

CONVERGENCE OF ADAPTED SEQUENCES OF PETTIS-INTEGRABLE FUNCTIONS

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When considering adapted sequences of Pettis-integrable functions with values in a Banach space we are dealing with the following problem: when do we have a strongly measurable Pettis-integrable limit? Here the limit can be taken in the strong or weak sense a.e. or in the sense of the Pettis-topology.

Not many results in this area are known so far.

In this paper we give some pointwise convergence results of martingales, amarts, weak sequential amarts and pramarts consisting of strongly measurable Pettis-integrable functions. Also the Pettis convergence result of Musiał for amarts is extended.

The results are preceded by a preliminary study of some vector measure notions such as Pettis uniform integrability and σ -bounded variation. We give a new proof of the result of Thomas stating that in every infinite dimensional Banach space one can find a vector measure which is not of σ -bounded variation.

1. Introduction, terminology and notation. In the sequel, E will be a Banach space and (Ω, \mathcal{F}, P) a fixed complete probability space. A function $X: \Omega \rightarrow E$ is called scalarly measurable if $\langle x', X \rangle$ is measurable for each $x' \in E'$.

A function X is called Pettis-integrable if it is scalarly integrable and if for each $A \in \mathcal{F}$, there exists $x_A \in E$ such that, for each $x' \in E'$

$$\langle x', x_A \rangle = \int_A \langle x', X \rangle dP.$$

x_A is denoted by $\int_A X dP$, the Pettis-integral of X over A . Let X and Y be two Pettis-integrable functions. We say that X is weakly equivalent with Y , denoted by $X \sim Y$, if for each $x' \in E'$

$$\langle x', X \rangle = \langle x', Y \rangle, \text{ a.e.}$$

Denote by P_E the space of all Pettis-integrable functions up to weak equivalence. Put on P_E the following norm, called Pettis-norm,

$$\|\cdot\|_P: P_E \rightarrow R^+, \quad X \rightarrow \|X\|_P = \sup_{\substack{\|x'\| \leq 1 \\ x' \in E'}} \int_{\Omega} |\langle x', X \rangle| dP.$$

The notions of strong measurability and Bochner-integrability we suppose to be familiar to the reader (see [D-U]).

It is trivial that every Bochner-integrable function is Pettis-integrable and that mean convergence (i.e. for the norm $\|\cdot\|_1$ on L^1_E , the space of Bochner-integrable functions) implies $\|\cdot\|_p$ convergence. The converse of these two implications is never true except in finite dimensional spaces as is well known. Even if we suppose that every strongly measurable Pettis-integrable function is Bochner-integrable we can prove that E must be finite dimensional. Indeed, suppose E is infinite dimensional. Using the theorem of Dvoretzky-Rogers (see [D-R]) there exists an unconditionally convergent series $\sum x_n$ such that $\sum \|x_n\| = \infty$. Let $(A_n)_{n \in \mathbb{N}}$ be a countable measurable partition of Ω such that $P(A_n) > 0$ for each $n \in \mathbb{N}$. Put

$$X = \sum_{n \in \mathbb{N}} \frac{x_n}{P(A_n)} \chi_{A_n}.$$

Then X is Pettis-integrable strongly measurable and not Bochner-integrable. Of course here one has $\|X\|_1 = \infty$ and $\|X\|_p < \infty$.

Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a stochastic basis, i.e. an increasing sequence of sub- σ -algebras of \mathcal{F} . Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in P_E . We say that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is an adapted sequence if each X_n is \mathcal{F}_n -scalarly measurable. If in addition each X_n is strongly measurable then X_n is \mathcal{F}_n -strongly measurable as follows easily from Pettis' measurability theorem (see [D-S]). With respect to a stochastic basis $(\mathcal{F}_n)_{n \in \mathbb{N}}$, we say that a function $\tau: \Omega \rightarrow \mathbb{N}$ is a stopping time if $\{\tau = n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

Let us denote by T the set of all bounded (i.e. finitely valued) stopping times. We order T in the natural (pointwise) way. For $\tau \in T$ and $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ an adapted sequence, define

$$X_\tau = \sum_{k=\min \tau}^{\max \tau} X_k \chi_{\{\tau=k\}}.$$

Also define $\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\tau = k\} \in \mathcal{F}_k \text{ each } k \in \mathbb{N}\}$.

It is obvious that X_τ is scalarly \mathcal{F}_τ -measurable. Adapted sequences of Bochner-integrable functions have been studied intensively. We refer to [Eg] which is a monograph on this subject which is about to appear at this moment. On the contrary, on adapted sequences of Pettis-integrable functions virtually nothing is known. This is of course due to the completely different nature of the Pettis-integral (see remark above). To illustrate this see also [T₃], [Mu₁], [Mu₂]. Before mentioning the results, let us give some well-known definitions. Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence in P_E .

It is called a martingale if for each $n \in N$ and each $A \in \mathcal{F}_n$:

$$\int_A X_n = \int_A X_{n+1}.$$

(Here the integrals are in the Pettis sense of course.) In other words, $X_n = \mathcal{E}^{\mathcal{F}_n} X_{n+1}$, where $\mathcal{E}^{\mathcal{F}_n}$ denotes the Pettis conditional expectation w.r.t. \mathcal{F}_n . It must be emphasized that the Pettis conditional expectation of a Pettis integrable function does not always exist, even if the function is strongly measurable; see [Mu₁] and also §2.

Of course, as is well-known and easy to prove, the conditional expectation of a Bochner integrable function always exists.

An adapted sequence $(X_n, \mathcal{F}_n)_{n \in N}$ is called an *amart* if the net $(\int_{\Omega} X_{\tau})_{\tau \in T}$ converges in E . It is obvious that every martingale is an amart. Finally a Banach space E is said to have the weak Radon-Nikodym-property (abbreviated (WRNP)) if for every complete probability space (Ω, \mathcal{F}, P) and for every vector measure $F: \mathcal{F} \rightarrow E$ of σ -bounded variation which is P -continuous, there exists $X \in P_E$ such that for each $A \in \mathcal{F}$:

$$F(A) = \int_A X.$$

An equivalent statement of (WRNP) is obtained if one replaces “ σ -bounded variation” by “bounded variation”.

However the first definition is more natural since a Pettis integral is always of σ -bounded variation.

For a discussion of (WRNP) we refer the reader to [Mu₁] or [Mu₂].

The first result we have to mention concerning convergence of adapted sequences of strongly measurable Pettis integrable functions is one of J. J. Uhl Jr.:

THEOREM ([U₂], THEOREM 3.1): *Let $(X_n, \mathcal{F}_n)_{n \in N}$ be a martingale consisting of strongly measurable Pettis-integrable functions.*

Suppose

- (i) $\sup_{n \in N} \|X_n\|_P < \infty$.
- (ii) *The set $\{ \int_A X_n \mid A \in \mathcal{F}_n, n \in N \}$ is weakly relatively compact.*
- (iii) *For each $\varepsilon > 0$ there is a weakly compact set $K \subset E$ such that for each $\delta > 0$, there exists $n_0 \in N$ and $A_0 \in \mathcal{F}_{n_0}$ for which $P(\Omega \setminus A_0) < \varepsilon$ such that $n \geq n_0$ implies $\int_A X_n \in P(A)K + \delta B_E$ for all $A \subset A_0, A \in \mathcal{F}_n$ (here $B_E = \{x \in E \mid \|x\| \leq 1\}$).*

Then $(X_n)_{n \in N}$ converges strongly a.e. to a function in P_E .

Condition (iii) is a Rieffel-type condition (see e.g. [Die] Theorem 1, pp. 204–205) and is satisfied if E has (RNP) and if (i) is strengthened to $\sup_{n \in N} \|X_n\|_1 < \infty$, in which case we have the Chatterji results on martingales consisting of Bochner integrable functions (see [C]) (condition (ii) can be dropped in this case; see Corollary 3.4 in [U₂]). So this is an interesting and nontrivial extension of the classical martingale convergence theorem.

Concerning weak convergence we refer to [Me]; we are not dealing with weak convergence of martingales in this paper.

Another result—one of Musiał—can be stated as follows:

THEOREM ([Mu₃], THEOREM 5). *The following assertions are equivalent:*

(i) E has (WRNP).

(ii) *For every complete probability space (Ω, \mathcal{F}, P) and every martingale $(X_n, \mathcal{F}_n)_{n \in N}$ in P_E on (Ω, \mathcal{F}, P) which is Pettis uniformly integrable and variationally bounded there is an $X \in P_E$ such that $\lim_{n \rightarrow \infty} \|X_n - X\|_P = 0$.*

Here, Pettis uniform integrability means

$$\lim_{P(A) \rightarrow 0} \sup_{n \in N} \left\| \int_A X_n \right\| = 0.$$

See also §2. Variationally bounded means that

$$\sup_{n \in N} |\nu_n|(\Omega) < \infty$$

where $|\nu_n|$ denotes the variation of

$$\nu_n: \mathcal{F}_n \rightarrow E, \quad A \rightarrow \int_A X_n.$$

The extension to amarts in the above theorem can easily be made in Musiał's proof.

However, Musiał's theorem, when stated for (RNP) spaces teaches nothing new, even when it is stated for amarts. Indeed, in this case, as is easily seen (see e.g. [J], p. 129), for each $n \in N$, there exists $Y_n \in P_E$, strongly measurable such that $Y_n \sim X_n$. From the definition of variation and from $Y_n \sim X_n$ for each $n \in N$ it now follows that

$$\sup_{n \in N} \int_{\Omega} \|Y_n\| = \sup_{n \in N} \int_{\Omega} \|X_n\| < \infty.$$

So we have here an L_E^1 -bounded amart satisfying also

$$\lim_{P(A) \rightarrow 0} \sup_{n \in N} \left\| \int_A Y_n \right\| = 0.$$

This is a known situation ([U₃], Corollary 4, p. 294) yielding a Bochner integrable function Y such that

$$\lim_{n \rightarrow \infty} \|Y_n - Y\|_p = 0.$$

But since $X_n \sim Y_n$ for each $n \in N$ we also have

$$\lim_{n \rightarrow \infty} \|X_n - Y\|_p = 0.$$

So, Musiał's theorem reduces—in case E has (RNP)—to Uhl's theorem [U₃], Corollary 4, p. 294, and is certainly only a convergence theorem for amarts consisting of Bochner integrable functions.

However, it would be interesting to have convergence results in (RNP) Banach spaces for strongly measurable Pettis-integrable amarts and martingales which are not Bochner integrable. This is one of the purposes of this article. The paper is divided as follows: In §2 we study the notions σ -bounded variation and Pettis-uniform integrability. Concerning the first notion a result of Thomas [T₂] (later reproved by Janicka-Kalton [J-K]) is reproved yielding a straightforward proof. In §3 we prove a martingale strong a.e. convergence theorem for strongly measurable functions in P_E where E has (RNP).

Section 4 reproves and generalizes the Pettis convergence theorems for amarts of Musiał and Uhl. A Riesz decomposition theorem is proved for amarts in P_E generalizing the one of Edgar-Sucheston.

Section 5 proves a weak a.e. convergence theorem for weak sequential amarts.

Finally §6 proves a strong a.e. convergence result for pramarts consisting of strongly measurable functions in P_E where E has (RNP), extending a theorem of Millet-Sucheston.

2. Study of some notions related to the Pettis integral.

2.1. Vector measures of σ -bounded variation.

DEFINITION 2.1.1. Let $F: \mathcal{F} \rightarrow E$ be a vector measure. We say that F is of σ -bounded variation if there is a disjoint sequence $(A_n)_{n \in N}$ in \mathcal{F} such that the restriction $F|_{A_n}$ to A_n is a vector measure of bounded variation, for each $n \in N$.

This is a notion which is very natural in connection with Pettis integrals. Indeed we have that every Pettis integral (considered as a vector measure) is of σ -bounded variation. This is a result of Musiał, see [Mu₁]. Compare this with the well known fact that every Bochner integral is a vector measure of bounded variation. As is the case with the Pettis integral, the notion of σ -bounded variation is not easy to study. For instance, if $F: \mathcal{F} \rightarrow E$ is of bounded variation and if \mathcal{G} is a sub- σ -algebra of \mathcal{F} then it is completely trivial that $F|_{\mathcal{G}}$ is of bounded variation. This is false for the σ -bounded variation notion. This is easy to prove and firstly remarked by Musiał [Mu₁]. Indeed we have:

PROPOSITION 2.1.2. *Suppose E has (WRNP). Let $F: \mathcal{F} \rightarrow E$ be given by a Pettis integral*

$$F(A) = \int_A X$$

for each $A \in \mathcal{F}$, and let \mathcal{G} be a sub- σ -algebra of \mathcal{F} . The $F|_{\mathcal{G}}$ is of σ -bounded variation iff $E^{\mathcal{G}}X$ exists in P_E .

Proof. The proof is very easy and included for the sake of completeness. Indeed, suppose $F|_{\mathcal{G}}$ is of σ -bounded variation.

Since $F \ll P|_{\mathcal{G}}$ we have also $F|_{\mathcal{G}} \ll P|_{\mathcal{G}}$.

So, using (WRNP), there exists a function $Y \in P_E(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ such that for each $A \in \mathcal{G}$

$$(F|_{\mathcal{G}})(A) = \int_A Y.$$

By the form of F , Y must be $E^{\mathcal{G}}X$.

Conversely, suppose that $E^{\mathcal{G}}X \in P_E(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ exists. Then for each $A \in \mathcal{G}$

$$(F|_{\mathcal{G}})(A) = \int_A E^{\mathcal{G}}X$$

and since this is a Pettis integral, it is of σ -bounded variation (see [Mu₁]). □

Now it is clear that a vector measure F of σ -bounded variation (w.r.t. a σ -algebra \mathcal{F}) exists such that $F|_{\mathcal{G}}$ is not of σ -bounded variation, where \mathcal{G} is a sub- σ -algebra of \mathcal{F} . It suffices to take a Pettis integrable function $f: \mathcal{F} \rightarrow E$ without a conditional expectation. see [Mu₁], [H], [R].

Although Proposition 2.1.2 is interesting it does not solve completely the problem of when a vector measure is of σ -bounded variation w.r.t. a sub- σ -algebra.

Another question one might ask is the following: Suppose a vector measure F is absolutely continuous w.r.t. Lebesgue-measure on $[0, 1]$. Is F of σ -bounded variation?

The property is obviously true in any finite dimensional Banach space. This was disproved in any infinite dimensional Banach space by G. Thomas in [T₂] and later by Janicka-Kalton [J-K]. Incidentally I found a new proof on this fact in a very straightforward way. It runs as follows:

THEOREM 2.1.3 (Thomas). *Suppose E is an infinite dimensional Banach space. Then there exists a probability space (Ω, \mathcal{F}, P) and a vector measure $F: \mathcal{F} \rightarrow E$ such that $F \ll P$ and such that F is not of σ -bounded variation.*

Proof. The characterization of infinite dimensional Banach spaces of Dvoretzky-Rogers [D-R] yields a sequence $(x_n)_{n \in \mathbb{N}}$ in E such that $\sum x_n$ converges unconditionally and such that $\sum \|x_n\| = \infty$.

Put $x_0 = \sum_{n=1}^{\infty} x_n$.

Write

$$(1) \quad x_1 = (x_1 - x_0) + \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \left[\frac{1}{2^n} (x_1 - x_0) + x_n \right],$$

$$(2) \quad x_2 = (x_2 - x_0) + \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \left[\frac{1}{2^n} (x_2 - x_0) + x_n \right].$$

Then do the same with the first term of the series appearing in (1), then with the second and so on. The same with (2) and so on. Every series appearing above converges unconditionally but not absolutely since (we take the series in (1)—an analogous proof for the other series)

$$\begin{aligned} \sum_{n=1}^{\infty} \left\| \frac{1}{2^n} (x_1 - x_0) + x_n \right\| &\geq \sum_{n=1}^{\infty} \left| \|x_n\| + \frac{1}{2^n} \|x_1 - x_0\| \right| \\ &\geq \sum_{n=1}^{\infty} \|x_n\| - \|x_1 - x_0\| = \infty. \end{aligned}$$

Let now $\sum y_n$ be any of the series introduced above. By assuming $x_0 \neq 0$ and $x_n \neq 0$ for each $n \in N$ and by making additions and relabelings we can assume that

$$y_n \neq 0, \quad \forall n \in N \quad \text{and} \quad \left\| \sum_{n=1}^{\infty} y_n \right\| > 2\|y_k\|, \quad \forall k \geq 2.$$

(Since $\sum_{k=1}^n y_k \rightarrow \sum_{n=1}^{\infty} y_n$ and since $y_n \rightarrow 0$ and $\|\sum_{n=1}^{\infty} y_n\| > 0$ there exists $n_0 \in N$ such that $\|\sum_{k=1}^{n_0} y_k\| > 2\|y_i\|, \forall i > n_0$. Relabel $y'_i = \sum_{k=1}^{n_0} y_k, y'_2 = y_{n_0+1}$, and so on.)

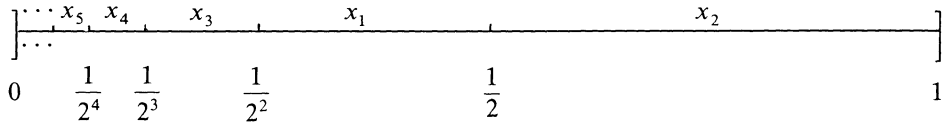
Take now $]0, 1]$ and divide it dyadically into

$$] \frac{1}{2}, 1],] \frac{1}{4}, \frac{1}{2}],] \frac{1}{8}, \frac{1}{4}], \dots$$

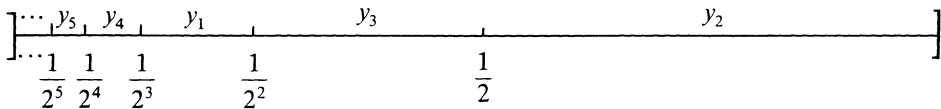
Each of these intervals are also divided dyadically in the same way (homothetically). This constitutes the second level. The same for the third level and so on.

We are now going to construct the vector measure F .

We put the following values for the first level intervals



i.e. x_i on $]1/2^i, 1/2^{i-1}]$ except for x_1 and x_2 or: We put first x_1 on the second place from the right and then put the vectors x_2, x_3, x_4, \dots consecutively from the right on the not yet used intervals. Let $\sum y_n$ be one of the series of the second level adding up to, say x_i . Here we put on the interval where we have put the value x_i :



(homothetic interval and divisions)

y_1 on the third place from the right (division $1/2^3$ relatively to the interval) and then put the vectors y_2, y_3, y_4, \dots consecutively from the right on the not yet used intervals.

Inductively, let $\sum y_n$ be one of the series from the n th level. Here, we put y_1 on the $(n + 1)$ th place from the right (division $1/2^{n+1}$ relatively to the interval in consideration) and then put the vectors y_2, y_3, y_4, \dots , consecutively from the right on the not yet used intervals.

Now we estimate from above the total length of the intervals on which the first terms y_1 are appearing.

For the first level: just one place: length $1/2^2$.

For the second level:

$$\frac{1}{2} \cdot \frac{1}{2^3} + \frac{1}{4} \cdot \frac{1}{2^3} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{2^3} = \frac{1}{2^3}$$

For the fourth level: $1/2^4$ and so on.

A majorization is obtained by adding these values: $\sum_{n=2}^{\infty} 1/2^n = 1/2$ (considering all the intervals on which the y_1 are appearing as disjoint which is not the case). Call the complement of this set A . So $\lambda(A) \geq \frac{1}{2}$ ($\lambda =$ Lebesgue measure).

Call F the constructed vector measure, restricted to A . Since on A , every vector appearing in the n th level has norm smaller than $\frac{1}{2}$ times the norm of the corresponding vector in the $(n - 1)$ th level, we have that F on A is $\lambda|_A$ -continuous.

Also since for every series $\sum y_n$, appearing in the construction we have that $\sum \|y_n\| = \infty$ it follows that F is not of σ -bounded variation. \square

We end with a problem of Musiał [Mu₁] which would (in case of an affirmative solution) completely settle the problem concerning σ -bounded variation w.r.t. a sub- σ -algebra as well as extend the result of Thomas (Theorem 2.1.3.)

Problem 2.1.4. Is it true that in any infinite dimensional Banach space E there exists a probability space (Ω, \mathcal{F}, P) , a P -continuous measure $F: \mathcal{F} \rightarrow E$ of σ -bounded variation and a sub- σ -algebra \mathcal{G} of \mathcal{F} such that $F|_{\mathcal{G}}$ is not of σ -bounded variation?

2.2. Pettis uniform integrability. If $(X_n)_{n \in N}$ is a sequence in L^1_E two well-known equivalent definitions of uniform integrability are in use:

(i) $\sup_{n \in N} \|X_n\|_1 < \infty$ and $\lim_{P(A) \rightarrow 0} \sup_{n \in N} \int_A \|X_n\| = 0$.

(ii) $\lim_{\lambda \rightarrow \infty} \sup_{n \in N} \int_{\{\|X_n\| > \lambda\}} \|X_n\| = 0$. Obviously, L^1_E -norms are involved here. When working with sequences $(X_n)_{n \in N}$ in P_E the following definition of Pettis uniform integrability is quite natural.

DEFINITION 2.2.1. The sequence $(X_n)_{n \in N}$ is said to be Pettis uniformly integrable if

$$\sup_{n \in N} \|X_n\|_P < \infty \quad \text{and} \quad \lim_{P(A) \rightarrow 0} \sup_{n \in N} \sup_{B \in \mathcal{F}} \left\| \int_{A \cap B} X_n \right\| = 0.$$

Indeed the Pettis norm is involved here since the norm $\sup_{A \in \mathcal{F}} \|\int_A X_n\|$ is equivalent with $\|X_n\|_P$.

Definition 2.2.1 is not only natural because of the replacement of $\|\cdot\|_1$ by $\|\cdot\|_P$ in the classical definition of uniform integrability. It is also a “uniform” notion of Pettis integrability. Indeed, in [P] it is proved that a Pettis integrable function satisfies

$$(1) \quad \lim_{P(A) \rightarrow 0} \sup_{B \in \mathcal{F}} \left\| \int_{A \cap B} X \right\| = 0.$$

(and of course $\|X\|_P < \infty$).

One question is coming up immediately: What about the following possible definition of Pettis uniform integrability:

$$\lim_{\lambda \rightarrow \infty} \sup_{n \in N} \sup_{B \in \mathcal{F}} \left\| \int_{\{B \cap \|X_n\| > \lambda\}} X_n \right\| = 0.$$

Certainly this is only meaningful if the sequence $(X_n)_{n \in N}$ consists of strongly measurable functions (only then $\{\|X_n\| > \lambda\}$ is surely in \mathcal{F}). In this case, what is the relation with Definition 2.2.1? We only have the following easy result:

THEOREM 2.2.2. *Let $(X_n)_{n \in N}$ be a sequence consisting of strongly measurable Pettis integrable functions. Suppose*

$$(1) \quad \lim_{\lambda \rightarrow \infty} \sup_{n \in N} P(\|X_n\| > \lambda) = 0$$

Then the following two assertions are equivalent:

- (i) $(X_n)_{n \in N}$ is Pettis uniformly integrable.
- (ii) $\lim_{\lambda \rightarrow \infty} \sup_{n \in N} \sup_{B \in \mathcal{F}} \left\| \int_{B \cap \{\|X_n\| > \lambda\}} X_n \right\| = 0.$

Proof. (i) \Rightarrow (ii). Is done much in the same way as in the case of (Bochner) uniform integrability and hence the proof is omitted. We do not use condition (1) here.

(ii) \Rightarrow (i). Is obvious, using condition (1). □

Theorem 2.2.2 is an extension of a remark in Bru-Heinich [B-H] proving that a strongly measurable function X is Pettis integrable if

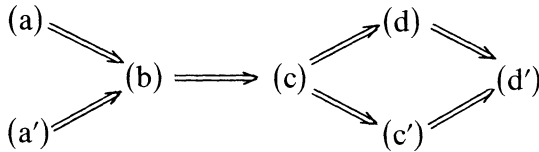
$$\lim_{\lambda \rightarrow \infty} \sup_{B \in \mathcal{F}} \left\| \int_{B \cap \{\|X\| > \lambda\}} X \right\| = 0.$$

Indeed, since $\|X\|$ is measurable we always have here $\lim_{\lambda \rightarrow \infty} P(\|X\| > \lambda) = 0$. To indicate some cases in which (1) is satisfied we prove the following

THEOREM 2.2.3. *Consider the following assertions for an adapted sequence $(X_n, \mathcal{F}_n)_{n \in N}$ consisting of strongly measurable Pettis integrable functions:*

- (a) $(X_n, \mathcal{F}_n)_{n \in N}$ is an L_E^1 -bounded martingale,
- (a') $(X_n, \mathcal{F}_n)_{n \in N}$ is of class (B); i.e. $\sup_{\tau \in T} \int_{\Omega} \|X_{\tau}\| < \infty$,
- (b) $\lim_{\lambda \rightarrow \infty} P(\sup_{n \in N} \|X_n\| > \lambda) = 0$,
- (c) $\lim_{\lambda \rightarrow \infty} \sup_{\tau \in T} P(\|X_{\tau}\| > \lambda) = 0$,
- (c') $\lim_{\lambda \rightarrow \infty} \sup_{n \in N} P(\|X_n\| > \lambda) = 0$,
- (d) $\lim_{\tau \in T} P(\|X_{\tau}\| > \tau) = 0$,
- (d') $\lim_{n \in N} P(\|X_n\| > n) = 0$.

Then we have



Proof. (a) \Rightarrow (b) and (a') \Rightarrow (b) follow from the classical maximal inequality for adapted sequences of Bochner integrable functions.

(b) \Rightarrow (c) \Rightarrow (c') is trivial.

(c) \Rightarrow (d). For each $\varepsilon > 0$, choose $\lambda_0 \in N$ such that if $\lambda \geq \lambda_0$, $\sup_{\tau \in T} P(\|X_{\tau}\| > \lambda) \leq \varepsilon$.

For each $\tau \geq \lambda_0, \tau \in T$ we so have

$$P(\|X_{\tau}\| > \tau) \leq P(\|X_{\tau}\| > \lambda_0) \leq \sup_{\tau \in T} P(\|X_{\tau}\| > \lambda_0) \leq \varepsilon.$$

(c') \Rightarrow (d') is similar

(d) \Rightarrow (d') is true since N is cofinal in T . □

EXAMPLES 2.2.4.

1. For any function $X \in P_E$ which is strongly measurable we have

$$\lim_{\lambda \rightarrow \infty} P(\|X\| > \lambda) = 0.$$

2. Let E be a σ -Dedekind complete Banach lattice and suppose (X_n, \mathcal{F}_n) is an adapted sequence of strongly measurable Pettis integrable functions.

Suppose also that, for each $n \in N$, $0 \leq X_n \leq Y$, a.e. where $Y \in P_E$. Then

$$\lim_{\lambda \rightarrow \infty} P\left(\sup_{n \in N} \|X_n\| > \lambda\right) = 0.$$

Proof. Since E is σ -Dedekind complete, $\sup_{n \in N} X_n(\omega) \in E$ exists, a.e. Hence

$$0 \leq X_n \leq \sup_{n \in N} X_n \leq Y, \quad \text{a.e.}$$

Now $\sup_{n \in N} X_n$ is strongly measurable and $Y \in P_E$. Hence by [J] pp. 19–20, Theorem 2.10, $\sup_{n \in N} X_n \in P_E$. In other words we may and do suppose that Y itself is strongly measurable. So $\|Y\|$ is measurable. Furthermore

$$\sup_{n \in N} \|X_n\| \leq \|Y\|, \quad \text{a.e.}$$

So (the inclusion is valid a.e.), for each $\lambda > 0$

$$\left\{\sup_{n \in N} \|X_n\| > \lambda\right\} \subset \{\|Y\| > \lambda\}.$$

Since $\|Y\|$ is measurable it follows that $\lim_{\lambda \rightarrow \infty} P(\|Y\| > \lambda) = 0$. Hence also

$$\lim_{\lambda \rightarrow \infty} P\left(\sup_{n \in N} \|X_n\| > \lambda\right) = 0. \quad \square$$

3. Pointwise convergence of martingales. In this section it is our purpose to extend Chatterji's martingale convergence theorem ([C]) stating that in every Banach space E with (RNP), each L_E^1 -bounded martingale converges strongly a.e. to an integrable function. As was done already in [U₂], the L_E^1 -boundedness condition was weakened as to be usable in the case of Pettis integrable functions (i.e. his condition (ii) in Theorem 3.1, p. 375 in [U₂], or see §1).

Of the same nature is our extension: instead of L_E^1 -boundedness, a notion of " σ - L_E^1 -boundedness" is required (see condition (ii) in the theorem below) which is natural since Pettis integrals are of σ -bounded variation ([Mu₁]).

THEOREM 3.1. *Let E be a Banach space such that c_0 is not isomorphic to a subspace of E (denoted $c_0 \not\subseteq E$). Let $(X_n, \mathcal{F}_n)_{n \in N}$ be a martingale*

consisting of strongly measurable Pettis integrable functions such that

- (i) $\sup_{n \in N} \|X_n\|_p < \infty$ (i.e. condition (i) in Theorem 3.1 in [U₂])
- (ii) There is a disjoint sequence $(A_n)_{n \in N}$ in $\bigcup_{n \in N} \mathcal{F}_n$ such that for each $m \in N$

$$\sup_{n \in N} \int_{A_m} \|X_n\| < \infty.$$

(iii) For each $\epsilon > 0$ there is a weakly compact set $K \subset E$ such that for each $\epsilon > 0$ there exists $n_0 \in N$ and a set $A_0 \in \mathcal{N}_{n_0}$ with $P(\Omega \setminus A_0) < \epsilon$ such that $n \geq n_0$ implies $\int_A X_n \in P(A)K + \delta B_E$ for all $A \subset A_0$, $A \in \mathcal{F}_n$, (i.e. condition (iii) in Theorem 3.1 in [U₂]). (This condition may in fact be localized on each A_m .)

Then there exists $X_\infty \in P_E$ (of course strongly measurable) such that $(X_n)_{n \in N}$ converges strongly a.e. to X_∞ .

Proof. For each $m \in N$, let $n(m) \in N$ be such that $A_m \in \mathcal{F}_{n(m)}$. Then

$$(X_n|_{A_m}, \mathcal{F}_n|_{A_m})_{n \geq n(m)}$$

is a martingale in $L^1_E(A_m)$ satisfying the conditions of [U₁], Theorem 4. So there exists $X_\infty^m \in L^1_E(A_m)$ such that $(X_n|_{A_m})_{n \in N}$ converges a.e. to X_∞^m for each $m \in N$. Put

$$X_\infty = \sum_{m=1}^\infty X_\infty^m \chi_{A_m}.$$

Obviously $(X_n)_{n \in N}$ converges strongly a.e. to X_∞ .

Furthermore $X_\infty \in P_E$. Indeed, from (i) it follows that for every $x' \in E'$, $(\langle x', X_n \rangle)_{n \in N}$ is an L^1 -bounded martingale so a.e. convergent to a function in L^1 . So X_∞ is scalarly integrable. Since $c_0 \not\subset E$ it now follows from [T₁] (see also [B-H], [J]), since X_∞ is strongly measurable, that $X_\infty \in P_E$. □

COROLLARY 3.2. *Let E have (RNP). Then every martingale $(X_n, \mathcal{F}_n)_{n \in N}$ consisting of strongly measurable Pettis integrable functions such that*

- (i) $\sup_{n \in N} \|X_n\|_p < \infty$.
- (ii) There is a disjoint sequence $(A_n)_{n \in N}$ in $\bigcup_{n \in N} \mathcal{F}_n$ such that for each $m \in N$

$$\sup_{n \in N} \int_{A_m} \|X_n\| < \infty$$

is strongly converging a.e. to a Pettis integrable function.

Proof. Indeed, for each $m \in N$, the martingale $(X_n|_{A_m}, \mathcal{F}_n|_{A_m})_{n \geq n(m)}$ ($n(m)$ was defined in the previous theorem) is $L_E^1(A_m)$ -bounded, with E (RNP). An appeal to Chatterji's theorem [C] now delivers X_∞^m as in the previous proof. The rest is now the same as in this proof since $c_0 \not\hookrightarrow E$. \square

This corollary is a meaningful extension of Chatterji's theorem (even in case $X_n \in L_E^1$ for each $n \in N$, it does not reduce to Chatterji's theorem).

We shall use the corollary later in this paper.

4. Pettis convergence of amarts. It is our purpose in this section to generalize and reprove Musiał's result on Pettis convergence of martingales (or amarts). This is done by first proving a Riesz decomposition theorem in this setting, generalizing the Riesz decomposition of Edgar-Sucheston [E-S].

THEOREM 4.1. *Let E be a Banach space with (WRNP) and $(X_n, \mathcal{F}_n)_{n \in N}$ be an amart in P_E (not necessarily consisting of strongly measurable functions).*

Suppose there is a disjoint sequence $(A_k)_{k \in N}$ in \mathcal{F}_1 and a strictly increasing sequence $(m_n)_{n \in N}$ in N such that

$$\sup_{n \in N} \int_{A_k} \|X_{m_n}\| < \infty$$

for each $k \in N$. Then there is a unique decomposition

$$X_n = Y_n + Z_n$$

where $(Y_n, \mathcal{F}_n)_{n \in N}$ is a martingale in P_E such that

$$\sup_{n \in N} \int_{A_k} \|Y_n\| < \infty$$

for each $k \in N$ and $(Z_n, \mathcal{F}_n)_{n \in N}$ is an adapted sequence such that

$$\lim_{\tau \in T} \|Z_\tau\|_P = 0.$$

Proof. From the definition of an amart we can see (in the same way as in the case of Bochner integrable functions—[B], Theorem 1, p. 279) that, for each $i \in N$

$$\mu_i(A) =: \lim_{n \rightarrow \infty} \int_A X_{m_n}$$

exists for each $A \in \mathcal{F}_i$. It is easy to check that μ_i is of σ -bounded variation w.r.t. (the sets $(A_k)_{k \in N}$ suffice) and that $\mu_i \ll P$. Using (WRNP) this yields $Y_i \in P_E$ such that

$$\int_A Y_i = \mu_i(A)$$

for each $A \in \mathcal{F}_i$ and each $i \in N$.

Now $(Y_n, \mathcal{F}_n)_{n \in N}$ is easily seen to be a martingale and furthermore since $\sup_{n \in N} |\mu_n|_{A_k}(A_k) < \infty$, for each $k \in N$, we see that

$$\sup_{n \in N} \int_{A_k} \|Y_n\| < \infty$$

for each $k \in N$. Furthermore, for each $\sigma \in T$ and each $A \in \mathcal{F}_\sigma$

$$\left\| \int_A (Y_\sigma - X_\sigma) \right\| = \left\| \lim_{\substack{n \rightarrow \infty \\ m_n \geq \sigma}} \int_A X_{m_n} - \int_A X_\sigma \right\|.$$

So

$$\begin{aligned} \sup_{A \in \mathcal{F}_\sigma} \left\| \int_A (Y_\sigma - X_\sigma) \right\| &= \sup_{A \in \mathcal{F}_\sigma} \left\| \lim_{\substack{n \rightarrow \infty \\ m_n \geq \sigma}} \int_A X_{m_n} - \int_A X_\sigma \right\| \\ &\leq \sup_{m_n \geq \sigma} \sup_{A \in \mathcal{F}_\sigma} \left\| \int_A X_{m_n} - \int_A X_\sigma \right\|. \end{aligned}$$

The term on the right-hand side goes to zero for σ going through T , again using [B] Theorem 1, p. 279, extended to our case. Hence

$$\lim_{\sigma \in T} \|Z_\sigma\|_P = \lim_{\sigma \in T} \|Y_\sigma - X_\sigma\|_P = 0.$$

The uniqueness of the decomposition is proved in exactly the same way as in the Bochner integrable case: suppose

$$X_n = Y_n + Z_n = Y'_n + Z'_n$$

where $(Y_n, \mathcal{F}_n)_{n \in N}$ and $(Y'_n, \mathcal{F}_n)_{n \in N}$ are martingales and where $\lim_{\tau \in T} \|Z_\tau\|_P = \lim_{\tau \in T} \|Z'_\tau\|_P = 0$.

Then, for each $A \in \bigcup_n \mathcal{F}_n$

$$\lim_{n \rightarrow \infty} \int_A (Z_n - Z'_n) = 0.$$

But $Y'_n - Y_n = Z'_n - Z_n$ forms also a martingale. So, for each $m \in N$ and $A \in \mathcal{F}_m$

$$\lim_{n \rightarrow \infty} \int_A (Z_n - Z'_n) = \int_A (Z_m - Z'_m) = 0.$$

So $Z_m = Z'_m$, a.e. and also $Y_m = Y'_m$, a.e. for each $m \in N$. \square

COROLLARY 4.2. *Let E have (WRNP) and $(X_n, \mathcal{F}_n)_{n \in N}$ be an amart in P_E such that there is a disjoint sequence $(A_k)_{k \in N}$ in \mathcal{F}_1 and a strictly increasing sequence $(m_n)_{n \in N}$ in N such that*

$$\sup_{n \in N} \int_{A_k} \|X_{m_n}\| < \infty$$

for each $k \in N$. Suppose that $(X_n)_{n \in N}$ is Pettis uniformly integrable. Then there exists a function $X_\infty \in P_E$ such that $\lim_{n \rightarrow \infty} \|X_n - X_\infty\|_P = 0$.

Proof. Using Theorem 4.1 above it suffices to prove the result for the martingale $(Y_n, \mathcal{F}_n)_{n \in N}$ constructed in the above proof. Obviously, since $\lim_{\sigma \in T} \|Z_\sigma\|_P = 0$, $(Y_n)_{n \in N}$ is Pettis uniformly integrable too.

The limit measure $\mu(A) = \lim_{n \rightarrow \infty} \int_A Y_n$ exists on $\bigcup_n \mathcal{F}_n$ and, due to the Pettis uniform integrability, on $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$. It now follows that μ is of σ -bounded variation on \mathcal{F}_∞ and that $\mu \ll P$.

So, the (WRNP) of E implies that $Y_\infty \in P_E$ exists such that

$$\int_A Y_\infty = \mu(A)$$

for each $A \in \mathcal{F}_\infty$. We have $\lim_{n \rightarrow \infty} \|Y_n - Y_\infty\|_P = 0$ since $Y_n = E^{\mathcal{F}_n} Y_\infty$ for each $n \in N$. This is seen in the next lemma.

LEMMA 4.2.1. *Let E have (WRNP). Let $Y \in P_E$ be such that w.r.t. $(\mathcal{F}_n)_{n \in N}$, $E^{\mathcal{F}_n} Y \in P_E$ exists, for each $n \in N$. Then $\lim_{n \rightarrow \infty} \|Y - E^{\mathcal{F}_n} Y\|_P = 0$.*

Proof. If Y is a simple function, the result is certainly true (and even $\lim_{n \rightarrow \infty} \|Y - E^{\mathcal{F}_n} Y\|_1 = 0$ in this case!).

The general case follows, using a $\|\cdot\|_P$ density argument. Simple functions are indeed dense in P_E since E has (WRNP), see [Mu₃], p. 330, Theorem 1. \square

REMARKS 4.2.2. (1) In the above lemma we in fact only used that E has the so-called ‘‘Pettis Compactness Property’’ i.e. In E has every Pettis integral norm relatively compact range.

(2) Corollary 4.2 extends the main result in [Mu₃] (Theorem 5, p. 334) and gives a simpler proof for it. Indeed we did not use Proposition 2 in [Mu₃].

The next corollary belongs to this section but is in fact only a corollary of Corollary 3.2.

COROLLARY 4.3. *Let E have (RNP) and $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be a martingale in P_E such that there is a disjoint sequence $(A_k)_{k \in \mathbb{N}}$ in $\bigcup_n \mathcal{F}_n$ such that for each $k \in \mathbb{N}$*

$$\sup_{n \in \mathbb{N}} \int_{A_k} \|X_n\| < \infty$$

and such that $(X_n)_{n \in \mathbb{N}}$ is Pettis uniformly integrable. Then there is a function $X_\infty \in P_E$ such that $\lim_{n \rightarrow \infty} \|X_n - X_\infty\|_p = 0$.

Proof. Since E has (RNP) and since we are dealing with $\|\cdot\|_p$ -convergence we may suppose (see [J], p. 129) that every X_n is strongly measurable (cf. the discussion in §1 concerning Musial’s theorem). From Corollary 3.2 we have $X_\infty \in P_E$ such that $\lim_{n \rightarrow \infty} X = X_\infty$, a.e. So, the Pettis uniform integrability of $(X_n)_{n \in \mathbb{N}}$ finishes the proof. □

5. Weak convergence of weak sequential amarts. It is our purpose to extend the theorem of Brunel-Sucheston [B – S] on weak convergence a.e. of weak sequential amarts to the case of strongly measurable Pettis integrable functions. First a definition.

DEFINITION 5.1. Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be an adapted sequence in P_E . We say that $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a weak sequential amart (abbreviated (WS) amart) if for every increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ in T the sequence $(\int_\Omega X_{\tau_n})_{n \in \mathbb{N}}$ converges weakly in E .

The same proof as in the case of Bochner integrable functions shows that every amart is a (WS) amart.

In [B – S], Brunel-Sucheston proved the following extension of a result on amarts of Chacon-Sucheston [C-S] to (WS) amarts.

THEOREM 5.2 (Brunel-Sucheston). *Let E be a Banach space with (RNP) and with separable dual. Let $(X_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ be a (WS) amart of class*

(B), i.e.

$$\sup_{\tau \in T} \int_{\omega} \|X_{\tau}\| < \infty.$$

Then there exists $X_{\infty} \in L_E^1$ such that X_n converges to X_{∞} , weakly a.e.

For (WS) amarts consisting of strongly measurable Pettis integrable functions we can prove

THEOREM 5.3. *Let E and E' have (RNP). Let $(X_n, \mathcal{F}_n)_{n \in N}$ be a (WS) amart in P_E , consisting of strongly measurable functions. Suppose that:*

- (i) $\lim_{\lambda \rightarrow \infty} P(\sup_{n \in N} \|X_n\| > \lambda) = 0$
- (ii) $\sup_{\tau \in T} \int_{\{\|X_{\tau}\| \leq \tau\}} \|X_{\tau}\| < \infty$
- (iii) $(X_{\tau})_{\tau \in T}$ is Pettis uniformly integrable.

Then there is a $X_{\infty} \in L_E^1$ such that $(X_n)_{n \in N}$ converges weakly a.e. to X_{∞} .

Proof. For each $n \in N$, define

$$Y_n = X_n \chi_{\{\|X_n\| \leq n\}}.$$

Then $(Y_n, \mathcal{F}_n)_{n \in N}$ is an adapted sequence in L_E^1 , due to the fact that every $\|X_n\|$ is measurable. $(Y_n, \mathcal{F}_n)_{n \in N}$ is furthermore a (WS) amart. Indeed, let $(\tau_n)_{n \in N}$ be any increasing sequence in T . We may assume that $\lim_{n \rightarrow \infty} \tau_n = \infty$. Now

$$\int_{\omega} (X_{\tau_n} - Y_{\tau_n}) = \int_{\{\|X_{\tau_n}\| > \tau_n\}} X_{\tau_n}.$$

Since $\lim_{\lambda \rightarrow \infty} P(\sup_{n \in N} \|X_n\| > \lambda) = 0$ it follows that

$$\lim_{n \rightarrow \infty} P(\|X_{\tau_n}\| > \tau_n) = 0$$

(even if $(\tau_n)_{n \in N}$ is not cofinal in T). Hence, using (iii) one sees easily that

$$\lim_{n \rightarrow \infty} \sup_{A \in \mathcal{F}} \left\| \int_{A \cap \{\|X_{\tau_n}\| > \tau_n\}} X_{\tau_n} \right\| = 0.$$

So

$$\lim_{n \rightarrow \infty} \left\| \int_{\{\|X_{\tau_n}\| > \tau_n\}} X_{\tau_n} \right\| = 0.$$

Since $(X_n, \mathcal{F}_n)_{n \in N}$ is a (WS) amart, there is an $x_0 \in E$ such that

$$\text{weak-lim}_{n \rightarrow \infty} \int_{\Omega} X_{\tau_n} = x_0.$$

So, also $\text{weak-lim}_{n \rightarrow \infty} \int_{\Omega} Y_{\tau_n} = x_0$.

Using (ii), $(Y_n, \mathcal{F}_n)_{n \in N}$ is of class (B).

Hence by Theorem 5.2, there is $Y_{\infty} \in L^1_E$ such that $\text{weak-lim}_{n \rightarrow \infty} Y_n = Y_{\infty}$, a.e. Now

$$\limsup_n \|X_n - Y_n\| = \limsup_n \|X_n\| \chi_{\{\|X_n\| > n\}}.$$

Since $\{\|X_n\| > n\} \subset \{\sup_n \|X_n\| > \lambda\}$ if $n \geq \lambda$ and by (i) we have now that $\limsup_n \|X_n - Y_n\| = 0$. Consequently $(X_n)_{n \in N}$ converges weakly a.e. to Y_{∞} . □

REMARK 5.4. From (ii) we knew already in advance that whenever $(X_n)_{n \in N}$ converges weakly a.e., the limit must be Bochner integrable. Indeed:

$$\sup_{\tau \in T} \int_{\{\|X_{\tau}\| \leq \tau\}} \|X_{\tau}\| \geq \limsup_{\tau \in T} \int_{\{\|X_{\tau}\| \leq \tau\}} \|X_{\tau}\| = \int_{\Omega} \limsup_n \|X_n\| \chi_{\{\|X_n\| \leq n\}}$$

(see [B-E], [Ed]).

Now, if we suppose weak convergence a.e. of $(X_n)_{n \in N}$, we have that $(X_n(\omega))_{n \in N}$ is weakly, hence strongly bounded a.e. So

$$\int_{\Omega} \limsup_n \|X_n\| \chi_{\{\|X_n\| \leq n\}} = \int_{\Omega} \limsup_n \|X_n\|.$$

So (ii) implies

$$\int_{\Omega} \left\| \text{weak lim}_{n \rightarrow \infty} X_n \right\| \leq \int_{\Omega} \liminf_n \|X_n\| \leq \int_{\Omega} \limsup_n \|X_n\| < \infty.$$

6. Strong convergence of pramarts.

DEFINITION 6.1. Let $(X_n, \mathcal{F}_n)_{n \in N}$ be an adapted sequence in P_E . Suppose that for every $\sigma, \tau \in T$ with $\sigma \leq \tau$ the conditional expectations $E^{\mathcal{F}_{\sigma}} X_{\tau} \in P_E$ exist.

$(X_n, \mathcal{F}_n)_{n \in N}$ is called a pramart if for every $\varepsilon > 0$ there exists $\sigma_0 \in T$ such that $\tau \geq \sigma \geq \sigma_0$, $\tau, \sigma \in T$ imply

$$P(\|E^{\mathcal{F}_\sigma} X_\tau - X_\sigma\| \geq \varepsilon) \leq \varepsilon.$$

In [M-S], Millet-Sucheston prove

THEOREM 6.2 (Millet-Sucheston). *Let E be a Banach space with (RNP). Then every pramart $(X_n, \mathcal{F}_n)_{n \in N}$ in L_E^1 such that $\sup_{\tau \in T} \int_{\Omega} \|X_\tau\| < \infty$ is convergent strongly, a.e. (to a Bochner integrable function).*

We can show now the following extension of Theorem 6.2

THEOREM 6.3. *Let E be a Banach space with (RNP). Let $(X_n, \mathcal{F}_n)_{n \in N}$ be a pramart consisting of strongly measurable Pettis integrable functions. Suppose that*

$$(i) \lim_{n \rightarrow \infty} P(\sup_{n \in N} \|X_n\| > \lambda) = 0$$

$$(ii) \sup_{\tau \in T} \int_{\{\|X_\tau\| < \tau\}} \|X_\tau\| < \infty.$$

Then there is a $X_\infty \in L_E^1$ such that $(X_n)_{n \in N}$ converges strongly a.e. to X_∞ .

Proof. We start in the same way as in the proof of Theorem 5.3. Indeed we also apply Theorem 6.2 on $(Y_n, \mathcal{F}_n)_{n \in N}$ where $Y_n = X_n \chi_{\{\|X_n\| \leq n\}}$ for every $n \in N$. $(Y_n, \mathcal{F}_n)_{n \in N}$ is indeed a pramart due to (i).

Since, as in Theorem 5.3 we also have

$$\limsup_n \|X_n - Y_n\| = 0$$

it follows that $(X_n)_{n \in N}$ converges strongly a.e. to $\lim_{n \rightarrow \infty} Y_n \in L_E^1$. \square

REMARKS 6.4.

(1) We have not used condition (iii) of Theorem 5.3 in the proof of the above theorem.

(2) In Theorem 6.2, (i) does not appear but is of course implied by the condition

$$\sup_{\tau \in T} \int_{\Omega} \|X_\tau\| < \infty$$

which appears there. So even if $X_n \in L_E^1$ for each $n \in N$, Theorem 6.3 is an extension of Theorem 6.2.

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