B_1^m \times dQ) \hookrightarrow I \times (\mathring{B}_2^n - B_{1/2}^n) \times B_1^m \times Q,$$

$$f(I \times \partial B_1^n \times (B_1^m - \{0\}) \times Q) \hookrightarrow I \times (\mathring{B}_2^n - B_{1/2}^n) \times (B_1^m - \{0\}) \times Q,$$

and

$$f(I \times \partial B_1^n \times \{0\} \times Q) \hookrightarrow I \times (\mathring{B}_2^n - B_{1/2}^n) \times \{0\} \times Q$$

are homotopy equivalences, and

(5) *there exists an open embedding*

$$\phi\colon I \times (\mathring{B}_5^n - B_{1/2}^n) \times \mathbf{R}^m \times Q \to I \times (\mathring{B}_5^n - B_{1/2}^n) \times B_1^m \times Q$$

such that $\phi = \text{Id}$ *on* $I \times (\mathring{B}_5^n - B_{1/2}^n) \times \{0\} \times Q$, *and*

(i) $$\phi(I \times (\mathring{B}_5^n - B_{1/2}^n) \times \mathbf{R}^m \times Q) \cap f(I \times \partial B_1^n \times B_1^m \times Q) = \phi(f(I \times \partial B_1^n \times \{0\} \times Q) \times \mathbf{R}^m),$$

(ii)

$$\phi(I \times (\mathring{B}_5^n - B_{1/2}^n) \times \mathbf{R}^m \times Q) \cap (\{0\} \times (\mathring{B}_5^n - B_{1/2}^n) \times B_1^m \times Q) = \phi(\{0\} \times (\mathring{B}_5^n - B_{1/2}^n) \times \mathbf{R}^m \times Q).$$

Then there is a homeomorphism

$$h\colon I \times \mathbf{R}^n \times (B_1^m, \{0\}) \times Q \to I \times \mathbf{R}^n \times (B_1^m, \{0\}) \times Q$$

such that $h = f$ *on* $I \times B_1^n \times B_1^m \times Q$ *and* h *is isotopic by pairs to* Id *relative to* $(\{0\} \times \mathbf{R}^n \times B_1^m \times Q) \cup (I \times (\mathbf{R}^n - \mathring{B}_3^n) \times B_1^m \times Q)$.

Proof. We will construct a homeomorphism

$$h\colon I \times \mathbf{R}^n \times (B_1^m, \{0\}) \times Q \to I \times \mathbf{R}^n \times (B_1^m, \{0\}) \times Q$$

such that $h = f$ on $I \times B_1^n \times B_1^m \times Q$ and $h = \mathrm{Id}$ on $(\{0\} \times \mathbf{R}^n \times B_1^m \times Q) \cup (I \times (\mathbf{R}^n - \mathring{B}_3^n) \times B_1^m \times Q)$. Then the isotopy follows from Lemma 5.2.

As in the proof of Lemma 3.1 of [5], there are homeomorphisms

$$\lambda_0\colon I \times \mathbf{R}^n \times B_1^m \times Q \to I \times \mathbf{R}^n \times B_1^m \times Q$$

and

$$\lambda_1\colon I \times \mathbf{R}^n \times \{0\} \times Q \to I \times \mathbf{R}^n \times \{0\} \times Q$$

such that $\lambda_0 = f$ on $I \times B_1^n \times B_1^m \times Q$, $\lambda_1 = f$ on $I \times B_1^n \times \{0\} \times Q$, $\lambda_0 = \mathrm{Id}$ on $(\{0\} \times \mathbf{R}^n \times B_1^m \times Q) \cup (I \times (\mathbf{R}^n - \mathring{B}_3^n) \times B_1^m \times Q)$, and $\lambda_1 = \mathrm{Id}$ on $(\{0\} \times \mathbf{R}^n \times \{0\} \times Q) \cup (I \times (\mathbf{R}^n - \mathring{B}_3^n) \times \{0\} \times Q)$. Unfortunately, $\lambda_0^{-1}((I \times B_3^n \times \{0\} \times Q) - f(I \times \mathring{B}_1^n \times \{0\} \times Q))$ may not be contained in $I \times (B_3^n - \mathring{B}_1^n) \times \{0\} \times Q$. In order to modify λ_0^{-1} to obtain h^{-1}, by using Lemma 5.4, we need to prove the following facts:

(I) the inclusion

$$\begin{aligned}&(I \times \partial B_3^n \times (B_1^m - \{0\}) \times Q)\\ &\qquad \cup \big(\{0\} \times (B_3^n - \mathring{B}_1^n) \times (B_1^m - \{0\}) \times Q\big)\\ &\qquad \cup f(I \times \partial B_1^n \times (B_1^m - \{0\}) \times Q)\\ &\qquad\qquad \hookrightarrow (I \times B_3^n \times (B_1^m - \{0\}) \times Q) - f(I \times \mathring{B}_1^n \times B_1^m \times Q)\end{aligned}$$

is a homotopy equivalence, and

(II) there is a neighborhood U of

$$\begin{aligned}A = {}&(I \times \partial B_3^n \times \{0\} \times Q) \cup \big(\{0\} \times (B_3^n - \mathring{B}_1^n) \times \{0\} \times Q\big)\\ &\cup f(I \times \partial B_1^n \times \{0\} \times Q)\end{aligned}$$

in $(I \times B_3^n \times \{0\} \times Q) - f(I \times \mathring{B}_1^n \times \{0\} \times Q)$ and an open embedding

$$\Phi\colon U \times \mathbf{R}^m \to (I \times B_3^n \times B_1^m \times Q) - f(I \times \mathring{B}_1^n \times B_1^m \times Q)$$

such that

(i) $\Phi = \text{Id}$ on $U \times \{0\}$,

(ii) $\Phi(U \times \mathbf{R}^m) \cap (A \times B_1^m) = \Phi(A \times \mathbf{R}^m)$, and

(iii) $\Phi = \text{Id}$ on $(I \times \partial B_3^n \times \mathring{B}_r^m \times Q) \cup (\{0\} \times (B_3^n - \mathring{B}_1^n) \times \mathring{B}_r^m \times dQ)$ and for each $(t, x, q, y) \in I \times \partial B_1^n \times Q \times \mathring{B}_r^m$ $\Phi(f(t, x, o, q), y) = f(t, x, y, q)$, for some $r > 0$.

Proof of (I). It follows from the fact that the inclusion

$$f(I \times \partial B_1^n \times (B_1^m - \{0\}) \times Q) \hookrightarrow I \times \left(\mathring{B}_2^n - B_{1/2}^n\right) \times (B_1^m - \{0\}) \times Q$$

is a homotopy equivalence.

Proof of (II). Using the fact that there is a homeomorphism

$$\varphi: I \times B_5^n \times B_1^m \times Q \to I \times B_5^n \times B_1^m \times Q$$

such that $\varphi^{-1} = \text{Id}$ on $(\{0\} \times B_5^n \times B_1^m \times Q) \cup (I \times \partial B_5^n \times B_1^m \times Q)$ and $\varphi^{-1} = f$ on $I \times B_1^n \times B_1^m \times Q$, it is not difficult to see that we may assume, without loss of generality, there exists $r > 0$ such that $\phi = \text{Id}$ on $\{0\} \times (B_4^n - \mathring{B}_{3/4}^n) \times \mathring{B}_r^m \times Q$ and, for each $(t, x, q, y) \in I \times \partial B_1^n \times Q \times \mathring{B}_r^m$, $\phi(f(t, x, o, q), y) = f(t, x, y, q)$ (see first part of the proof of Lemma 5.3).

The desired open embedding Φ can be obtained from ϕ by observing that there is a homeomorphism

$$\tau: I \times B_5^n \times (B_1^m, \{0\}) \times Q \to I \times B_3^n \times (B_1^m, \{0\}) \times Q$$

such that $\tau = \text{Id}$ on $I \times B_2^n \times B_1^m \times Q$,

$$\tau\left(\{0\} \times \left(B_4^n - \mathring{B}_2^n\right) \times B_1^m \times Q\right) = \left(\{0\} \times \left(B_3^n - \mathring{B}_2^n\right) \times B_1^m \times Q\right) \cup (I \times \partial B_3^n \times B_1^m \times Q),$$

and

$$\tau\left(\{0\} \times \left(B_5^n - \mathring{B}_4^n\right) \times B_1^m \times Q\right) = \left(\{1\} \times \left(B_3^n - \mathring{B}_{5/2}^n\right) \times B_1^m \times Q\right).$$

This completes the proof of Lemma 5.5.

Proof of Theorem 5.1. The proof of Theorem 5.1 is virtually identical to the proof of Theorem 2 of **[5]** but using our Lemma 5.1 instead of their Lemma 2.3, our Lemma 5.5 instead of their Lemmas 3.1 and 5.3.

Our next step is to prove the following theorem.

THEOREM 5.2. *Let* (M^{n+m+1}, M_0^{n+1}) *and* (N^{n+m}, N_0^n) *be flat compact PL manifold pairs and let* $\alpha: (N, N_0) \to (\partial M, \partial M_0)$ *be a PL embedding. Let* $h: I \times (N, N_0) \times Q \to (M, M_0) \times Q$ *be a homeomorphism such that* $h = \alpha \times \mathrm{Id}_Q$ *on* $\{0\} \times N \times Q$. *Then there exists an* $l \geq 0$ *and a PL homeomorphism* $g: I \times (N, N_0) \times I^l \to (M, M_0) \times I^l$ *such that* $g = \alpha \times \mathrm{Id}_{I^l}$ *on* $\{0\} \times N \times I^l$ *and h is isotopic by pairs to* $g \times \mathrm{Id}_{Q_{l+1}}$ *relative to* $\{0\} \times N \times I^l$.

The proof of Theorem 5.2 requires two more lemmas.

LEMMA 5.6. *Let* M^{n+k+1} *be a PL manifold,* $\alpha: \mathbf{R}^n \times B_1^k \to \partial M$ *a PL embedding and* $h: I \times \mathbf{R}^n \times B_1^k \times Q \to M \times Q$ *an open embedding such that* $h = \alpha \times \mathrm{Id}_Q$ *on* $\{0\} \times \mathbf{R}^n \times B_1^k \times Q$. *Then there exists a straight submanifold* $A \subset M \times I^l$, *a PL embedding* $g: I \times B_1^n \times B_1^k \times I^l \to A$ *such that* $g = \alpha \times \mathrm{Id}_{I^l}$ *on* $\{0\} \times B_1^n \times B_1^k \times I^l$, *and an open embedding* $f: I \times \mathbf{R}^n \times B_1^k \times Q \to M \times Q$ *such that*

(1) $f = \alpha \times \mathrm{Id}_Q$ *on* $\{0\} \times \mathbf{R}^n \times B_1^k \times Q$,

(2) $f = g \times \mathrm{Id}_{Q_{l+1}}$ *on* $I \times B_1^n \times B_1^k \times Q$, *and*

(3) *f is isotopic to h relative to* $(\{0\} \times \mathbf{R}^n \times B_1^k \times Q) \cup (I \times (\mathbf{R}^n - \mathring{B}_2^n) \times B_1^k \times Q)$.

Proof. The proof is contained in the proof of Theorem 2 of [**5**].

LEMMA 5.7. *Let* M^{n+2} *be a PL manifold,* N^n *a compact PL manifold,* $\alpha: \mathbf{R} \times N \to \partial M$ *a PL embedding and let* $h: I \times \mathbf{R} \times N \times Q \to M \times Q$ *be an open embedding such that* $h = \alpha \times \mathrm{Id}_Q$ *on* $\{0\} \times \mathbf{R} \times N \times Q$. *Then there exists a bicollared submanifold* $A \subset M \times I^l$, *a PL homeomorphism* $g: I \times \{0\} \times N \times I^l \to A$ *such that* $g = \alpha \times \mathrm{Id}_{I^l}$ *on* $\{0\} \times \{0\} \times N \times I^l$, *and an open embedding* $f: I \times \mathbf{R} \times N \times Q \to M \times Q$ *such that*

(1) $f = \alpha \times \mathrm{Id}_Q$ *on* $\{0\} \times \mathbf{R} \times N \times Q$,

(2) $f = g \times \mathrm{Id}_{Q_{l+1}}$ *on* $I \times \{0\} \times N \times Q$, *and*

(3) *f is isotopic to h relative to* $(\{0\} \times \mathbf{R} \times N \times Q) \cup (I \times (\mathbf{R} - \mathring{B}_1^1) \times N \times Q)$.

Proof. We consider a PL handle decomposition of N, $N_{-1} \subset N_0 \subset N_1 \subset \cdots \subset N_n = N$, where N_{-1} is a regular neighborhood of ∂N in N and each N_i is obtained from N_{i-1} by adding disjoint handles of index i.

Set

$$X_i = \left([0, \tfrac{1}{2}] \times \mathbf{R} \times N\right) \cup \left(I \times \mathbf{R} \times N_i\right) \subset I \times \mathbf{R} \times N$$

and

$$Y_i = \left([0, \tfrac{1}{2}] \times B^1_{n-i+1} \times N\right) \cup \left(I \times B^1_{n-i+1} \times N_i\right)$$
$$\subset I \times B^1_{n-i+1} \times N, \qquad -1 \le i \le n.$$

It is clear that there exists a collection $\{\varphi_j\}_1^{p_i}$ of PL embeddings φ_j: $I \times \mathbf{R} \times \mathbf{R}^{m-i} \times B^i_1 \to (I \times \mathbf{R} \times N) - \operatorname{Int} X_{i-1}$ such that

(a) $\varphi_j(I \times \mathbf{R} \times \mathbf{R}^{n-i} \times B^i) \times \partial X_{i-1} = \varphi_j(\{0\} \times \mathbf{R} \times \mathbf{R}^{n-i} \times B^i_1)$,

(b) $\varphi_j(I \times \mathbf{R} \times \mathbf{R}^{n-i} \times B^i_1) \cap \partial Y_{i-1} = \varphi_j(\{0\} \times B^1_{n-i+2} \times \mathbf{R}^{n-i} \times B^i_1)$,

(c)

$$X_i = X_{i-1} \cup \bigcup_1^{p_i} \varphi_j(I \times \mathbf{R} \times B^{n-i}_1 \times B^i_1),$$

(d)

$$Y_i = (Y_{i-1} \cap (I \times B^1_{n-i+1} \times N)) \cup \bigcup_1^{p_i} \varphi_j(I \times B^1_{n-i+1} \times B^{n-i}_1 \times B^i_1),$$

and

(e) the $\varphi_j(I \times \mathbf{R} \times \mathbf{R}^{n-i} \times B^i_1)$'s are pairwise disjoint.

By inductively working through these "handles" we will prove the following statement.

$S_i(-1 \le i \le n)$: There exists a straight submanifold $A_i \subset M \times I^{l_i}$, a PL embedding g_i: $Y_i \times I^{l_i} \to A_i$ such that $g = \alpha \times \mathrm{Id}_{I^{l_i}}$ on $\{0\} \times B^1_{n-i+1} \times N \times I^{l_i}$, and an open embedding f_i: $I \times \mathbf{R} \times N \times Q \to M \times Q$ such that $f_i = g_i \times \mathrm{Id}_{Q_{l_i+1}}$ on $Y_i \times Q$, and f_i is isotopic to f relative to $(\{0\} \times \mathbf{R} \times N \times Q) \cup (I \times (\mathbf{R} - B^1_{n+3}) \times N \times Q)$.

It is easy to establish S_{-1} (see proof of Theorem 2 of [5]). Furthermore, S_{i+1} can be obtained from S_i by applying Lemma 5.6 to the open embeddings

$$f_i\varphi_j\colon I \times \left(\mathring{B}^1_{n-i+1} \times \mathbf{R}^{n-i-1}\right) \times \left(B^{i+1}_1 \times I^{l_i}\right) \times Q_{l_i+1}$$
$$\to \left((M \times I^{l_i}) - \operatorname{Int} A_i\right) \times Q_{l_i+1}.$$

We finish the proof of Lemma 5.7 by letting h be h_n and A be

$$g_n\left(I \times \{0\} \times N \times I^{l_n}\right).$$

Proof of Theorem 5.2. Let us assume $N_0 \times \mathbf{R}^m$ is contained in N as an open subset. By Lemma 5.7 we may assume without loss of generality that there exists a straight submanifold A of M, containing M_0 as a flat submanifold, and a PL homeomorphism β: $I \times N_0 \times \partial B^m_1 \times Q \to \operatorname{Bd} A$ such that $\beta = \alpha$ on $\{0\} \times N_0 \times \partial B^m_1$, $h = \beta \times \mathrm{Id}_Q$ on $I \times N_0 \times \partial B^m_1 \times Q$, and $h(I \times N_0 \times (B_1^m, \{0\}) \times Q) = (A, M_0) \times Q$. Theorem 5.2 now follows by applying Theorem 5.1 to $h | I \times N_0 \times B_1^m \times Q$ and Theorem 2 of **[5]** to $h | I \times (N - (N_0 \times \mathring{B}_1^m)) \times Q$. This completes the proof of Theorem 5.2.

We will now prove Theorem 3, which can be restated as follows.

THEOREM 3. *Let (M^{n+m}, M_0^n) and (N^{n+m}, N_0^n) be flat compact PL manifold pairs. Let α: $(N, N_0) \to (M, M_0)$ be a PL homeomorphism and let h_t: $(N, N - N_0, N_0) \times Q \to (M, M - M_0, M_0) \times Q$ be a homotopy such that h_0 is a homeomorphism and $h_1 = \alpha \times \mathrm{Id}_Q$. Then h_0 is isotopic by pairs to a QPL homeomorphism.*

Proof. By Theorem 3.2 of **[10]**, h_0 is isotopic by pairs to a homeomorphism h: $(N, N_0) \times Q \to (M, M_0) \times Q$ such that $h = \alpha \times \mathrm{Id}_{Q_2}$ on $N \times \{0\} \times Q_2$. Hence, by Theorem 5.2, there exists an $l \geq 0$ and a PL homeomorphism g: $(N, N_0) \times I^l \to (M, M_0) \times I^l$ such that h is isotopic by pairs to $g \times \mathrm{Id}_{Q_{l+1}}$. This concludes the proof of Theorem 3.

REMARK. The hypothesis $h_t((N - N_0) \times Q) \subset (M - M_0) \times Q$, in the homotopy of Theorem 3, is necessary. To see this, let h: $\partial B_1^n \times Q \to \partial B_1^n \times Q$ be a homeomorphism which is not ambient isotopic and hence not homotopic to a QPL homeomorphism. Using a coordinate-switching technique, it is not difficult to construct a homeomorphism $\tilde{h}$: $B_1^n \times Q \to B_1^n \times Q$ such that $\tilde{h} = \mathrm{Id}$ on $\{0\} \times Q$ and $\tilde{h} = h$ on $\partial B_1^n \times Q$ (see proof of Lemma 3.1 of [3]). It is clear that there is a homotopy h_t: $(B_1^n, \{0\}) \times Q \to (B_1^n, \{0\}) \times Q$ such that $h_0 = \tilde{h}$ and $h_1 = \mathrm{Id}$. Nevertheless, $\tilde{h}$ is not ambient isotopic by pairs to a QPL homeomorphism, otherwise, h would be homotopic to a QPL homeomorphism, contradicting the choice of h.

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