PACIFIC JOURNAL OF MATHEMATICS
Vol. 114, No. 2, 1984

PROJECTIVE SPACE AS A BRANCHED COVERING OF THE SPHERE WITH ORIENTABLE BRANCH SET

ROBERT D. LITTLE

If $\mathbf{R}P^n$ is a branched covering of S^n with locally flat, orientable branch set, then $n = 1, 3$, or 7.

1. Introduction. Let M be a closed, orientable PL n-manifold. A theorem of Alexander [**2**] states that every such manifold is a piecewise linear branched covering of the n-sphere, S^n, i.e. there is a finite-to-one open PL map f: $M \to S^n$. The subset of M where f fails to be a local homeomorphism is called the singular set and the image of the singular set is called the branch set. Brand [**4**] suggests the problem of determining the values of n for which $\mathbf{R}P^n$, real projective n-space, is a branched covering of S^n with branch set a locally flat submanifold of S^n, and he shows that if such a covering exists, then $n = 2^t \pm 1$. We show that the values of n can be further limited if the branch set is orientable.

THEOREM 1.1. *If $\mathbf{R}P^n$ is a branched covering of S^n with locally flat, orientable branch set, then $n = 1, 3$, or 7.*

The converse of Theorem 1.1 is true in the cases $n = 1$ or 3: the identity map provides a branched covering of S^1 and Hilden and Montesinos have shown, independently, that every closed, orientable 3-manifold is a branched covering of S^3 with branch set a locally flat 1-manifold, [**6**] and [**9**]. Theorem 1.1 shows that if the branch set is required to be orientable, $n = 7$ is the only open case.

2. Normalized branched coverings. In [**5**], Brand proves a normalization theorem for smooth branched coverings. He uses his normalization theorem to show that there is a certain K-theoretic necessary condition for the existence of smooth branched coverings. In [**4**], Brand extended his normalization theorem to branched coverings with locally flat branch sets. He then showed that a branched covering with locally flat branch set is the pull-back of a universal smooth branched covering and hence must satisfy the same K-theoretic necessary conditions as a smooth branched covering.

If η is a 2-plane bundle over a complex X, let $\mu_k(\eta)$ be the 2-plane bundle obtained from η by the homomorphism μ_k: $O(2) \to O(2)$ given by

$\mu_k(z) = z^k$ for $z \in \mathrm{SO}(2)$ and $\mu_k(\tau) = \tau$ for

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If k is odd, μ_k and the Adams operation ψ^k agree on 2-plane bundles, ([**1**], p. 193). There is a map, $\mu_k\colon \eta \to \mu_k(\eta)$, given in local coördinates by $\mu_k(x, z) = (x, z^k)$, where $x \in X$ and $z \in \mathbf{R}^2 = \mathbf{C}$. The definition which follows is due to Brand, [**4**].

DEFINITION 2.1. A branched covering $f\colon M \to N$, with locally flat branch set B and branch cover $\tilde{B} = f^{-1}(B)$, is called a normalized branched covering if

(2.2) $f \mid M - \tilde{B}$ is a local PL isomorphism,

(2.3) $f \mid \tilde{B}$ is a local PL isomorphism, and

(2.4) if $\tilde{\xi}$ = normal bundle of $\tilde{B} \subset M$ and ξ = normal bundle of $B \subset N$, then for each component $\tilde{B}_i$ of $\tilde{B}$, there is an integer k_i such that $\mu_{k_i}(\tilde{\xi}) \mid \tilde{B}_i \cong f^*\xi \mid \tilde{B}_i$ and the map f maps a tubular neighborhood of $\tilde{B}$ into a tubular neighborhood of B via $D\tilde{\xi} \mid \tilde{B}_i \to D\mu_{k_i}(\tilde{\xi}) \mid \tilde{B}_i \cong Df^*\xi \mid \tilde{B}_i \to D\xi$.

If $f\colon M \to N$ is a branched covering, then $f \mid M - \tilde{B}$ is a covering space map and, if the degree of this covering is k, f is called a k-fold branched covering. It is possible to visualize a k-fold, normalized branched covering in the following way. For each component B_j of the branch set B, there is associated a partition of k, $\sum_{i=1}^{n_j} l_i k_i = k$, such that on part of $f^{-1}(B_j)$, the map f has the form $f(x, z) = (f(x), z^{k_i})$ in the local coördinates of the disc normal bundle, and this part of $f^{-1}(B_j)$ is an l_i-fold covering of B_j, [**4**]. The integers k_i are called the exponents of the branched covering. Each exponent is less than or equal to k, and so a normalized branched covering has only a finite number of distinct exponents.

For each integer $k \geq 2$, Brand defines a K-theoretic characteristic class η_k in $\mathrm{KO}(\mathrm{MO}(2))$ with the property that $\eta_k \mid \mathrm{BO}(2) = \gamma - \mu_k(\gamma)$, where γ is the universal 2-plane bundle. Let $\tilde{B}_{k_i} = \{x \in \tilde{B}\colon f$ maps the fibre over x of the tubular neighborhood of $\tilde{B}$ in M to the fibre over $f(x)$ of the tubular neighborhood of B in N by the map $z \to z^{k_i}\}$, where k_i is one of the exponents of f, $1 \leq i \leq m$. For each i, $1 \leq i \leq m$, let g_i be the composite map

$$M \xrightarrow{c_i} T(\tilde{\xi} \mid \tilde{B}_{k_i}) \xrightarrow{h_i} \mathrm{MO}(2), \tag{2.5}$$

where c_i is the collapsing map onto the Thom space and h_i is induced by the classifying map of the bundle $\tilde{\xi} \mid \tilde{B}_{k_i}$.

The theorem below is due to Brand, ([**4**], Theorems 0.1 and 6.4). The proof involves Brand's theorem that locally flat branched coverings of a fixed dimension are all pull-backs of a universal smooth branched covering, ([**4**], Lemma 6.3), the naturality of the stable block bundle $\tau M - f^*\tau N$ with respect to pull-backs, and Brand's K-theoretic necessary condition for smooth coverings, ([**5**], Theorem 1). Our statement is slightly different than Brand's original statement. We wish to focus attention on the exponents of the branched covering, and so we have observed that it clearly follows from the construction of the universal smooth covering, ([**4**], Lemma 6.3), that the realization of a PL covering preserves local degree.

THEOREM 2.6. (*Brand*, [**4**].) *If f: $M \to N$ is a branched covering with locally flat branch set, then, given a PL structure on N, there exists a unique PL structure on M making f a normalized branched covering. The stable block bundle $\tau M - f^*\tau N$ determined by this unique PL structure is related to the exponents of f by the equation*

$$\tau M - f^*\tau N = \sum_{i=1}^{m} g_i^*\eta_{k_i}. \tag{2.7}$$

A natural question now arises: Does formula (2.7) depend on the special PL structure for M which makes f a normalized branched covering? If one wants to use (2.7) to obtain information about branched coverings, there will typically be a strongly preferred stable tangent block bundle τM that one will want to use in (2.7). For example, if M and N are smooth manifolds, one might like to use the stable tangent bundle determined by a preferred smooth structure for M in (2.7). A form of (2.7) which does not depend on the PL structure of M can be obtained if stable block bundles are replaced by stable spherical fibrations. Before we can establish this fact, we need a preliminary lemma.

If M is a PL n-manifold, a homotopy PL structure for M is a PL n-manifold X together with a simple homotopy equivalence ϕ: $M \to X$. It is well-known that a block bundle determines a spherical fibration, ([**10**], p. 23). If β is a stable block bundle over M, let $J(\beta)$ denote the stable spherical fibration determined by β. In particular, if τX is the stable tangent block bundle of X, $J(\phi^*\tau X)$ denotes the stable spherical fibration determined by $\phi^*\tau X$.

LEMMA 2.8. *If ϕ: $M \to X$ and ψ: $M \to Y$ are two homotopy PL structures for M, then $J(\phi^*\tau X) = J(\psi^*\tau Y)$.*

Proof. Set $\nu = \phi^*\nu_X$, where ν_X is the stable normal block bundle of X, and note that the homotopy PL structure ϕ: $M \to X$ determines an element of the PL bordism group $\Omega_n(M, \nu)$ ([**11**], p. 32), of degree $+1$. The homotopy PL structure ψ: $M \to Y$ determines an element of degree $+1$ in $\Omega_n(M, \nu')$, where $\nu' = \psi^*\nu_Y$. Let ν_M be the Spivak normal fibration of M, ([**11**], p. 105). According to Spivak's theorem, $\Omega_n(M, \nu)$ contains an element of degree $+1$ if, and only if, ν is stably homotopy equivalent to ν_M, that is, if, and only if, $J(\nu) = \nu_M$, ([**11**], p. 105). If follows that $J(\nu) = J(\nu')$ and so $J(\phi^*\tau X) = J(\psi^*\tau Y)$ since in the group of stable fibre homotopy equivalence classes, we have $J(\phi^*\tau X \oplus \nu) = J(\psi^*\tau Y \oplus \nu')$, and Whitney sums are preserved.

If X is a complex, taking spherical fibrations yields a homomorphism $\widetilde{\mathrm{KO}}(X) \to J(X)$, ([7], p. 211). The terms on the right-hand side of (2.7) are stable block bundles determined by elements of $\widetilde{\mathrm{KO}}(M)$. It follows immediately from Lemma 2.8 and the fact that J preserves Whitney sums, that the image of the right-hand side of (2.7) under the J-homomorphism, J: $\widetilde{\mathrm{KO}}(M) \to J(M)$, is equal to $J(\tau M - f^*\tau N)$, where $\tau M - f^*\tau N$ is the stable block bundle determined by any choice of PL structures for M and N.

THEOREM 2.9. *If there exists a branched covering f: $M \to N$ with locally flat branch set, then the stable block bundle $\tau M - f^*\tau N$, determined by any choice of PL structures for M and N, is related to the exponents of f by an equation in the group $J(M)$,*

$$J(\tau M - f^*\tau N) = \sum_{i=1}^{m} J(g_i^*\eta_{k_i}). \tag{2.10}$$

3. Products of the Brand characteristic classes. If f: $M \to N$ is a branched covering with locally flat branch set and exponents k_i, $1 \leq i \leq m$, let

$$\eta_{k_i}(M) = g_i^*\eta_{k_i}, \tag{3.1}$$

be the Brand characteristic classes in $\widetilde{\mathrm{KO}}(M)$ described above. If X is any complex, the tensor product operation on vector bundles makes $\mathrm{KO}(X)$ a ring with ideal $\widetilde{\mathrm{KO}}(X)$, ([7], p. 116). Our next theorem asserts that the product of any two Brand characteristic classes corresponding to distinct exponents must be zero.

THEOREM 3.2. *If f: $M \to N$ is a branched covering with locally flat branch set and exponents k_i, $1 \leq i \leq m$, then, if $i \neq j$,*

$$\eta_{k_i}(M)\eta_{k_j}(M) = 0. \tag{3.3}$$

Theorem 3.2 will follow immediately from the lemma below. If X_i, $1 \leq i \leq m$, is a family of complexes, let pr_i: $X_1 \vee X_2 \vee \cdots \vee X_m \to X_i$ denote projection on the ith summand.

LEMMA 3.4. *If $c_i \in \widetilde{\mathrm{KO}}(X_i)$ and $c_j \in \widetilde{\mathrm{KO}}(X_j)$, $i \neq j$, then in the ring $\widetilde{\mathrm{KO}}(X_1 \vee X_2 \vee \cdots \vee X_m)$*

$$(\mathrm{pr}_i^* c_i)(\mathrm{pr}_j^* c_j) = 0. \tag{3.5}$$

Proof. If i_j: $X_j \to X_1 \vee X_2 \vee \cdots \vee X_m$ is the inclusion map, then there is an isomorphism $(i_1^*, i_2^*, \ldots, i_m^*)$: $\widetilde{\mathrm{KO}}(X_1 \vee X_2 \vee \cdots \vee X_m) \to \sum_{i=1}^m \widetilde{\mathrm{KO}}(X_i)$, ([7], p. 116). To establish (3.5), it is therefore enough to show that $i_k^* [(\mathrm{pr}_i^* c_i)(\mathrm{pr}_j^* c_j)] = 0$, $i \neq j$, $1 \leq i, j, k \leq m$. This is clear since $\mathrm{pr}_i \circ i_k$ is the constant map at the basepoint if $i \neq k$ and the identity if $i = k$.

To prove Theorem 3.2, let c: $M \to \vee_{i=1}^m T(\tilde{\xi} \,|\, \tilde{B}_{k_i})$ be the collapsing map into the one point union of the Thom spaces determined by the exponents of the branched covering and note that

$$\eta_{k_i}(M) = c^* \, \mathrm{pr}_i^* \, h_i^* \eta_{k_i}, \tag{3.6}$$

where h_i is the map in (2.5). Theorem 3.2 now follows from Lemma 3.4 if we set $X_i = T(\tilde{\xi} \,|\, \tilde{B}_{k_i})$, $1 \leq i \leq m$.

It is interesting to compare Theorem 3.2 with a theorem concerning branched coverings and products of Stiefel-Whitney classes. Berstein and Edmonds showed that if M is spin ($w_2(M) = 0$) and if there exists a branched covering f: $M \to S^n$ with locally flat branched set, then all the products of the Stiefel-Whitney classes of M are zero, [3]. Theorem 3.2 says that if M is a branched covering of any manifold with locally flat branch set, then the product of any two Brand classes corresponding to distinct exponents is zero.

4. The proof of Theorem 1.1. We will prove Theorem 1.1 by using (2.10) to show that if $\mathbf{R}P^n$ is a branched covering with locally flat, orientable branch set and $n \neq 1, 3$, or 7, then the branched covering must have more than two distinct exponents. It will then follow from Theorem 3.2 that the product of two Brand classes corresponding to a pair of distinct exponents must be zero and we will show that this leads to a contradiction in view of the structure of the ring $\widetilde{\mathrm{KO}}(\mathbf{R}P^n)$ for every $n \neq 1, 3$, or 7 except $n = 5$. The next lemma eliminates this one additional case. The proof of the lemma involves the techniques used by Brand to prove his theorem that if $\mathbf{R}P^n$ is a branched covering with locally flat branch set of any manifold with trivial total Stiefel-Whitney class, then $n = 2^t \pm 1$, [**4**] and [**5**].

LEMMA 4.1. *If* $\mathbf{R}P^n$ *is a branched covering with locally flat, orientable branch set of any manifold with trivial total Stiefel-Whitney class, then* $n = 2^t - 1$.

Proof. If $\tilde{\xi}_{\text{even}}$ denotes the union of all $\tilde{\xi}_k$ for k even, we have Brand's formula, [**4**] and [**5**], for the total Stiefel-Whitney class of $\tau M - f^*\tau N$ where $f\colon M \to N$ is a branched covering with locally flat branch set,

$$(4.2) \quad w(\tau M - f^*\tau N) = 1 + c^*\phi\left(1 + w_1\left(\tilde{\xi}_{\text{even}}\right) + w_1\left(\tilde{\xi}_{\text{even}}\right)^2 + \cdots\right).$$

In formula (4.2), $c\colon M \to T(\tilde{\xi}_{\text{even}})$ is the collapsing map and ϕ is the Thom isomorphism. If $M = \mathbf{R}P^n$, $n = 2^t + 1$, and N has trivial total Stiefel-Whitney class, it follows from (4.2) that $0 \neq \alpha^{2^t} = c^*\phi(w_1(\tilde{\xi}_{\text{even}})^{2^t-2})$ where α in $H^1(\mathbf{R}P^n; \mathbf{Z}_2)$ is the generator. If $t \geq 2$, this means $w_1(\tilde{\xi}_{\text{even}}) \neq 0$ and so the branch set is not orientable.

It follows from the work of Brand that there are orientable characteristic classes η_k in KO(MSO(2)) such that $\eta_k \mid \text{BSO}(2) = \gamma - \mu_k(\gamma)$, where γ is the universal orientable 2-plane bundle, ([**5**], p. 2). Using the same symbols in the orientable and non-orientable cases will not cause any confusion in what follows.

If O(2) is replaced by SO(2) in Brand's construction, ([**4**], Theorem 2.2 and Lemma 6.3), it is clear that it is possible to construct a universal smooth branched covering with orientable branch set. It follows that if $f\colon M \to N$ is a branched covering with locally flat, orientable branch set, then each of the mappings g_i in (2.5) lifts to MSO(2), that is, each Brand class of M is the pull-back of a universal orientable Brand class in KO(MSO(2)). The space MSO(2) is homotopically simple. In fact, MSO(2) is a $K(\mathbf{Z}, 2)$. It is therefore possible to compute the orientable Brand classes in $\widetilde{\text{KO}}(\mathbf{R}P^n)$. The group $\widetilde{\text{KO}}(\mathbf{R}P^n)$ is a finite cyclic group generated by $\lambda - 1$, where λ is the canonical line bundle over $\mathbf{R}P^n$, ([7], p. 223). The proof of the following lemma is found in [**8**].

LEMMA 4.3. *If there exists a branched covering* $f\colon \mathbf{R}P^n \to N$, *with locally flat, orientable branch set, then*

$$(4.4) \qquad \eta_k(\mathbf{R}P^n) = \begin{cases} 0, & k \text{ odd}, \\ 2(\lambda - 1), & k \text{ even}. \end{cases}$$

Before we can proceed, we need to know more about the ring $\widetilde{\mathrm{KO}}(\mathbf{R}P^n)$. The order of $\widetilde{\mathrm{KO}}(\mathbf{R}P^n)$ is $2^{\phi(n)}$, where $\phi(n)$ is the number of integers in the set $\{s\colon 0 < s \leq n,\ s \equiv 0, 1, 2, 4 \pmod 8\}$, ([7], p. 223), and the ring structure is given by $(\lambda - 1)^2 = -2(\lambda - 1)$, ([7], p. 225). It is known that $(\tau \mathbf{R}P^n) = (n+1)\lambda$, where the left-hand side of the equation is the stable isomorphism class of the tangent bundle determined by the standard smooth structure on $\mathbf{R}P^n$, ([7], p. 17).

In the theorem below, the term π-manifold means a smooth, closed n-manifold with stably trivial tangent bundle, such as S^n.

THEOREM 4.5. *If $\mathbf{R}P^n$ is a branched covering of a π-manifold with locally flat, orientable branch set, then $n = 1, 3$, or 7.*

Proof. Suppose $\mathbf{R}P^n$ is a branched covering of a π-manifold N with locally flat, orientable branch set. It is known that N immerses in $\mathbf{R}^{n+1}$ with trivial normal bundle, i.e. $\tau N \oplus 1$ is the trivial $n+1$ bundle. It therefore follows immediately from Theorem 2.9, the fact that the J-homomorphism $J\colon \widetilde{\mathrm{KO}}(\mathbf{R}P^n) \to \widetilde{J}(\mathbf{R}P^n)$ is an isomorphism, ([7], p. 225), and the comments above that, in $\widetilde{\mathrm{KO}}(\mathbf{R}P^n)$,

$$(4.6) \qquad (n+1)(\lambda - 1) = \sum_{i=1}^{m} \eta_{k_i}(\mathbf{R}P^n),$$

where k_i, $1 \leq i \leq m$, are the exponents in a normalization of the branched covering.

Let $\mathfrak{p}$ be the number of distinct even exponents with essential classifying map. It follows from Lemma 4.3 and formula (4.6) that n is odd and

$$(4.7) \qquad \mathfrak{p} \equiv \tfrac{1}{2}(n+1) \pmod{2^{\phi(n)-1}}.$$

If $n \neq 1, 3$, or 7, it is clear that $n + 1 \not\equiv 0 \pmod{2^{\phi(n)}}$, ([7], pp. 219–221). This is just the fact that $\mathbf{R}P^n$ is a π-manifold if, and only if, $n = 1, 3$, or 7. In fact, $\mathbf{R}P^n$ is parallelizable if $n = 1, 3$, or 7. Therefore, if $n \neq 1, 3$, or 7, $\mathfrak{p} \equiv \frac{1}{2}(n+1) \not\equiv 0 \pmod{2^{\phi(n)-1}}$, and so $\mathfrak{p} \geq 3$. This means that there are at least three distinct even exponents with essential classifying maps. But Lemma 4.3 and the ring structure of $\widetilde{\mathrm{KO}}(\mathbf{R}P^n)$, together with Theorem 3.2 imply that two distinct even exponents is an impossibility. If k_i and k_j are distinct, then by Theorem 3.2, $\eta_{k_i}(\mathbf{R}P^n)\eta_{k_j}(\mathbf{R}P^n) = 0$. If k_i and k_j are distinct and even with essential classifying map,

$$\eta_{k_i}(\mathbf{R}P^n)\eta_{k_j}(\mathbf{R}P^n) = [2(\lambda - 1)][2(\lambda - 1)] = -8(\lambda - 1) = 0,$$

but $8 \not\equiv 0 \pmod{2^{\phi(n)}}$ if $n \neq 1, 3, 5$, or 7, ([7], p. 219). Since Lemma 4.1 eliminates $n = 5$, the proof is complete.

The author would like to thank Robert E. Stong for his helpful observations. In particular, Stong suggested Lemma 4.1.

REFERENCES

[1] J. F. Adams, *On the groups $J(X)I$*, Topology, **2** (1963), 181–195.

[2] J. W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc., **26** (1920), 370–370.

[3] I. Berstein and A. L. Edmonds, *The degree and branch set of a branched covering*, Invent. Math., **45** (1978), 213–220.

[4] N. Brand, *Classifying spaces for branched coverings*, Indiana Univ. Math. J., **29** (1980), 229–248.

[5] _____, *Necessary conditions for the existence of branched coverings*, Invent. Math., **54** (1979), 1–10.

[6] H. M. Hilden, *Three-fold branched coverings of S^3*, Amer. J. Math., **98** (1976), 989–997.

[7] D. Husemoller, *Fibre Bundles*, New York: McGraw-Hill, 1966.

[8] R. D. Little, *Projective space as a branched covering with orientable branch set*, Pacific J. Math., **99** (1982), 89–96.

[9] J. M. Montesinos, *Three-manifolds as 3-fold branched covers of S^3*, Quart. J. Math. Oxford, Ser. (2) **27** (1976), 85–94.

[10] C. P. Rourke and B. J. Sanderson, *Block bundles I*, Ann. of Math., **87** (1968), 1–28.

[11] C. T. C. Wall, *Surgery on Compact Manifolds*, London: Academic Press, 1970.

Received October 13, 1982

UNIVERSITY OF HAWAII
HONOLULU, HI 96822