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BIMEASURE ALGEBRAS ON LCA GROUPS

COLIN C. GRAHAM AND BERTRAM M. SCHREIBER

For locally compact abelian groups G_1 and G_2, with character groups Γ_1 and Γ_2, respectively, let $BM(G_1, G_2)$ denote the Banach space of bounded bilinear forms on $C_0(G_1) \times C_0(G_2)$. Using a consequence of the fundamental inequality of A. Grothendieck, a multiplication and an adjoint operation are introduced on $BM(G_1, G_2)$ which generalize the convolution structure of $M(G \times H)$ and which make $BM(G_1, G_2)$ into a K_G^2-Banach *-algebra, where K_G is Grothendieck's universal constant. The Fourier transforms of elements of $BM(G_1, G_2)$ are defined and characterized in terms of certain unitary representations of Γ_1 and Γ_2. Various aspects of the harmonic analysis of the algebras $BM(G_1, G_2)$ are studied.

Introduction. Let S be the space of all doubly-indexed, bilateral, complex sequences of the form $\langle U_1^m\xi, U_2^n\eta\rangle$, where U_1 and U_2 are unitary operators on a Hilbert space H and $\xi, \eta \in H$. In **[17]** it was shown that, under coordinatewise addition and multiplication, S is an algebra containing all sequences of Fourier-Stieltjes coefficients of complex Borel measures on the torus T^2. It was also show that if

$$V = C(T) \hat{\otimes} C(T)$$

denotes the projective tensor product of the space $C(T)$ with itself, then there is a natural embedding of S in the dual V^* of V. Namely, if $(\alpha_{mn}) \in S$ there is a unique element $u \in V^*$ such that

$$\alpha_{mn} = \left\langle e^{-in\theta} \otimes e^{-im\phi}, u\right\rangle, \qquad -\infty < m, n < \infty.$$

The question whether every element of V^* arises from S in this way was left open in **[17]**. However, it was pointed out to us by G. Pisier that a positive answer to this question follows easily from the Fundamental Theorem of the Metric Theory of Tensor Products (Theorem 1.2 below) of A. Grothendieck. (See Theorem 2.4(i).) It is a pleasure to express our gratitude to him here for having communicated this fact to us and to T. Ito for a number of helpful conversations with the second author. The purpose of this paper is to extend these ideas to the context of all locally compact abelian (LCA) groups and to examine some of their ramifications. That is, we wish to initiate the study of the harmonic analysis of the space $[C_0(G_1) \hat{\otimes} C_0(G_2)]^*$ of bimeasures on a pair G_1, G_2 of LCA groups.

We shall begin with some preliminary observations about bimeasures on locally compact spaces and spaces of Fourier-Stieltjes transforms. In particular, we introduce in §1 the concepts of discrete and continuous bimeasures and show that every bimeasure is the sum of its discrete and continuous parts. These results are related to those of [25], although our methods differ from those of Saeki. The Fourier transform a bimeasure on the LCA group $G_1 \times G_2$ is also introduced here. We prove in §2 that if G_1 and G_2 are LCA groups, then the space of bimeasures on $G_1 \times G_2$ has a natural Banach-algebra structure which agrees with convolution on the space of measures on $G_1 \times G_2$. There follows a short section on subgroups, quotients and so on. The main result obtained in §3 is an extension to the present context of the well-known result of W. F. Eberlein characterizing Fourier-Stieltjes transforms on a LCA group Γ as the continuous functions on Γ which are Fourier-Stieltjes transforms on the discrete group Γ_d. In §4 we study the closure of $L^1(G_1 \times G_2)$ as a subalgebra of the bimeasure algebra on $G_1 \times G_2$. Section 5 is devoted to the subject of idempotent bimeasures and homomorphisms between bimeasure algebras, as well as some consequences of the fact (Theorem 5.8) that when $G_1 = G_2 = G$ and Δ denotes the diagonal in the dual of $G \times G$, then every bounded, uniformly continuous function on Δ is the restriction of the Fourier transform of a bimeasure. We conclude in §6 with a short discussion of Sidon sets in the context of bimeasures.

A number of authors have studied tensor products in the context of Banach algebras. See [1], [2], [8], [9], [11] and [20] and the references cited in those papers. However, it should be observed that it is an easy consequence of our Theorem 5.8 cited above that, at least when $G_1 = G_2 = G$, the space of bimeasures on $G \times G$ does not arise as the completion of $M(G) \otimes M(G)$ with respect to any tensorial norm. Our approach to the algebra structure on the space of bimeasures on $G_1 \times G_2$ is via generalization of the arguments appearing in [17].

NOTATION. The symbols G, G_1, G_2 will always stand for LCA groups. The character group and the Bohr compactification of G, G_i will be denoted by Γ, Γ_i and bG, bG_i, respectively, and Γ_d, Γ_{id} denote Γ, Γ_i with the discrete topology. As is customary, $L^1(G)$, $M(G)$, $A(G)$, $B(\Gamma)$ and $PM(\Gamma)$ denote, respectively, the group and measure (*-)algebras of G and the algebras of Fourier transforms, Fourier-Stieltjes transforms and pseudo-measures on Γ. For $\mu \in M(G)$, the Fourier-Stieltjes transform of μ is given by $\hat{\mu}(\gamma) = \int_G (-x, \gamma)\, d\mu(x)$. X and Y will represent locally compact spaces, and for such a space X, $\mathfrak{L}^\infty(X)$, $C(X)$, $C_0(X)$ and $C_{00}(X)$ are the spaces of bounded functions on x which are, respectively, Borel measurable, continuous, continuous with limit zero at infinity, and continuous with compact support. The norm in $C(X)$ is denoted by $\| \|_X$.

1. Tensor algebras and bimeasures.

DEFINITION 1.1. Given locally compact spaces X and Y, let $V_0 = V_0(X, Y) = C_0(X) \hat{\otimes} C_0(Y)$ be the projective tensor product of the indicated spaces of functions. Elements of the dual space of $V_0(X, Y)$ have been referred to traditionally as *bimeasures* on $X \times Y$. Following **[28]**, for example, we denote the dual of $V_0(X, Y)$ by $BM(X, Y)$. Let $V(X, Y) = C(X) \hat{\otimes} C(Y)$ and set

$$N(X, Y) = \{f \in C(X \times Y): fg \in V_0(X, Y) \text{ for all } g \in V_0(X, Y)\}.$$

The version of the Fundamental Theorem of Grothendieck which we need is the following one. For proofs and applications in a contemporary setting, we refer the reader to **[18]**, **[8]**, **[9]**, and **[11]**. In particular, see **[11**, Theorem 3.1].

THEOREM 1.2. *Let $u \in BM(X, Y)$. There exist regular Borel probability measures λ_X on X and λ_Y on Y and $C > 0$ such that*

$$(1) \qquad |\langle f \otimes g, u\rangle| \leq C\|f\|_2 \|g\|_2, \qquad f \in C_0(X), g \in C_0(Y),$$

where the L^2-norms refer to $L^2(X, \lambda_X)$ and $L^2(Y, \lambda_Y)$, respectively. Let $|||u||| = \inf\{C$: (1) holds for some $\lambda_X, \lambda_Y\}$. Then there is a universal constant K_G such that

$$\|u\| \leq |||u||| \leq K_G\|u\|, \qquad u \in BM(X, Y).$$

Thus u can be extended to a bounded linear functional on $L^2(X, \lambda_X) \hat{\otimes} L^2(Y, \lambda_Y)$.

If u, λ_X and λ_Y are as in Theorem 1.2, we shall refer to λ_X, λ_Y as a *Grothendieck measure pair* for u and denote the norm in $L^2(X, \lambda_X) \hat{\otimes} L^2(Y, \lambda_Y)$ by $\| \|_{2,2}$. Similarly, we denote the norm in $\mathfrak{L}^\infty(X) \hat{\otimes} \mathfrak{L}^\infty(Y)$ by $\| \|_{\infty,\infty}$.

COROLLARY 1.3. *If $u \in BM(X, Y)$ there is a unique extension of u to $\mathfrak{L}^\infty(X) \hat{\otimes} \mathfrak{L}^\infty(Y)$ such that for every pair λ_X, λ_Y of Grothendieck measures for u with constant C as in* (1),

$$|\langle \phi \otimes \psi, u\rangle| \leq C\|\phi\|_2 \|\psi\|_2, \qquad \phi \in \mathfrak{L}^\infty(X), \psi \in \mathfrak{L}^\infty(Y).$$

Proof. Let λ_X, λ_Y and λ'_X, λ'_Y be two Grothendieck measure pairs for u. Set $\mu_X = \frac{1}{2}(\lambda_X + \lambda'_X)$ and $\mu_Y = \frac{1}{2}(\lambda_Y + \lambda'_Y)$. Then there is a constant $C > 0$ such that (1) holds with reference to $L^2(X, \mu_X)$ and $L^2(Y, \mu_Y)$. Given $\phi \in \mathfrak{L}^\infty(X)$ and $\psi \in \mathfrak{L}^\infty(Y)$, choose $\{f_n\}_{n=1}^\infty \subset C_0(X)$ and $\{g_n\}_{n=1}^\infty \subset C_0(Y)$ such that $f_n \to \phi$ in $L^2(X, \mu_X)$ and $g_n \to \psi$ in $L^2(Y, \mu_Y)$. Then $f_n \otimes g_n \to \phi \otimes \psi$ in both $L^2(X, \lambda_X) \hat{\otimes} L^2(Y, \lambda_Y)$ and $L^2(X, \lambda'_X) \hat{\otimes} L^2(Y, \lambda'_Y)$, and the corollary follows.

Our first application of Theorem 1.2 concerns bimeasures with compact support. Recall that the support of a bimeasure u on $X \times Y$ is the smallest closed set F in $X \times Y$ for which $\langle f, u \rangle = 0$ for all $f \in V_0(X, Y)$ such that $f \equiv 0$ on a neighborhood of F.

LEMMA 1.4. *The bimeasures with compact support are dense in $BM(X, Y)$.*

Proof. For $\phi_X \in \mathfrak{L}^\infty(X)$, $\phi_Y \in \mathfrak{L}^\infty(Y)$, $u \in BM(X, Y)$ and $\phi = \phi_X \otimes \phi_Y$, we define

$$\langle f, \phi u \rangle = \langle \phi f, u \rangle, \qquad f \in V_0(X, Y).$$

Since $\phi f \in \mathfrak{L}^\infty(X) \hat{\otimes} \mathfrak{L}^\infty(Y)$, Corollary 1.3 applies, and $\langle \phi f, u \rangle$ is well defined. Thus $\phi u \in BM(X, Y)$. If ϕ_X and ϕ_Y are continuous, then

$$\|\phi u\| \le \|\phi_X\|_X \|\phi_Y\|_Y \|u\|.$$

Let $u \in BM(X, Y)$, and let λ_X, λ_Y be a Grothendieck measure pair for u. Given $\varepsilon > 0$, choose a compact set $K_X \subset X$ such that $\lambda_X(K_X^c) < \varepsilon^2$ and $\phi_X \in C_{00}(X)$ such that $0 \le \phi_X \le 1$ and $\phi_X \equiv 1$ on K_X. Similarly choose K_Y and ϕ_Y. Then ϕu has support in the compact set $(\operatorname{supp} \phi_X) \times (\operatorname{supp} \phi_Y)$, and

$$\begin{aligned} \|u - \phi u\| &= \sup_{\|f\|_{V_0} \le 1} |\langle f, u - \phi u \rangle| = \sup_{\|f\|_{V_0} \le 1} |\langle f - \phi f, u \rangle| \\ &\le C \sup_{\|f\|_{V_0} \le 1} \|(1 - \phi) f\|_{2,2} \le C \|1 - \phi\|_{2,2} \\ &\le C\big(2\lambda_X(K_X^c)^{1/2} + \lambda_Y(K_Y^c)^{1/2}\big) < 3C\varepsilon. \end{aligned}$$

The lemma follows.

LEMMA 1.5. *Let E and F be closed subsets of X and Y, respectively. There is a projection of norm one from $BM(X, Y)$ onto $BM(E, F)$.*

Proof. For $u \in BM(X, Y)$, let $v = (\chi_E \otimes \chi_F)u$, χ_E and χ_F being the characteristic functions of E and F and the product defined as in Lemma 1.4. Then v is supported on $E \times F$, and it is clear that the mapping $u \to v$ is a linear projection onto $BM(E, F)$. To see that that mapping has norm at most one, choose a Grothendieck measure pair λ_X, λ_Y for u. For each positive integer n, choose an open set U_n in X such that $E \subset U_n$ and $\lambda_X(U_n) < \lambda_X(E) + n^{-1}$ and choose a function $f_n \in C(X)$ such that $f_n \equiv 1$ on E, $f_n \equiv 0$ on U_n^c and $0 \le f_n \le 1$. Then $f_n \to \chi_E$ in $L^2(X, \lambda_X)$. Similarly choose $\{g_n\}_{n=1}^{\infty} \subset C(Y)$ so that $g_n \to \chi_F$ in $L^2(Y, \lambda_Y)$. Then $\|(f_n \otimes g_n)u\| \le \|u\|$, $n \ge 1$, and $(f_n \otimes g_n)u \to v$ weak-* in $BM(X, Y)$. Thus $\|v\| \le \|u\|$, and the lemma is proved.

DEFINITION 1.6. Let E and F be closed subsets of X and Y, respectively. If $u \in BM(X, Y)$, then the image of u under the projection of $BM(X, Y)$ onto $BM(E, F)$ will be called the restriction of u to $E \times F$ and denoted $u|_{E\times F}$. Note that there is a natural isometric isomorphism between $BM(E, F)$ and the subspace of $BM(X, Y)$ of bimeasures supported on $E \times F$. If $u \in BM(X, Y)$ satisfies $u|_{E\times F} = 0$ for all finite subsets E of X and F of Y, then we shall call u a *continuous* bimeasure. If there exist increasing sequences $\{E_n\}$ and $\{F_n\}$ of finite subsets of X and Y, respectively, such that $u = \lim_{n\to\infty} u|_{E_n\times F_n}$ (norm limit), then we call u a *discrete* bimeasure. The spaces of all continuous and all discrete bimeasures on $X \times Y$ will be denoted, respectively, by $BM_c(X, Y)$ and $BM_d(X, Y)$.

LEMMA 1.7. *Let $\mathfrak{S}$ be the family of all finite sets of the form $E \times F$, where $E \subset X$ and $F \subset Y$, and let $\mathfrak{S}$ be directed by inclusion. Let $u \in BM(X, Y)$. Then $u \in BM_d(X, Y)$ if and only if $u = \lim_{E\times F\in\mathfrak{S}} u|_{E\times F}$ in the norm topology.*

Proof. If $u = \lim_{E\times F\in\mathfrak{S}} u|_{E\times F}$, then it is clear that $u \in BM_d(X, Y)$. The opposite implication follows from the observation that if $E \subset E'$ and $F \subset F'$ then

$$\|u - u|_{E'\times F'}\| \le 3\|u - u|_{E\times F}\|. \tag{2}$$

To see that this is true, set $A = E' \cap E^c$, $B = F' \cap F^c$ and $v = u - u|_{E\times F}$. Then

$$\begin{aligned} u - u|_{E'\times F'} &= u - u|_{E\times F} - u|_{A\times F} - u|_{E'\times B} \\ &= v - v|_{A\times F} - v|_{E'\times B}. \end{aligned}$$

Since restrictions are norm decreasing, the inequality (2) holds. The proof of our lemma is complete.

THEOREM 1.8. (1) *$BM_c(X, Y)$ and $BM_d(X, Y)$ are closed linear subspaces of $BM(X, Y)$.*

(2) *There is a norm-reducing projection from $BM(X, Y)$ onto $BM_d(X, Y)$ whose kernel is $BM_c(X, Y)$. Thus topologically,*

$$BM(X, Y) = BM_c(X, Y) \oplus BM_d(X, Y).$$

Proof. (1) That $BM_c(X, Y)$ is a closed linear subspace follows at once from Lemma 1.5. That $BM_d(X, Y)$ is a linear space follows easily from Lemma 1.7. To show that $BM_d(X, Y)$ is closed, we argue as follows. Let $\{u_n\} \subset BM_d(X, Y)$ be such that $u_n \to u \in BM(X, Y)$. Given $\varepsilon > 0$, choose n_0 so that $\|u_{n_0} - u\| < \varepsilon$ and finite sets $E_0 \subset X$ and $F_0 \subset Y$ such that

$$\left\|u_{n_0} - u_{n_0}|_{E\times F}\right\| < \varepsilon \quad \text{whenever } E_0 \subset E,\ F_0 \subset F.$$

Then for such E and F,

$$\begin{aligned}\left\|u - u|_{E\times F}\right\| &\leq \|u - u_{n_0}\| + \left\|u_{n_0} - u_{n_0}|_{E\times F}\right\| + \left\|u_{n_0}|_{E\times F} - u|_{E\times F}\right\| \\ &\leq 2\|u - u_{n_0}\| + \left\|u_{n_0} - u_{n_0}|_{E\times F}\right\| < 3\varepsilon,\end{aligned}$$

so $u \in BM_d(X, Y)$.

(2) Let $u \in BM(X, Y)$ with Grothendieck measure pair λ_X and λ_Y. Let $E = \{x \in X: \lambda_X(\{x\}) > 0\}$ and $F = \{y \in Y: \lambda_Y(\{y\}) > 0\}$. Let $E_1 \subset E_2 \subset \cdots \subset E$ be finite sets whose union is E, and similarly choose $F_1 \subset F_2 \subset \cdots \subset F$. Set $v = (\chi_E \otimes \chi_F)u$ and $v_n = (\chi_{E_n} \otimes \chi_{F_n})u = u|_{E_n\times F_n}$. Given $\varepsilon > 0$, choose n so that

$$\lambda_X(E \cap E_n^c) < \varepsilon^2, \qquad \lambda_Y(F \cap F_n^c) < \varepsilon^2.$$

Then a calculation similar to that in the proof of Lemma 1.4 gives

$$\begin{aligned}\|v - v_n\| &= \sup_{\|f\|_{V_0}\leq 1} \left|\left\langle f(\chi_E \otimes \chi_F - \chi_{E_n} \otimes \chi_{F_n}), u\right\rangle\right| \\ &\leq C\left\|\chi_{\chi_E} \otimes \chi_F - \chi_{E_n} \otimes \chi_{F_n}\right\|_{2,2} < 3C\varepsilon.\end{aligned}$$

Thus $v \in BM_d(X, Y)$ and $\|v\| \leq \|u\|$. If P and Q are finite sets in X and Y, respectively, such that $E \cap P = F \cap Q = \varnothing$, then $u|_{P\times Q} = 0$. It is now easy to check that $w = u - v \in BM_c(X, Y)$. So it remains only to check that

$$BM_c(X, Y) \cap BM_d(X, Y) = \{0\},$$

which follows immediately from the definitions. This ends the proof of Theorem 1.8.

EXAMPLE 1.9. It is not true in general that if $u \in BM_c(X, Y)$ and $v \in BM_d(X, Y)$ then $\|u + v\| = \|u\| + \|v\|$. Suppose X supports a continuous probability measure μ. We may assume $\operatorname{supp}\mu$ is not all of X. Choose $x \in X \backslash \operatorname{supp}\mu$ and $y_1, y_2 \in Y$. For $f \in C_0(X)$ and $g \in C_0(Y)$, set

$$\langle f \otimes g, u \rangle = (g(y_1) - g(y_2)) \int_X f\, d\mu$$

and

$$\langle f \otimes g, v \rangle = f(x)(g(y_1) + g(y_2)).$$

Then $u \in BM_c(X, Y)$, $v \in BM_d(X, Y)$, $\|u\| = \|v\| = 2$, but $\|u + v\| = 2^{3/2}$, as straightforward calculations show.

DEFINITION 1.10. Let G_1 and G_2 be LCA groups with character groups Γ_1 and Γ_2, respectively. Set $G = G_1 \times G_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Let $u \in BM(G_1, G_2)$. We define the *Fourier transform* $\hat{u}$ of u by the formula

$$\hat{u}(\gamma, \delta) = \langle \bar{\gamma} \otimes \bar{\delta}, u \rangle, \qquad \gamma \in \Gamma_1, \delta \in \Gamma_2.$$

That the formula for $\hat{u}$ makes sense follows from Corollary 1.3, which also implies

$$\|\hat{u}\|_{\Gamma_1 \times \Gamma_2} \leq K_G \|u\|.$$

However, since the compactly-supported bimeasures are dense in $BM(G_1, G_2)$, it is easy to see that

$$\|\hat{u}\|_{\Gamma_1 \times \Gamma_2} \leq \|u\|, \qquad u \in BM(G_1, G_2),$$

since, when u has compact support, each character agrees on the support of u with some $f \hat{\otimes} g \in V_0$, where $\|f \otimes g\|_\infty \leq 1$.

REMARKS 1.11. (1) Preserving the notation of Definition 1.10, let us recall the relationships between the spaces of Fourier transforms and Fourier-Stieltjes transforms on Γ and the spaces introduced above. Namely:

(i) $A(\Gamma) \subset V_0(\Gamma_1, \Gamma_2)$.

(ii) $B(\Gamma) \subset N(\Gamma_1, \Gamma_2)$.

Moreover, each of these containments represents an embedding of norm one. Indeed, it is a well-known result of Grothendieck **[13]** that isometrically

$$L^1(G) = L^1(G_1; L^1(G_2)) = L^1(G_1) \hat{\otimes} L^1(G_2).$$

So if $f \in L^1(G)$ and $\varepsilon > 0$ are given, then there exist $\{f_n\}_{n=1}^{\infty} \subset L^1(G_1)$ and $\{g_n\} \subset L^1(G_2)$ such that

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y) \qquad \text{a.e. on } G$$

and

$$\sum_{n=1}^{\infty} \|f_n\|_1 \, \|g_n\|_1 < (1 + \varepsilon)\|f\|_1.$$

Thus

$$\hat{f}(\gamma, \delta) = \sum_{n=1}^{\infty} \hat{f}_n(\gamma) \hat{g}_n(\delta), \qquad \gamma \in \Gamma_1, \delta \in \Gamma_2,$$

so $\hat{f} \in V_0(\Gamma_1, \Gamma_2)$, and we have

$$\|\hat{f}\|_{V_0} \leq \sum_{n=1}^{\infty} \|\hat{f}_n\|_{\Gamma_1} \|\hat{g}_n\|_{\Gamma_2} \leq (1 + \varepsilon)\|f\|_1.$$

Assertion (i) follows.

To see that (ii) holds, let $\{f_\alpha\}$ be an approximate identity in $L^1(G)$ such that for each compact set $K \subset \Gamma$, $\hat{f}_\alpha \equiv 1$ on K for some α and $\|f_\alpha\|_1 \leq 1$. For each α as above and $\mu \in M(G)$, $(f_\alpha * \mu)\hat{} \in V_0(\Gamma_1, \Gamma_2)$ and $\|(f_\alpha * \mu)\hat{}\|_{V_0} \leq \|\mu\|$ by (i). Since $C_{00}(\Gamma_1) \otimes C_{00}(\Gamma_2)$ is dense in $V_0(\Gamma_1, \Gamma_2)$, it follows easily that $\hat{\mu} \in N(\Gamma_1, \Gamma_2)$ with $\|\hat{\mu}\|_N \leq \|\mu\|$. (The norm on N is the multiplier norm of operators on $V_0(\Gamma_1, \Gamma_2)$.)

(2) Suppose G is compact, and let P: $C(G \times G) \to C(G)$ be the Herz P-mapping, namely,

$$Pf(y) = \int_G f(y - x, x)\, dx, \qquad y \in G.$$

Then, as is well known, P maps $C(G \times G)$ onto $C(G)$, and if $A(G)$ is given its usual norm via $l^1(\Gamma)$ then the restriction of P to $V(G, G)$ is a norm-reducing map of $V(G, G)$ onto $A(G)$. In this case, $PM(G) = A(G)^*$ $\cong l^\infty(\Gamma)$. The following observation will be used in several places in the present work to transfer statements about $PM(G)$ to analogous assertions concerning $BM(G, G)$ when G is compact. The details are straightforward using, say, [**12**, 11.1.1] and will be left to the reader.

LEMMA 1.12. *Let G be compact and P: $V(G, G) \to A(G)$ be the Herz P-map. Then P^*: $PM(G) \to BM(G, G)$ is an isometric embedding on $PM(G)$ carrying $M(G)$ into $M(G \times G)$ and $L^1(G)$ into $L^1(G \times G)$, and $\|P^*\mu\|_{M(G\times G)} = \|\mu\|_{M(G)}$, $\mu \in M(G)$. More specifically, for $S \in PM(G)$,*

$$(P^*S)\hat{}(\gamma, \delta) = \begin{cases} \hat{S}(\gamma), & \gamma = \delta, \\ 0, & \gamma \neq \delta. \end{cases}$$

We conclude this section with another lemma which we shall need later.

LEMMA 1.13. *Let $u \in BM(G_1, G_2)$. There is a net $\{u_\alpha\}$ of finitely-supported bimeasures on $G_1 \times G_2$ such that $\|u_\alpha\| \le \|u\|$,*

$$\lim_\alpha \langle f, u_\alpha \rangle = \langle f, u \rangle, \qquad f \in V(G_1, G_2),$$

and $\hat{u}_\alpha \to \hat{u}$ uniformly on compact sets in $\Gamma_1 \times \Gamma_2$.

Proof. By Lemma 1.4, Theorem 1.2 and Corollary 1.3, we may assume u has compact support contained in $K = K_1 \times K_2$, where K_i is compact in G_i, $i = 1, 2$. Let E be a compact subset of $\Gamma = \Gamma_1 \times \Gamma_2$. Then $\{\gamma|_K: \gamma \in E\}$ has compact (norm) closure in $V(K_1, K_2)$. Thus to prove our lemma it will suffice to prove the (more general) assertion that *given $u \in BM(K_1, K_2)$ there exists a net $\{u_\alpha\}$ of finitely-supported bimeasures such that $\|u_\alpha\| \le \|u\|$ for all α, and $u_\alpha \to u$ in the topology of uniform convergence on compact sets of $V(K_1, K_2)$.*

A standard two-epsilon argument shows that it will suffice to find a net $\{u_\alpha\}$ of finitely supported bimeasures such that $\|u_\alpha\| \le \|u\|$ for all α and such that for each finite set $\{f_1, \ldots, f_n\} \subset V(K_1, K_2)$, $\langle f_j, u_\alpha \rangle \to \langle f_j, u \rangle$, $j = 1, \ldots, n$. The existence of such a net $\{u_\alpha\}$ is the content of [**12**, Lemma 11.1.6]. The proof of the lemma is complete.

2. Bimeasure algebras and unitary representations.

DEFINITION 2.1. If Γ_1 and Γ_2 are LCA groups, let $S(\Gamma_1, \Gamma_2)$ be the set of all functions α on $\Gamma_1 \times \Gamma_2$ of the form

$$\alpha(\gamma, \delta) = \langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle, \tag{3}$$

where π_1 and π_2 are strongly-continuous unitary representations of Γ_1 and Γ_2, respectively, on a Hilbert space H and ξ, $\eta \in H$. Clearly $S(\Gamma_1, \Gamma_2)$ consists of bounded, uniformly continuous functions on $\Gamma_1 \times \Gamma_2$.

LEMMA 2.2. *$S(\Gamma_1, \Gamma_2)$ is a conjugate-closed algebra of functions on $\Gamma_1 \times \Gamma_2$.*

Proof. Clearly $S(\Gamma_1, \Gamma_2)$ is closed under complex conjugation and scalar multiplication. Let $\alpha \in S(\Gamma_1, \Gamma_2)$ be given by (3) on the Hilbert space H and $\beta \in S(\Gamma_1, \Gamma_2)$ be given as in (3) by representations π_1' and π_2' and vectors ξ' and η' in the Hilbert space K. On the Hilbert space $H \oplus K$, the representations $\pi_1 \oplus \pi_1'$ and $\pi_2 \oplus \pi_2'$ are unitary, and

$$\begin{aligned}\alpha(\gamma,\delta) + \beta(\gamma,\delta) &= \langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle_H + \langle \pi_1'(\gamma)\xi', \pi_2'(\delta)\eta' \rangle_K \\ &= \langle (\pi_1(\gamma)\xi, \pi_1'(\gamma)\xi'), (\pi_2(\delta)\eta, \pi_2'(\delta)\eta') \rangle_{H\oplus K} \\ &= \langle (\pi_1 \otimes \pi_1')(\gamma)(\xi,\xi'), (\pi_2 \oplus \pi_2')(\delta)(\eta,\eta') \rangle_{H\oplus K}.\end{aligned}$$

Thus $\alpha + \beta \in S(\Gamma_1, \Gamma_2)$. To show $S(\Gamma_1, \Gamma_2)$ is closed under multiplication, consider the Hilbert-space tensor product $H \overline{\otimes} K$ of H and K. The representations $\pi_1 \otimes \pi_1'$ and $\pi_2 \otimes \pi_2'$ are unitary on $H \overline{\otimes} K$, and

$$\begin{aligned}\alpha(\gamma,\delta)\beta(\gamma,\delta) &= \langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle_H \langle \pi_1'(\gamma)\xi', \pi_2'(\delta)\eta' \rangle_K \\ &= \langle \pi_1(\gamma)\xi \otimes \pi_1'(\gamma)\xi', \pi_2(\delta)\eta \otimes \pi_2'(\delta)\eta' \rangle_{H\overline{\otimes}K} \\ &= \langle (\pi_1 \otimes \pi_1')(\gamma)(\xi \otimes \xi'), (\pi_2 \otimes \pi_2')(\delta)(\eta \otimes \eta') \rangle_{H\overline{\otimes}K},\end{aligned}$$

so $\alpha\beta \in S(\Gamma_1, \Gamma_2)$. That ends the proof of Lemma 2.2.

The following lemma is a special case of a well-known result about representations of *-algebras. (See [**19**, p. 147].)

LEMMA 2.3. *Let π be a strongly-continuous unitary representation of Γ on the Hilbert space H, and let $f \to \pi(f)$ be the induced *-representation of $L^1(\Gamma)$, namely*

$$\pi(f)\xi = \int_\Gamma f(\gamma)\pi(\gamma)\xi \, d\gamma, \qquad f \in L^1(\Gamma), \xi \in H.$$

Then $\|\pi(f)\| \le \|\hat{f}\|_G$, $f \in L^1(\Gamma)$.

Proof. Let A denote the closure of $\{\pi(f): f \in L^1(\Gamma)\}$ in $\mathfrak{B}(H)$. Then A is a commutative C^*-algebra; let $\mathfrak{M}$ denotes its maximal ideal space. There is a natural continuous embedding Φ of $\mathfrak{M}$ into G such that $\pi(f)\hat{}(\phi) = \hat{f}(\Phi(\phi))$, $\phi \in \mathfrak{M}$. Thus $\|\pi(f)\| = \|\pi(f)\hat{}\|_{\mathfrak{M}} \le \|\hat{f}\|_G$, and our lemma is proved.

THEOREM 2.4. (i) *If* $u \in BM(G_1, G_2)$, *then* $\hat{u} \in S(\Gamma_1, \Gamma_2)$.

(ii) *If* $\alpha \in S(\Gamma_1, \Gamma_2)$, *then there exists a unique* $u \in BM(G_1, G_2)$ *such that* $\hat{u} = \alpha$.

(iii) *If* $\alpha \in S(\Gamma_1, \Gamma_2)$ *is represented as in* (3) *and* $u \in BM(G_1, G_2)$ *is such that* $\hat{u} = \alpha$, *then* $\|u\| \leq \|\xi\| \|\eta\|$.

Proof. (i) We may assume $u \neq 0$. Let λ_1, λ_2 be a Grothendieck measure pair for u, so that (1) holds. Then there is an operator

$$T\colon L^2(G_1, \lambda_1) \to L^2(G_2, \lambda_2)$$

such that $\langle f \otimes g, u \rangle = \langle Tf, \bar{g} \rangle$. Let π_1 and π_2 be the representations on $L^2(G_1, \lambda_1)$ and $L^2(G_2, \lambda_2)$ given, respectively, by

$$\pi_1(\gamma) f = \bar{\gamma} f, \qquad \pi_2(\delta) g = \delta g.$$

Then π_1 and π_2 are strongly-continuous unitary representations of Γ_1 and Γ_2, respectively, and

$$\hat{u}(\gamma, \delta) = \langle \bar{\gamma} \otimes \bar{\delta}, u \rangle = \langle T\bar{\gamma}, \delta \rangle = \langle T\pi_1(\gamma)1, \pi_2(\delta)1 \rangle.$$

Let H be the Hilbert space $L^2(G_1, \lambda_1) \oplus L^2(G_2, \lambda_2)$, and let the extension of T to H with matrix $\left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$ also be denoted by T. Following [**17**, p. 638], let $c = \|T\|$ and let W be a unitary dilation of $c^{-1}T$ on the Hilbert space K containing H. Writing

$$K = L^2(G_1, \lambda_1) \oplus L^2(G_2, \lambda_2) \oplus H^{\perp},$$

set

$$\begin{aligned} \tilde{\pi}_1 &= \pi_1 \oplus I \oplus I, \\ \pi_2 &= W^*(I \oplus \pi_2 \oplus I)W, \\ \xi &= (c \cdot 1, 0, 0) \end{aligned}$$

and

$$\eta = W^*(0, 1, 0).$$

Then in K,

$$\begin{aligned} \hat{u}(\gamma, \delta) &= \langle cW(\pi_1(\gamma)1, 0, 0), (0, \pi_2(\delta)1, 0) \rangle \\ &= \langle c(\pi_1(\gamma)1, 0, 0), W^*(0, \pi_2(\delta)1, 0) \rangle \\ &= \langle \tilde{\pi}_1(\gamma)\xi, \tilde{\pi}_2(\delta)\eta \rangle. \end{aligned}$$

(ii)–(iii). Let $\alpha \in S(\Gamma_1, \Gamma_2)$. For $f \in A(G_1)$, $g \in A(G_2)$, let $\phi \in L^1(\Gamma_1)$, $\psi \in L^1(\Gamma_2)$ be such that $\hat{\phi} = f$ and $\hat{\psi} = g$. Define

$$\langle f \otimes g, u \rangle = \int_{\Gamma_2} \int_{\Gamma_1} \phi(-\gamma)\psi(-\delta)\alpha(\gamma, \delta)\, d\gamma\, d\delta.$$

If α is represented as in (3), then by Lemma 2.3,

$$\begin{aligned} |\langle f \otimes g, u \rangle| &= \left| \int_{\Gamma_2} \int_{\Gamma_1} \phi(-\gamma)\psi(-\delta)\langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle\, d\gamma\, d\delta \right| \\ &= \left| \left\langle \int_{\Gamma_1} \phi(-\gamma)\pi(\gamma)\xi\, d\gamma, \int_{\Gamma_2} \overline{\psi(-\delta)}\pi_2(\delta)\eta\, d\delta \right\rangle \right| \\ &= |\langle \pi_1(\overline{\phi^*})\xi, \pi_2(\psi^*)\eta \rangle| \\ &\leq \|\pi_1(\overline{\phi^*})\|\, \|\xi\|\, \|\pi_2(\psi^*)\|\, \|\eta\| \leq \|\xi\|\, \|\eta\|\, \|f\|_{G_1} \|g\|_{G_2}. \end{aligned}$$

Since u is clearly bilinear on $A(G_1) \times A(G_2)$ and $A(G_i)$ is dense in $C_0(G_i)$ $(i = 1, 2)$, u extends to a bimeasure on $G_1 \times G_2$ of norm at most $\|\xi\|\|\eta\|$, yielding (iii) once (ii) is proven.

Extend u to $\mathcal{L}^\infty(G_1) \hat{\otimes} \mathcal{L}^\infty(G_2)$ via Corollary 1.3. We now show that $\hat{u} = \alpha$. Let $\gamma_0 \in \Gamma_1$, $\delta_0 \in \Gamma_2$. For $i = 1, 2$ and for each neighborhood U_i of 0 in Γ_i, let ϕ_{U_i} be a nonnegative function in $L^1(\Gamma_i)$ such that $\phi_{U_i} = 0$ outside U_i, $\phi_{U_i} = \phi^*_{U_i}$, and $\int_{U_i} \phi_{U_i}(\gamma)\, d\gamma = 1$. If λ_1, λ_2 is a Grothendieck measure pair for u, then for $\sigma \in \Gamma_i$,

$$((\phi_{U_i})_\sigma)\hat{}(x) = (x, \sigma)\hat{\phi}_{U_i}(x) \to (x, \sigma)$$

uniformly on compact sets in G_i, hence in $L^2(G_i, \lambda_i)$. (Here $\phi_\sigma(\gamma) = \phi(\gamma - \sigma)$.) For $\chi, \zeta \in H$, the strong continuity of π_1 and π_2 gives

$$\begin{aligned} &\left| \int_{\Gamma_2} \int_{\Gamma_1} \phi_{U_1}(\gamma)\phi_{U_2}(\delta)\langle \pi_1(\gamma)\chi, \pi_2(\delta)\zeta \rangle\, d\gamma\, d\delta - \langle \chi, \zeta \rangle \right| \\ &\quad \leq \int_{U_2} \int_{U_1} \phi_{U_1}(\gamma)\phi_{U_2}(\delta) |\langle \pi_1(\gamma)\chi, \pi_2(\delta)\zeta \rangle - \langle \chi, \zeta \rangle|\, d\gamma\, d\delta \\ &\quad \leq \sup_{\gamma \in U_1} \sup_{\delta \in U_2} |\langle \pi_1(\gamma)\chi, \pi_2(\delta)\zeta \rangle - \langle \chi, \zeta \rangle|, \end{aligned}$$

which tends to 0 as U_1 and U_2 decrease to $\{0\}$ in Γ_1 and Γ_2. Thus, setting $U = (U_1, U_2)$,

$$\begin{aligned}
\langle \bar{\gamma}_0 \otimes \bar{\delta}_0, u \rangle &= \left\langle \lim_U \left((\phi_{U_1})_{-\gamma_0}\right)\hat{}\otimes \left((\phi_{U_2})_{-\delta_0}\right)\hat{}, u \right\rangle \\
&= \lim_U \left\langle \left((\phi_{U_1})_{-\gamma_0}\right)\hat{}\otimes \left((\phi_{U_2})_{-\delta_0}\right)\hat{}, u \right\rangle \\
&= \lim_U \int_{\Gamma_2}\int_{\Gamma_1} (\phi_{U_1})_{-\gamma_0}(-\gamma)(\phi_{U_2})_{-\delta_0}(-\delta)\alpha(\gamma, \delta)\, d\gamma\, d\delta \\
&= \lim_U \int_{\Gamma_2}\int_{\Gamma_1} \phi_{U_1}(\gamma_0 - \gamma)\phi_{U_2}(\delta_0 - \delta)\langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle\, d\gamma\, d\delta \\
&= \lim_U \int_{\Gamma_2}\int_{\Gamma_1} \phi_{U_1}(\gamma - \gamma_0)\phi_{U_2}(\delta - \delta_0)\langle \pi_1(\gamma)\xi, \pi_2(\delta)\eta \rangle\, d\gamma\, d\delta \\
&= \lim_U \int_{\Gamma_2}\int_{\Gamma_1} \phi_{U_1}(\gamma)\phi_{U_2}(\delta)\langle \pi_1(\gamma)\pi_1(\gamma_0)\xi, \pi_2(\delta)\pi_2(\delta_0)\eta \rangle\, d\gamma\, d\delta \\
&= \langle \pi_1(\gamma_0)\xi, \pi_2(\delta_0)\eta \rangle = \alpha(\gamma_0, \delta_0).
\end{aligned}$$

Suppose u_1, $u_2 \in BM(G_1, G_2)$ are such that $\hat{u}_1 = \hat{u}_2$. Let λ_{11}, λ_{12} be a Grothendieck measure pair for u_1 and λ_{21}, λ_{22} be a Grothendieck measure pair for u_2. As in the proof of Corollary 1.3, let $\lambda_i = \frac{1}{2}(\lambda_{1i} + \lambda_{2i})$, $i = 1, 2$. Then

$$L^2(G_i, \lambda_i) = L^2(G_i, \lambda_{1i}) \cap L^2(G_i, \lambda_{2i}), \qquad i = 1, 2,$$

so both u_1 and u_2 can be extended to $L^2(G_1, \lambda_1) \hat{\otimes} L^2(G_2, \lambda_2)$. Since the trigonometric polynomials are dense in $L^2(G_i, \lambda_i)$, $i = 1, 2$, we have $u_1 = u_2$ on $L^2(G_1, \lambda_1) \hat{\otimes} L^2(G_2, \lambda_2)$, so in particular on $V_0(G_1, G_2)$, which proves the uniqueness assertion.

The proof of Theorem 2.4 is complete.

DEFINITION 2.5. For u, $v \in BM(G_1, G_2)$, define $u * v \in BM(G_1, G_2)$ by $(u * v)\hat{} = \hat{u}\hat{v}$. By Lemmas 2.2, 2.3 and Theorem 2.4, this defines a commutative algebra structure on $BM(G_1, G_2)$ which extends the algebra structure of $M(G_1 \times G_2)$. When G is compact and $PM(G)$ is embedded in $BM(G, G)$ as in Lemma 1.12, then $PM(G)$ may be considered as a closed ideal in $BM(G, G)$. In fact, $BM(G_1, G_2)$ is a K_G^2-Banach algebra, as the following theorem asserts.

THEOREM 2.6. *For* u, $v \in BM(G_1, G_2)$,

$$\|u * v\| \le K_G^2 \|u\|\, \|v\|.$$

Proof. For $u \in BM(G_1, G_2)$, let λ_1, λ_2 be Grothendieck measures for u such that (1) holds with $C = K_G$. If T is the operator defined in the proof of Theorem 2.4(i), then (1) implies $\|T\| \leq K_G\|u\|$. Now let $\tilde{\pi}_1, \tilde{\pi}_2, \xi$, η, w be as in the proof of Theorem 2.4(i). Then

$$\|\xi\| \leq K_G\|u\| \quad \text{and} \quad \|\eta\| \leq \|W^*\| = 1.$$

Thus, applying Theorem 2.4(ii) and (iii), there exists a Hilbert space H_u, unitary representations π_i^u ($i = 1,2$) on H_u and $\xi_u, \eta_u \in H_u$ such that

$$\hat{u}(\gamma, \delta) = \langle \pi_1^u(\gamma)\xi_u, \pi_2^u(\delta)\eta_u \rangle, \qquad \gamma \in \Gamma_1, \delta \in \Gamma_2,$$

and

$$\|u\| \leq \|\xi_u\| \, \|\eta_u\| \leq K_G\|u\|. \tag{4}$$

Let $u, v \in BM(G_1, G_2)$. If we set $H = H_u \overline{\otimes} H_v$, $\xi = \xi_u \otimes \xi_v$, $\eta = \eta_u \otimes \eta_v$ and $\pi_i = \pi_i^u \otimes \pi_i^v$ ($i = 1,2$), then as in the proof of Lemma 2.2, $\alpha = \hat{u}\hat{v}$ can be represented as in (3) by means of π_1, π_2, ξ, η on H. So by (4) and Theorem 2.4,

$$\|u * v\| \leq \|\xi\| \, \|\eta\| = \|\xi_u\| \, \|\xi_v\| \, \|\eta_u\| \, \|\eta_v\| \leq K_G^2\|u\| \, \|v\|.$$

Theorem 2.6 is proved.

COROLLARY 2.7. *$BM_d(G_1, G_2)$ is a closed subalgebra of $BM(G_1, G_2)$. If $u \in BM_d(G_1, G_2)$ and $v \in BM_c(G_1, G_2)$, then $u * v \in BM_c(G_1, G_1)$.*

Proof. By Theorem 1.8, $BM_d(G_1, G_2)$ contains any bimeasure which is the limit of bimeasures (measures) with finite support. It now follows easily from Theorem 2.6 that $BM_d(G_1, G_2)$ is closed under multiplication. It is also easy to see from Definition 2.5 and the injectivity of the Fourier transform that if u is the bimeasure defined by evaluation at a point of $G_1 \times G_2$ and $v \in BM(G_1, G_2)$, then $u * v$ is a translate of v. Since the translate of a continuous bimeasure is clearly continuous, the second assertion of our corollary follows also.

REMARK 2.8. It is also true that the product of two continuous bimeasures on $G_1 \times G_2$ is again continuous, i.e., $BM_c(G_1, G_2)$ is an ideal in $BM(G_1, G_2)$, as in the case for the analogous classes of measures on $G_1 \times G_2$. The proof will appear in the forthcoming paper by J. E. Gilbert, T. Ito and B. M. Schreiber [**10**]. It is easy to see via Theorem 2.10 below that the product is continuous when at least one of the factors is in the closure of $M(G_1 \times G_2)$. However it is not true, in general, that the continuous measures on $G_1 \times G_2$ are dense in $BM_c(G_1, G_2)$. (See Corollary 5.10 below.)

DEFINITION 2.9. For $f \in V_0(G_1, G_2)$ and $u \in BM(G_1, G_2)$ set

$$\langle f, \check{u} \rangle = \langle f(-x, -y), u \rangle$$

and

$$f * u(x, y) = \left\langle f(x - x', y - y'), u_{(x', y')} \right\rangle,$$

where $u_{(x',y')}$ means that u is to be applied to the function $(x', y') \to f(x - x', y - y')$, which is clearly in $V_0(G_1, G_2)$ and has the same V_0-norm as f. Clearly, $\|\check{u}\| = \|u\|$.

THEOREM 2.10. *Let $f \in V_0(G_1, G_2)$ and u, $v \in BM(G_1, G_2)$. Then $f * u \in V_0(G_1, G_2)$,*

$$\|f * u\|_{V_0} \leq K_G^2 \|f\|_{V_0} \|u\|, \tag{5}$$

and

$$\langle f, u * v \rangle = \langle f * \check{u}, v \rangle = \left\langle \left\langle f(x + x', y + y'), u_{(x', y')} \right\rangle, v_{(x, y)} \right\rangle. \tag{6}$$

Proof. Suppose first that $f \in A(G_1 \times G_2)$, and write $f = \hat{\phi}$ for $\phi \in L^1(\Gamma_1 \times \Gamma_2)$. Then

$$\begin{aligned} f * u(x, y) &= \langle [\phi(-\gamma, -\delta)(x, \gamma)(y, \delta)]\hat{}, u \rangle \\ &= \int_{\Gamma_2} \int_{\Gamma_1} \phi(-\gamma, -\delta)(x, \gamma)(y, \delta) \hat{u}(\gamma, \delta) \, d\gamma \, d\delta \\ &= [\phi(\gamma, \delta) \hat{u}(-\gamma, -\delta)]\hat{}(x, y). \end{aligned}$$

So $f * u \in V_0(G_1, G_2)$ by Remark 1.11(1)(i). Since $\hat{\check{u}}(\gamma, \delta) = \hat{u}(-\gamma, -\delta)$, we see that

$$f * \check{u} = (\phi \hat{u})\hat{}.$$

Thus

$$\begin{aligned} \langle f * \check{u}, v \rangle = \langle (\phi \hat{u})\hat{}, v \rangle &= \int_{\Gamma_2} \int_{\Gamma_1} \phi(\gamma, \delta) \hat{u}(\gamma, \delta) \hat{v}(\gamma, \delta) \, d\gamma \, d\delta \\ &= \langle f, u * v \rangle. \end{aligned}$$

By Theorem 2.6,

$$\|f * u\|_{V_0} = \sup_{\|v\| \le 1} |\langle f * u, v \rangle| = \sup_{\|v\| \le 1} |\langle f, \check{u} * v \rangle|$$

$$\le \|f\|_{V_0} \sup_{\|v\| \le 1} \|\check{u} * v\| \le K_G^2 \|f\|_{V_0} \|u\|.$$

Finally, let $f \in V_0(G_1, G_2)$ and choose $\{f_n\} \subset A(G_1, G_2)$ such that $\|f_n - f\|_{V_0} \to 0$. Then since (5) holds for $f \in A(G_1, G_2)$, $\{f_n * u\}$ is a Cauchy sequence in $V_0(G_1, G_2)$ whose limit is clearly $f * u$. Hence (5) and (6) follow for f also. That completes the proof of Theorem 2.10.

There are a number of observations regarding pseudo-measures which can be translated using Lemma 1.12 into assertions about bimeasures. Theorem 2.11 is one such observation; Corollary 2.12 its bimeasure version. Theorem 2.13 contains another observation and its bimeasure version.

THEOREM 2.11. *Let G be compact and infinite. For each integer $n \ge 1$ there exists $S \in PM(G)$ such that $S, S^2, \ldots, S^n \notin M(G)$ and $S^{n+1} \in L^1(G)$.*

Proof. Fix $n \ge 1$. For each $j \ge 1$ let g_j and h_j be trigonometric polynomials on G such that

$$\|g_j^k\|_1 \ge 2^j, \qquad 1 \le k \le n, \tag{7}$$

$$\|g_j^{n+1}\|_1 \le 2^{-j}, \tag{8}$$

$$\|h_j\|_1 \le 2, \qquad 0 \le \hat{h}_j \le 1, \tag{9}$$

$$\hat{h}_j = 1 \quad \text{on supp}\, \hat{g}_j, \tag{10}$$

$$(\text{supp}\, \hat{h}_i) \cap (\text{supp}\, \hat{h}_j) = \varnothing \quad \text{if } i \ne j. \tag{11}$$

The g_j can be constructed by adapting, for example, [**12**, 9.5.3]; the existence of the h_j is well known, since the g_j are constructed to have disjoint supports.

By (8), $\|\hat{g}_j\|_\Gamma \le 1$. Therefore there exists $S \in PM(G)$ such that $\hat{S} = \Sigma_j \hat{g}_j$. For $k = 1, \ldots, n + 1$, (10) and (11) give

$$h_j * S^k = g_j^k. \tag{12}$$

So (7) implies that if $S^k \in M(G)$ for some $k \le n$ we would have

$$2^j \le \|h_j * S^k\|_M \le \|h_j\|_1 \|S^k\|_M \le 2\|S^k\|_M,$$

so that $\|S^k\| \geq 2^j$ for all j. Thus $S^k \notin M(G)$. On the other hand, since $\Sigma_j \|g_j^{n+1}\|_1 < \infty$, we see from (12) that

$$S^{n+1} = \sum_{j=1}^{\infty} S^{n+1} * h_j = \sum_{j=1}^{\infty} g_j^{n+1} \in L^1(G).$$

The proof of Theorem 2.11 is complete.

If we now set $u = P^*S$, where P^* is as in Lemma 1.12, we obtain the following corollary.

COROLLARY 2.12. *Let G be a compact, infinite abelian group. For each integer $n \geq 1$ there exists $u \in BM(G, G)$ such that $u, u^2, \ldots, u^n \notin M(G \times G)$ and $u^{n+1} \in L^1(G \times G)$.*

Our final application of Lemma 1.12 in this section concerns the relationship between the maximal ideal spaces $\mathfrak{M}(M(G \times G))$ and $\mathfrak{M}(BM(G, G))$ for G compact. Let

$$\rho' \colon \mathfrak{M}(BM(G, G)) \to \mathfrak{M}(M(G \times G))$$

and

$$\rho \colon \mathfrak{M}(M(G)) \to \mathfrak{M}(PM(G)) = \beta\Gamma$$

be the canonical (restriction) mappings induced by the corresponding algebra embeddings P^*. Let $\partial(G)$ and $\partial'(G)$ denote the Šilov boundary of $M(G)$ and $M(G \times G)$, respectively. The symbol $\tilde{}$ will denote a Gelfand transform.

THEOREM 2.13. *Let G be compact and infinite. Then the image of ρ does not contain $\partial(G)$ and hence the image of ρ' does not contain $\partial'(G)$.*

Proof. Let $\mu \in M(G)$ be such that $\|\hat{\mu}\|_\Gamma \leq 1$ and the spectral radius $\|\tilde{\mu}\|_{\mathfrak{M}(M(G))} = 2$. Then $\|\tilde{\mu}\|_{\mathfrak{M}(PM(G))} \leq 1$, but there exists $\phi \in \partial(G)$ such that $|\tilde{\mu}(\phi)| = 2$. Hence $\phi \notin \rho(\mathfrak{M}(PM(G)))$. If P^* is the map defined in Lemma 1.12, then P^* is an isometric algebra isomorphism of $PM(G)$ and $M(G)$ onto closed ideals in $BM(G, G)$ and $M(G \times G)$, respectively. Thus

$$\|(P^*\mu)\tilde{}\|_{\mathfrak{M}(M(G \times G))} = 2, \qquad \|(P^*\mu)\tilde{}\|_{\mathfrak{M}(BM(G,G))} \leq 1$$

also, so the second assertion of our theorem follows.

3. Subgroups, quotients and Bohr compactifications.

THEOREM 3.1. *Let* Λ_i *be a closed subgroup of* Γ_i, $i = 1, 2$. *Then*

$$S(\Lambda_1, \Lambda_2) = \left\{\alpha|_{\Lambda_1 \times \Lambda_2}: \alpha \in S(\Gamma_1, \Gamma_2)\right\}.$$

Proof. If $\alpha \in S(\Gamma_1, \Gamma_2)$, then clearly

$$\alpha|_{\Lambda_1 \times \Lambda_2} \in S(\Lambda_1, \Lambda_2).$$

Let $H_i = \Lambda_i^{\perp}$, $i = 1, 2$, and pick $u \in BM(G_1/H_1, G_2/H_2)$. We must show there exists $v \in BM(G_1, G_2)$ such that

$$\hat{v}|_{\Lambda_1 \times \Lambda_2} = \hat{u}. \tag{13}$$

By Lemma 1.13 there is a net

$$\{u_\alpha\} \subset BM(G_1/H_1, G_2/H_2)$$

such that each u_α has finite support and norm bounded by $\|u\|$, and

$$\lim_\alpha \langle f, u_\alpha \rangle = \langle f, u \rangle, \qquad f \in V(G_1/H_1, G_2/H_2).$$

If

$$\phi: G_1 \times G_2 \to (G_1/H_1) \times (G_2/H_2)$$

is the canonical map, for each α let $F_\alpha = F_{\alpha 1} \times F_{\alpha 2}$ be a finite set in $G_1 \times G_2$ such that $\phi(F_\alpha) = \operatorname{supp} u_\alpha$ and $F_{\alpha i} \to G_i/H_i$ is one-to-one. Then it is easy to see that there is a bimeasure (measure) v_α on $G_1 \times G_2$ with support F_α such that $\|v_\alpha\| = \|u_\alpha\|$ and

$$\langle f, u_\alpha \rangle = \langle f \circ \phi, v_\alpha \rangle, \qquad f \in V_0(G_1/H_1, G_2/H_2). \tag{14}$$

Let v be a weak-* limit point of $\{v_\alpha\}$ so $\|v\| \le \|u\|$. If u has compact support, then we may assume all the F lie in some compact set in $G_1 \times G_2$ and, hence, v has compact support. (Cf. Def. 1.6.) In this case (14) persists with u and v replacing u_α and v_α, and then (13) follows.

In general, it follows from Lemma 1.4 that there is a sequence $\{u_n\}$ of bimeasures with compact supports such that

$$u = \sum_{n=1}^{\infty} u_n \quad \text{and} \quad \sum_{n=1}^{\infty} \|u_n\| < \infty.$$

For each n choose $v_n \in BM(G_1, G_2)$ such that v_n has compact support, (14) holds with u_n and v_n replacing u_α and v_α, and $\|v_n\| \le \|u_n\|$. Set

$v = \Sigma\, v_n$. The it follows from Corollary 1.3 and Theorem 1.2 that

$$\langle f, v\rangle = \sum_{n=1}^{\infty} \langle f, v_n\rangle, \qquad f \in V(G_1, G_2).$$

Thus we can again replace u_α and v_α in (14) by u and v and obtain (13).

LEMMA 3.2. *The canonical embedding of $G_1 \times G_2$ in $bG_1 \times bG_2$ induces an isometric algebra embedding of $BM(G_1, G_2)$ into $BM(bG_1, bG_2)$.*

Proof. If $u \in BM(G_1, G_2)$ then

$$\|u\| = \sup|\langle f, u\rangle|,$$

the supremum being taken over all trigonometric polynomials f such that $\|f\|_{\infty,\infty} \leq 1$. This follows easily from Corollary 1.3 and Lemma 1.4. The remainder of the argument needed to prove the Lemma is clear.

The following theorem is the extension to the present context of the well-known characterization of Fourier-Stieltjes transforms due to W. F. Eberlein **[6]**, **[24**, Thm. 1.9.1].

THEOREM 3.3. *Let $u \in BM(bG_1, bG_2)$. Then $u \in BM(G_1, G_2)$ if and only if $\hat{u}$ is continuous on $\Gamma_1 \times \Gamma_2$. Equivalently,*

$$S(\Gamma_1, \Gamma_2) = S(\Gamma_{1d}, \Gamma_{2d}) \cap C(\Gamma_1 \times \Gamma_2). \tag{15}$$

Proof. If $u \in BM(G_1, G_2)$, then $\hat{u}$ is continuous and u may be considered as an element of $BM(bG_1, bG_2)$ by Lemma 3.2. Hence the left-hand side of (15) is contained in the right-hand side. (This may also be observed by neglecting the continuity of representations at first.) The reverse containment remains to be established.

Let $u \in BM(bG_1, bG_2)$ be such that $\hat{u}$ is continuous on $\Gamma_1 \times \Gamma_2$. We shall show that for $f \in L^1(\Gamma_1)$ and $g \in L^1(\Gamma_2)$,

$$\left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma)g(\delta)\hat{u}(\gamma, \delta)\, d\gamma\, d\delta\right| \leq \|\hat{f}\|_{G_1}\, \|\hat{g}\|_{G_2}\, \|u\|. \tag{16}$$

It will then follow that there exists an element $v \in BM(G_1, G_2)$ such that

$$\langle \hat{f}, v\rangle = \int_{\Gamma_2}\int_{\Gamma_1} f(\gamma, \delta)\hat{u}(\gamma, \delta)\, d\gamma\, d\delta$$

for all $f \in L^1(\Gamma_1 \times \Gamma_2)$. Since $A(G_1 \times G_2)$ is dense in $V_0(G_1, G_2)$, $\hat{v} = \hat{u}$, so $u \in BM(G_1, G_2)$ as asserted. It remains to establish (16).

Note that if p and q are trigonometric polynomials on G_1 and G_2, respectively, then

$$|\langle p \otimes q, u \rangle| \leq \|p\|_{G_1} \|q\|_{G_2} \|u\|,$$

since $u \in BM(bG_1, bG_2)$. Of course, since q is a trigonometric polynomial and $\hat{u}$ is continuous, the function $\gamma \to \langle \gamma \otimes q, u \rangle$ is continuous. By Eberlein's Theorem, there exists a regular Borel measure μ_q on G_1 such that

$$\langle p \otimes q, u \rangle = \int_{G_1} p \, d\mu_q$$

for all trigonometric polynomials p on G_1. Therefore, if $f \in L^1(\Gamma_1)$, then

$$(17) \quad \left| \int_{\Gamma_1} f(\gamma) \sum_{\delta \in \Gamma_2} \hat{q}(\delta) \hat{u}(\gamma, \delta) \, d\gamma \right| = \left| \int_{\Gamma_1} f(\gamma) \hat{\mu}_q(\gamma) \, d\gamma \right|$$

$$= \left| \int_{G_1} \hat{f} \, d\mu_q \right| \leq \|\hat{f}\|_{G_1} \|\mu_q\| \leq \|\hat{f}\|_{G_1} \|q\|_{G_2} \|u\|.$$

For $f \in L^1(\Gamma_1)$ and $\delta \in \Gamma_2$, set

$$\phi_f(\delta) = \int_{\Gamma_1} f(\gamma) \hat{u}(\gamma, \delta) \, d\gamma.$$

Then ϕ_f is continuous on Γ_2 by a standard argument based on the fact that $\delta \to \hat{u}(\gamma, \delta)$ is equicontinuous over compact sets of $\gamma \in \Gamma_1$. Furthermore, the function $q \to \Sigma_\delta \, \hat{q}(\delta) \phi_f(\delta)$ is linear on trigonometric polynomials on G_2, and by (17),

$$\left| \sum_{\delta} \hat{q}(\delta) \phi_f(\delta) \right| \leq \|\hat{f}\|_{G_1} \|q\|_{G_2} \|u\|.$$

Hence we may apply Eberlein's Theorem again to obtain a measure $\nu_f \in M(G_2)$ such that for all trigonometric polynomials q on G_2,

$$\sum_{\delta} \hat{q}(\delta) \phi_f(\delta) = \int_{G_2} q \, d\nu_f$$

and

$$\|\nu_f\| \leq \|\hat{f}\|_{G_1} \|u\|.$$

Therefore, if $g \in L^1(\Gamma_2)$, then

$$\left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma)g(\delta)\hat{u}(\gamma,\delta)\,d\gamma\,d\delta\right| = \left|\int_{\Gamma_2} g(\delta)\hat{\nu}_f(\delta)\,d\delta\right|$$

$$= \left|\int_{G_2} \hat{g}\,d\nu_f\right| \leq \|\hat{g}\|_{G_2}\|\nu_f\| \leq \|\hat{g}\|_{G_2}\|\hat{f}\|_{G_1}\|u\|.$$

This establishes (16), and Theorem 3.3 is proved.

COROLLARY 3.4. *Let H_i be a closed subgroup of G_i and $\Lambda_i = H_i^{\perp} \subset \Gamma_i$, $i = 1, 2$. Let $u \in BM(G_1, G_2)$. Then u is supported on $H_1 \times H_2$ if and only if $\hat{u}$ is constant on cosets of $\Lambda_1 \times \Lambda_2$.*

Proof. If $u \in BM(H_1, H_2)$ and u is considered as a bimeasure on $G_1 \times G_2$, then clearly $\hat{u}$ is constant on cosets of $\Lambda_1 \times \Lambda_2$. On the other hand, assume $\hat{u} \in BM(G_1, G_2)$ and $\hat{u}$ is constant on cosets of $\Lambda_1 \times \Lambda_2$. If G_1 and G_2 are compact, then clearly u defines a bimeasure $\tilde{u}$ on $H_1 \times H_2$ by

$$\langle \tilde{\gamma} \otimes \tilde{\delta}, \tilde{u} \rangle = \langle \gamma \otimes \delta, u \rangle,$$

where $\tilde{\gamma}$ and $\tilde{\delta}$ are the restrictions of γ to H_1 and δ to H_2, respectively. If G_1 and G_2 are not both compact, we pass to the Bohr compactifications. Since the embeddings $bH_i \hookrightarrow bG_i$, $i = 1, 2$, induce an embedding of $BM(bH_1, bH_2)$ in $BM(bG_1, bG_2)$, we shall consider u as an element of $BM(bG_1, bG_2)$ such that $\hat{u}$ is continuous on $\Gamma_1 \times \Gamma_2$ and is constant on cosets of $\Lambda_1 \times \Lambda_2$. Let $\tilde{u}$ be the induced bimeasure, as above, on $bH_1 \times bH_2$. Then $\hat{\tilde{u}}$ is continuous on $(\Gamma_1/\Lambda_1) \times (\Gamma_2/\Lambda_2)$, so $\tilde{u} \in BM(H_1, H_2)$ by Theorem 3.3, and the Corollary is proved.

4. The closure of L^1.

DEFINITION 4.1. Let $BM_a(G_1, G_2)$ denote the closure of $L^1(G_1 \times G_2)$ (considered as the space of absolutely continuous measures on $G_1 \times G_2$) in $BM(G_1, G_2)$. Elements of $BM_a(G_1, G_2)$ might be called "absolutely continuous bimeasures", for as we shall see below, $BM_a(G_1, G_2)$ plays a role in $BM(G_1, G_2)$ similar to that played by $L^1(G_1 \times G_2)$ in $M(G_1 \times G_2)$. For $f \in L^1(G_1 \times G_2)$, let u_f denote the bimeasure determined by integration against f.

REMARKS 4.2. (i) Clearly $BM_a(G_1, G_2)$ is the closure of each of the spaces $C_{00}(G_1 \times G_2)$, $C_{00}(G_1) \otimes C_{00}(G_2)$, $L^1(G_1 \times G_2)$ or the trigonometric polynomials when G_1 and G_2 are compact.

(ii) $BM_a(G_1, G_2)$ is a closed subalgebra of $BM(G_1, G_2)$ on which the Gelfand transform is the Fourier transform and whose maximal ideal space is $\Gamma_1 \times \Gamma_2$ with its usual topology. All this is easy to see from the results in §2. For example, each character in $\Gamma_1 \times \Gamma_2$ defines a nonzero complex homomorphism on $BM(G_1, G_2)$ by the definition of the multiplication on $BM(G_1, G_2)$, and every nonzero complex homomorphism on $BM_a(G_1, G_2)$ is nonzero on $L^1(G_1 \times G_2)$ and hence is given by a character.

(iii) From (ii) we have $\hat{u} \in C_0(\Gamma_1 \times \Gamma_2)$ for all $u \in BM_a(G_1, G_2)$.

LEMMA 4.3. *Let $v \in BM(G_1, G_2)$ and $\phi \in V_0(G_1, G_2)$ both have compact support. Then $\phi * v \in C_{00}(G_1 \times G_2)$ and*

$$u_\phi * v = u_{\phi * v}.$$

Proof. By Theorem 2.10, $\phi * v \in V_0(G_1, G_2)$. Definition 2.9 easily implies that $\varphi * v = 0$ off support φ + support v, so $\varphi * v \in C_{00}(G_1 \times G_2)$. It is a straightforward calculation to check that

$$(u_{\phi * v})\hat{} = \hat{\phi}\hat{v} = (u_\phi * v)\hat{},$$

from which the Lemma follows by Theorem 2.4(ii).

THEOREM 4.4. *$BM_a(G_1, G_2)$ is a (closed) ideal in $BM(G_1, G_2)$.*

Proof. By Lemma 1.4 and Remark 4.2(i) it suffices to check that

$$u_\phi * v \in BM_a(G_1, G_2)$$

for $\phi \in C_{00}(G_1) \otimes C_{00}(G_2)$ and v with compact support. But then $u_\phi * v = u_\psi$ for $\psi \in C_{00}(G_1 \times G_2)$ by Lemma 4.3, and the Theorem follows.

THEOREM 4.5. *Let $u \in BM(G_1, G_2)$. Then $u \in BM_a(G_1, G_2)$ if and only if the function $(x, y) \to u_{(x,y)}$ (translation) is norm-continuous.*

Proof. Since translation by (x, y) induces an isometry on $BM(G_1, G_2)$, and translation is continuous on $L^1(G_1 \times G_2)$ in the L^1-norm which dominates the bimeasure norm, it is clear that translation is continuous on elements of $BM_a(G_1, G_2)$. Thus let $v \in BM(G_1, G_2)$ be such that $(x, y) \to v_{(x,y)}$ is norm-continuous. Given $\varepsilon > 0$, choose a neighborhood U of 0 in $G_1 \times G_2$ with compact closure such that $\|v - v_{(x,y)}\| < \varepsilon$ if $(x, y) \in U$, and let $\phi \in C_{00}(G_1) \otimes C_{00}(G_2)$ be such that $\phi \geq 0$, ϕ vanishes off U, and

$$\int_{G_2} \int_{G_1} \phi(x, y)\, dx\, dy = 1.$$

Then considering functions $f \in C_{00}(G_1) \otimes C_{00}(G_2)$,

$$\begin{aligned}
\|v - u_\phi * v\| &= \sup_{\|f\|_{V_0} \leq 1} \left|\langle f, v - u_{\phi * v}\rangle\right| \\
&= \sup_{\|f\|_{V_0} \leq 1} \left| \int_{G_2} \int_{G_1} \langle f, v\rangle \phi(x, y)\, dx\, dy \right. \\
&\qquad \left. - \left\langle \int_{G_2} \int_{G_1} f_{(-x,-y)} \phi(x, y)\, dx\, dy, v \right\rangle \right| \\
&= \sup_{\|f\|_{V_0} \leq 1} \left| \int_{G_2} \int_{G_1} \left[\langle f, v\rangle - \langle f_{(-x,-y)}, v\rangle\right] \phi(x, y)\, dx\, dy \right| \\
&= \sup_{\|f\|_{V_0} \leq 1} \left| \int_{G_2} \int_{G_1} \langle f, v - v_{(x,y)}\rangle \phi(x, y)\, dx\, dy \right| \\
&\leq \int_U \|v - v_{(x,y)}\| \phi(x, y)\, dx\, dy < \varepsilon.
\end{aligned}$$

By Theorem 4.4, $v \in BM_a(G_1, G_2)$.

Our next result is the analog of the multiplier theorem of Helson and Edwards [**15**], [**7**], [**24**, Thm. 3.8.1].

THEOREM 4.6. *Let T: $BM_a(G_1, G_2) \to BM(G_1, G_2)$ be a multiplier. Then there exists $v \in BM(G_1, G_2)$ such that $Tu = u * v$ for all $u \in BM_a(G_1, G_2)$, and $\|v\| \leq \|T\|$.*

Proof. There is a continuous function ϕ on $\Gamma_1 \times \Gamma_2$ such that $(Tu)\hat{} = \phi\hat{u}$ for all $u \in BM_a(G_1, G_2)$. Given $f \in C_{00}(\Gamma_1, \Gamma_2)$ and $\varepsilon > 0$, choose

$h \in L^1(G_1 \times G_2)$ such that $\hat{h} = 1$ on supp f and $\|h\|_1 < 1 + \varepsilon$. Then

$$\left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma,\delta)\phi(\gamma,\delta)\,d\gamma\,d\delta\right| = \left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma,\delta)\hat{h}(\gamma,\delta)\phi(\gamma,\delta)\,d\gamma\,d\delta\right|$$

$$= \left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma,\delta)\hat{u}_h(\gamma,\delta)\phi(\gamma,\delta)\,d\gamma\,d\delta\right|$$

$$= \left|\int_{\Gamma_2}\int_{\Gamma_1} f(\gamma,\delta)(Tu_h)\hat{\ }(\gamma,\delta)\,d\gamma\,d\delta\right| = |\langle \hat{f}, Tu_h\rangle| \le \|\hat{f}\|_{V_0}\|Tu_h\|$$

$$\le \|\hat{f}\|_{V_0}\|T\|\,\|h\|_1 < \|\hat{f}\|_{V_0}\|T\|(1+\varepsilon).$$

Thus there exists $v \in BM(G_1, G_2)$ with $\hat{v} = \phi$ and $\|v\| \le \|T\|$. Theorem 4.6 is proved.

There is one significant way in which the role played by $BM_a(G_1, G_2)$ in $BM(G_1, G_2)$ differs from its analog in $M(G_1 \times G_2)$. Namely, as the following theorem shows, there is, in general, no decomposition of $BM(G_1, G_2)$ into what might be called "absolutely continuous" and "singular" bimeasures.

THEOREM 4.7. *Let G be a nondiscrete LCA group. There is no bounded projection from $BM(G, G)$ onto $BM_a(G, G)$.*

Proof. Suppose first that G is compact. If $P^*: PM(G) \to BM(G, G)$ is the map considered in Lemma 1.12, then $P^*(L^1(G)) \subset BM_a(G, G)$. Of course, for $S \in PM(G)$, $\hat{S} \in C_0(\Gamma)$ if and only if $S \in L^1(G)$ (cf. Thm. 5.8). Suppose there exists a bounded projection

$$Q: BM(G,G) \to BM_a(G,G). \tag{18}$$

For $S \in PM(G)$, set

$$\tilde{Q}(\hat{S})(\gamma) = (QP^*S)\hat{\ }(\gamma,\gamma), \qquad \gamma \in \Gamma.$$

Then it is easy to see from Lemma 1.12 and our previous observation that $\tilde{Q}$ is a bounded projection from $l^\infty(\Gamma)$ onto $c_0(\Gamma)$. But by **[21]** no such projection exists, so no Q as above could exist.

If G is neither discrete nor compact, then the Main Structure Theorem for LCA groups says that there exists an open subgroup H of G such that $H \cong \mathbf{R}^n \times K$ for some $n \ge 0$ and compact group K. If ι: $BM(H, H) \to BM(G, G)$ is the natural injection of $BM(H, H)$ onto the bimeasures supported on $H \times H$, and ρ: $BM(G, G) \to BM(H, H)$ is the operator of restriction to $H \times H$, then $\rho \circ \iota = I_{BM(H, H)}$. It follows that if Q, as in (18), is a projection, then $\rho Q \iota$ is a bounded projection from $BM(H, H)$ onto $BM_a(H, H)$. Thus to prove that no such Q exists, we may assume $G = \mathbf{R}^n \times K$.

If $n = 0$ then we are in the first case above. Otherwise, we shall show next that it suffices to prove our assertion for the case $G = \mathbf{R}^n$. For $f \in V_0(G, G)$, let us suppress some parentheses and write

$$(19) \qquad (\sigma f)(\xi, \eta) = \int_K \int_K f(\xi, k, \eta, k')\, dk\, dk', \qquad \xi, \eta \in \mathbf{R}^n.$$

It is well known that if $f,\ g \in C_0(G)$ then $\sigma(f \otimes g) = \phi \otimes \psi$, where ϕ, $\psi \in C_0(\mathbf{R}^n)$, $\|\phi\| \leq \|f\|$ and $\|\psi\| \leq \|g\|$. Hence one can verify directly that σ is a norm-reducing operator from $V_0(G, G)$ to $V_0(\mathbf{R}^n, \mathbf{R}^n)$. On the other hand, if $g \in V_0(\mathbf{R}^n, \mathbf{R}^n)$, define $\tau(g) = \tilde{g}$ by

$$(20) \qquad \tilde{g}(\xi, k, \eta, k') = g(\xi, \eta), \qquad \xi, \eta \in \mathbf{R}^n, k, k' \in K.$$

Then $\tau(g) \in V_0(G, G)$ with

$$\|\tau(g)\|_{V_0(G,G)} \leq \|g\|_{V_0(\mathbf{R}^n, \mathbf{R}^n)}.$$

Consider σ^*: $BM(\mathbf{R}^n, \mathbf{R}^n) \to BM(G, G)$. If $g \in L^1(\mathbf{R}^{2n})$, let $\tilde{g}$ be defined by (20) so that $\tilde{g} \in L^1(G \times G)$. For $\phi, \psi \in C_0(G)$, a straightforward computation shows that

$$\begin{aligned}\langle \phi \otimes \psi, \sigma^*(u_g)\rangle &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \sigma(\phi \otimes \psi)(\xi, \eta) g(\xi, \eta)\, d\xi\, d\eta \\ &= \int_G \int_G \phi(x)\psi(y)\tilde{g}(x, y)\, dx\, dy = \langle \phi \otimes \psi, u_{\tilde{g}}\rangle,\end{aligned}$$

so that $\sigma^*(u_g) = u_{\tilde{g}}$. In particular,

$$\sigma^*(BM_a(\mathbf{R}^n, \mathbf{R}^n)) \subset BM_a(G, G).$$

Now consider τ^*: $BM(G, G) \to BM(\mathbf{R}^n, \mathbf{R}^n)$. On $M(G \times G)\tau^*$ is the classical norm-reducing operator onto $M(\mathbf{R}^{2n})$; in particular, if $f \in L^1(G \times G)$ then $\tau^*(u_f) = u_g$, where $g \in L^1(\mathbf{R}^{2n})$, and is given by the integral in (19). Thus if $g \in L^1(\mathbf{R}^{2n})$, then

$$\tau^*\sigma^*(u_g) = \tau^*(u_{\tilde{g}}) = u_g.$$

Hence if Q as in (18) is a bounded projection, then $\tau^* Q \sigma^*$ is a bounded projection from $BM(\mathbf{R}^n, \mathbf{R}^n)$ onto $BM_a(\mathbf{R}^n, \mathbf{R}^n)$. Thus we have shown that we may assume $G = \mathbf{R}^n$.

We shall show that for $G = \mathbf{R}^n$ no projection as in (18) exists. To accomplish this we shall show that if Q is such a projection, then Q

induces a projection from $BM(\mathbf{T}^n, \mathbf{T}^n)$ onto $BM_a(\mathbf{T}^n, \mathbf{T}^n)$. An appeal to the compact case discussed above then completes the proof of Theorem 4.7.

Let I^n denote the closed unit cube, and define ψ: $I^n \to \mathbf{T}^n$ by

$$\psi(x_1,\ldots,x_n) = \left(e^{2\pi i x_1},\ldots,e^{2\pi i x_n}\right).$$

Let $J^n = \{x \in I^n\colon x_i \neq 1 \text{ for all } i\}$. Then $\psi|_{J^n}$ is injective; let us denote its inverse by ψ^{-1}. If $f \in V_0(G, G)$ then

$$f \circ \left(\psi^{-1} \times \psi^{-1}\right) \in \mathfrak{L}^\infty(\mathbf{T}^n) \hat{\otimes} \mathfrak{L}^\infty(\mathbf{T}^n).$$

So appealing to Corollary 1.3, if $u \in BM(\mathbf{T}^n, \mathbf{T}^n)$ there is a unique bimeasure $\Psi(u) \in BM(G, G)$ supported on I^n of norm at most $K_G\|u\|$ such that

$$\langle f, \Psi(u)\rangle = \left\langle f \circ \left(\psi^{-1} \times \psi^{-1}\right), u\right\rangle, \qquad f \in V_0(G, G).$$

The bimeasure u lies in $BM_a(\mathbf{T}^n, \mathbf{T}^n)$ if and only if $\Psi(u) \in BM_a(G, G)$, and the restriction Φ of Ψ to $BM_a(\mathbf{T}^n, \mathbf{T}^n)$ is a Banach-space isomorphism onto $\{u \in BM_a(G, G)\colon u \text{ is supported on } I^n \times I^n\}$. If ρ denotes the operator of restriction of bimeasures on $G \times G$ to $I^n \times I^n$, then it is clear that $\Phi^{-1}\rho Q\Psi$ is the required projection on $BM(\mathbf{T}^n, \mathbf{T}^n)$, and our Theorem is proved.

5. Idempotents, homomorphisms and restriction to diagonals. We begin this section by showing that the connection between homomorphisms between measure algebras and idempotents observed by P. J. Cohen [**3**], [**24**, Thm. 4.4.3] also holds for homomorphisms between bimeasure algebras. However, one must use some care in defining the "graph" associated with a given homomorphism.

DEFINITION 5.1. Let Ψ: $BM_a(G_1, G_2) \to BM(G_1', G_2')$ be an algebra homomorphism. Since $BM(G_1', G_2')$ is semi-simple, Ψ is bounded. Let

$$N_\Psi = \left\{(\gamma', \delta') \in \Gamma_1' \times \Gamma_2'\colon \Psi(u)\hat{\ }(\gamma', \delta') \neq 0 \text{ for some } u \in BM(G_1, G_2)\right\},$$

so N_Ψ is an open set in $\Gamma_1' \times \Gamma_2'$. Let

$$\Psi^*\colon N_\Psi \to \Gamma_1 \times \Gamma_2$$

be the continuous map defined by

$$\Psi(u)\hat{\ }(\gamma', \delta) = \hat{u}(\Psi^*(\gamma', \delta')), \tag{21}$$
$$u \in BM_a(G_1, G_2), \quad \gamma' \in \Gamma_1', \delta' \in \Gamma_2'.$$

LEMMA 5.2. *Let Ψ be as in Definition* 5.1. *There is a (unique) extension of Ψ to a homomorphism of $BM(G_1, G_2)$ into $BM(G_1', G_2')$ with norm at most $K_G^2\|\Psi\|$ such that* (21) *holds for all $u \in BM(G_1, G_2)$.*

Proof. The proof follows lines similar to those exploited in the proof of Theorem 4.6. Let $u \in BM(G_1, G_2)$, and set

$$\phi(\gamma', \delta') = \begin{cases} \hat{u} \circ \Psi^*(\gamma', \delta'), & (\gamma', \delta') \in N_\Psi, \\ 0, & (\gamma', \delta') \notin N_\Psi. \end{cases} \tag{22}$$

Given $f \in C_{00}(\Gamma_1' \times \Gamma_2')$ and $\varepsilon > 0$, choose $h \in L^1(G_1 \times G_2)$ such that $\hat{h} = 1$ on $\Psi^*(N_\Psi \cap \operatorname{supp} f)$ and $\|h\|_1 < 1 + \varepsilon$. Let $v = u * u_h$. Then $v \in BM_a(G_1, G_2)$, and

$$\hat{v} \circ \Psi^*(\gamma', \delta') = \hat{h} \circ \Psi^*(\gamma', \delta')\phi(\gamma', \delta'), \qquad \gamma' \in \Gamma_1', \delta' \in \Gamma_2'.$$

Thus

$$\begin{aligned} &\int_{\Gamma_2'} \int_{\Gamma_1'} f(\gamma', \delta')\phi(\gamma', \delta')\, d\gamma'\, d\delta' \\ &\quad = \int_{\Gamma_2'} \int_{\Gamma_1'} f(\gamma', \delta')\hat{v} \circ \Psi^*(\gamma', \delta')\, d\gamma'\, d\delta' = \left\langle \hat{f}, \Psi(v) \right\rangle. \end{aligned}$$

So by Theorem 4.6,

$$\begin{aligned} \left| \int_{\Gamma_2'} \int_{\Gamma_1'} f(\gamma', \delta')\phi(\gamma', \delta')\, d\gamma'\, d\delta' \right| &\leq \|\Psi(v)\|\, \|\hat{f}\|_{V_0} \\ &\leq \|\Psi\|\, \|v\|\, \|\hat{f}\|_{V_0} < K_G^2(1 + \varepsilon)\|\Psi\|\, \|u\|\, \|\hat{f}\|_{V_0}. \end{aligned}$$

Letting $\varepsilon \to 0$ we see there exists $w \in BM(G_1', G_2')$ such that $\hat{w} = \phi$ and $\|w\| \leq K_G^2\|\Psi\|\|u\|$, which proves our Lemma.

COROLLARY 5.3. *N_Ψ is an open and closed subset of $\Gamma_1' \times \Gamma_2'$ which is the support of the Fourier transform of an idempotent in $BM(G_1', G_2')$.*

Proof. We already observed that N_Ψ is open, which is clear from its definition. On the other hand, $\Psi(\delta_{(0,0)})$ is an idempotent in $BM(G_1', G_2')$ which acts as the identity for $\Psi(BM(G_1, G_2))$, from which it follows that

$$N_\Psi = \left\{(\gamma', \delta') \in \Gamma_1' \times \Gamma_2' \colon \Psi\left(\delta_{(0,0)}\right)\hat{}\,(\gamma', \delta') \neq 0\right\}.$$

Since $\Psi(\delta_{(0,0)})\hat{}$ is continuous, N_Ψ is closed.

THEOREM 5.4. *Let* Ψ *be as in Definition* 5.1. *There is a* (*unique*) *extension of* Ψ *to a homomorphism of* $BM(bG_1, bG_2)$ *into* $BM(bG_1', bG_2')$ *of norm at most* $K_G^2\|\Psi\|$ *such that* (21) *holds for all* $u \in BM(bG_1, bG_2)$.

Proof. Let $u \in BM(bG_1, bG_2)$ and let ϕ be defined by (22). Note that, since G_i is dense in bG_i, $i = 1, 2$, the points $x_1, \ldots, x_m \in bG_1$ and $y_1, \ldots, y_n \in bG_2$ in the proof of Lemma 1.13 could in this case be chosen in G_1 and G_2, respectively. Hence given $\varepsilon > 0$ and $(\gamma_1', \delta_1'), \ldots, (\gamma_n', \delta_n') \in \Gamma_2'$, there exists $v \in BM(G_1, G_2)$ with $\|v\| \leq \|u\|$ such that if ψ is defined as in (22) for v then

$$|\phi(\gamma_i', \delta_i') - \psi(\gamma_i', \delta_i')| < \varepsilon, \qquad 1 \leq i \leq n.$$

By Lemma 5.2, $\psi \in S(\Gamma', \Gamma') \subset S(\Gamma_{1,d}', \Gamma_{2,d}')$ and $\psi = \hat{w}$, where $w \in BM(G_1', G_2')$ with $\|w\| \leq K_G^2\|\Psi\|\|u\|$. Since ε is arbitrary, it is easy to see (by considering trigonometric polynomials on bG_1' and bG_2') there exists $z \in BM(bG_1', bG_2')$ such that $\hat{z} = \phi$ and $\|z\| \leq K_G^2\|\Psi\|\|u\|$, completing the proof.

DEFINITION 5.5. Let G_1, G_2, G_1' and G_2' be compact groups, and let Ψ: $BM(G_1, G_2) \to BM(G_1', G_2')$ be an algebra homomorphism. Let Ψ^* and N_Ψ be defined as in Definition 5.1. By the *graph* of Ψ^* we shall mean the set

$$\{(\gamma, \gamma', \delta, \delta') \in \Gamma_1 \times \Gamma_1' \times \Gamma_2 \times \Gamma_2' \colon (\gamma', \delta') \in N_\Psi, (\gamma, \delta) = \Psi^*(\gamma', \delta')\}.$$

For $x \in G_1, y \in G_2$, set $u(x, y) = \Psi(\delta_{(x,y)})$. Then $u(x, y) \in BM(G_1', G_2')$, $\|u(x, y)\| \leq \|\Psi\|$ and

$$u(x, y)\hat{\ }(\gamma', \delta') = \begin{cases} ((-x, -y), \Psi^*(\gamma', \delta')), & (\gamma', \delta') \in N_\Psi, \\ 0, & (\gamma', \delta') \notin N_\Psi. \end{cases}$$

THEOREM 5.6. *Let* G_1, G_2, G_1', G_2' *and* Ψ *be as in Definition* 5.5, *and let* χ *denote the characteristic function of the graph of* Ψ^*. *Then there exists* $u \in BM(G_1 \times G_1', G_2 \times G_2')$ *such that* $\|u\| \leq K_G^2\|\Psi\|$ *and* $\hat{u} = \chi$.

Proof. The proof is an adaptation of the argument on pp. 84–85 of [**24**]. Let p be a trigonometric polynomial on $G_1' \times G_2'$. Consider p as a trigonometric polynomial on $G_1 \times G_1' \times G_2 \times G_2'$ which does not depend on the first and third variables. So now

$$\hat{p}(\gamma, \gamma', \delta, \delta') = \hat{p}(\gamma', \delta').$$

For $x \in G_1, x' \in G_1', y \in G_2, y' \in G_2'$, set

$$\phi(x, x', y, y') = \sum_{(\gamma', \delta') \in N_\Psi} \hat{p}(\gamma', \delta')((x, y), \Psi^*(\gamma', \delta'))(x', \gamma')(y', \delta').$$

Then ϕ is a trigonometric polynomial whose Fourier transform is $\hat{p}\chi$. Defining $u(x, y)$ as in Definition 5.5, we see from Definition 2.9 that

$$\begin{aligned}\phi(x, x', y, y') &= \sum_{\delta' \in \Gamma_2'} \sum_{\gamma' \in \Gamma_1'} \hat{p}(\gamma', \delta')(x', \gamma')(y', \delta')u(-x, -y)\hat{\ }(\gamma', \delta') \\ &= \sum_{\delta' \in \Gamma_2'} \sum_{\gamma' \in \Gamma_1'} \hat{p}(\gamma', \delta')\langle (x' - s', \gamma')(y' - t', \delta'), u(-x, -y)_{(s', t')} \rangle \\ &= p * u(-x, -y)(x', y').\end{aligned}$$

Applying Lemma 4.3, we see that

$$u_{p * u(-x, -y)} = u_p * u(-x, -y)$$

as elements of $BM(G_1', G_2')$. Hence in $BM(G_1', G_2')$

$$\|u_{p * u(-x, -y)}\| \leq K_G^2 \|u_p\| \, \|\Psi\| \leq K_G^2 \|\Psi\| \, \|p\|_1.$$

Let $\gamma_1, \ldots, \gamma_m \in \Gamma_1, \gamma_1', \ldots, \gamma_m' \in \Gamma_1', \delta_1, \ldots, \delta_n \in \Gamma_2, \delta_1', \ldots, \delta_n' \in \Gamma_2'$, and let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be complex numbers. Set

$$q_1(x, x') = \sum_{i=1}^{m} a_i(-x, \gamma_i)(-x', \gamma_i')$$

and

$$q_2(y, y') = \sum_{j=1}^{n} b_j(-y, \delta_j)(-y', \delta_j').$$

Given $\varepsilon > 0$, let p be a trigonometric polynomial on $G_1' \times G_2'$ such that $\|p\|_1 < 1 + \varepsilon$ and $\hat{p}(\gamma_i', \delta_j') = 1$, $1 \leq i \leq m$, $1 \leq j \leq n$. Then

$$\begin{aligned}
&\left|\sum_{j=1}^{n} \sum_{i=1}^{m} a_i b_j \chi(\gamma_i, \gamma_i', \delta_j, \delta_j')\right| \\
&\quad = \left|\sum_{j=1}^{n} \sum_{i=1}^{m} a_i b_j \hat{p}(\gamma_i, \gamma_i', \delta_j, \delta_j')\chi(\gamma_i, \gamma_i', \delta_j, \delta_j')\right| \\
&\quad = \left|\sum_{j=1}^{n} \sum_{i=1}^{m} a_i b_j \hat{\phi}(\gamma_i, \gamma_i', \delta_j, \delta_j')\right| \\
&\quad = \left|\int_{G_1}\int_{G_2}\int_{G_1'}\int_{G_2'} q_1(x, x')q_2(y, y')\phi(x, x', y, y')\, dy'\, dx'\, dy\, dx\right| \\
&\quad = \left|\int_{G_1}\int_{G_2} \left\langle q_1(x, x') \otimes q_2(y, y'), (u_{p * u(-x,-y)})_{(x',y')}\right\rangle dy\, dx\right| \\
&\quad \leq \int_{G_1}\int_{G_2} \|q_1(x, \cdot)\|_{G_1'} \|u_{p * u(-x,-y)}\| dy\, dx \\
&\quad \leq \|q_1\|_{G_1 \times G_1'} \|q_2\|_{G_2 \times G_2'} K_G^2 \|\Psi\|\, \|p\|_1 \\
&\quad < \|q_1\|_{G_1 \times G_1'} \|q_2\|_{G_2 \times G_2'} K_G^2 \|\Psi\|(1 + \varepsilon).
\end{aligned}$$

Since ε is arbitrary, our Theorem is proved.

REMARKS 5.7. Theorem 5.6 tells us that in order to characterize homomorphisms between bimeasure algebras, it behooves us to identify the supports of the Fourier transforms of idempotent bimeasures. However, our next results shows that, in fact, this is not an easy task. For if Γ is totally disconnected, then $BM(G, G)$ has many idempotents which are not measures.

For any LCA group Γ, let Δ denote the diagonal in $\Gamma \times \Gamma$, i.e.,

$$\Delta = \{(\gamma, \gamma) : \gamma \in \Gamma\}.$$

For the group $\mathbf{Z}$, the following theorem appears in [**17**]. In fact, when Γ is discrete, it is just a restatement of Lemma 1.12.

THEOREM 5.8. *Given any bounded uniformly continuous function f on Γ, there exists $\alpha \in S(\Gamma, \Gamma)$ such that*

$$\alpha(\gamma, \gamma) = f(\gamma), \qquad \gamma \in \Gamma. \tag{23}$$

There exists $u \in BM_a(G, G)$ such that $\alpha = \hat{u}$ satisfies (23) *if and only if $f \in C_0(\Gamma)$. If Γ is discrete, we can also require that $\alpha(\gamma, \delta) = 0$ if $(\gamma, \delta) \notin \Delta$.*

Proof. Let $C_u(\Gamma)$ denote the bounded uniformly continuous functions on Γ. By [**16**, (32.45)(b)], $L^1(\Gamma) * L^\infty(\Gamma) = C_u(\Gamma)$. So if $f \in C_u(\Gamma)$, choose $g \in L^1(\Gamma)$ and $h \in L^\infty(\Gamma)$ such that $f = g * h$. Write $g^* = g_1 g_2$ with $g_1 g_2 \in L^2(\Gamma)$. Let $\pi(\gamma)$ be the operator of translation by γ on $L^2(\Gamma)$: $\pi(\gamma)\phi(\delta) = \phi(\delta - \gamma)$, γ, $\delta \in \Gamma$, and let M_h denote the operator of multiplication by h: $M_h\phi = h\phi$. Then for all $\gamma \in \Gamma$,

$$f(\gamma) = g * h(\gamma) = \int_\Gamma h(\delta) g(\gamma - \delta)\, d\delta = \int_\Gamma h(\delta) \overline{g^*(\delta - \gamma)} d\delta$$

$$= \int_\Gamma h(\delta) \overline{g_1(\delta - \gamma)}\, \overline{g_2(\delta - \gamma)} d\delta = \langle M_h \pi(\gamma) \bar{g}_1, \pi(\gamma) g_2 \rangle.$$

Proceeding as in the proof of Lemma 2.6, we can pass to a Hilbert space containing $L^2(\Gamma)$ and a unitary dilation of $\|h\|_\infty^{-1} M_h$ to obtain a function $\alpha \in S(\Gamma, \Gamma)$ satisfying (23).

If Γ is discrete, then since Δ is a subgroup of Γ there is an idempotent measure μ on G such that $\hat{\mu}$ is the characteristic function of Δ. Thus $\hat{\mu}\alpha$ is the desired function in this case. However, a more direct proof in case Γ is discrete comes from consideration of the function

$$\alpha(\gamma, \lambda) = \langle M_f \pi(\gamma) \delta_0, \pi(\lambda) \delta_0 \rangle,$$

which has the desired properties.

Returning now the general case, let $f \in C_0(\Gamma)$, and write $f = \sum_{n=1}^\infty f_n$, where $f_n \in C_{00}(\Gamma)$ and $\sum_{n=1}^\infty \|f_n\| < \infty$. By the Open Mapping Theorem there exists $C > 0$ and $u_n \in BM(G, G)$ such that $\hat{u}_n(\gamma, \gamma) = f_n(\gamma)$, $\gamma \in \Gamma$ and $\|u_n\| \le C\|f_n\|$, $n = 1, 2, \ldots$. For each n, let $h_n \in L^1(G \times G)$ such that $\|h_n\| < 2$ and $\hat{h}_n(\gamma, \gamma) = 1$ if $f_n(\gamma) \neq 0$. Set $v_n = u_n * u_{h_n}$. Then $v_n \in BM_a(G, G)$, $\|v_n\| \le 2CK_G^2 \|f_n\|$, and $\hat{v}_n(\gamma, \gamma) = f_n(\gamma)$, $\gamma \in \Gamma$. Thus

$$v = \sum_{n=1}^\infty v_n \in BM_a(G, G),$$

and $\alpha = \hat{v}$ satisfies (23).

COROLLARY 5.9. *Let Γ be a LCA group, and U an open and closed subset of Γ such that the characteristic function χ_U is uniformly continuous on Γ. Let*

$$\tilde{U} = \{(\gamma, \gamma) : \gamma \in U\} \subset \Delta.$$

Then there exists an idempotent $\alpha \in S(\Gamma, \Gamma)$ such that $\alpha|_\Delta = \chi_{\tilde{U}}$.

Proof. Note that the uniform continuity of χ_U is equivalent to the existence of an open subgroup Λ of Γ such that U is a union of cosets of Λ. For U and U^c must be unions of cosets of the identity component of Γ, so it suffices to consider the case when Γ is totally disconnected. In this case it is easy to see that uniform continuity means there is a compact open subgroup Λ of Γ such that U is a union of cosets of Λ.

If $\phi\colon \Gamma \to \Gamma/\Lambda$ is the canonical homomorphism, then

$$\bar{\alpha} = \chi_{(\phi\times\phi)(\tilde{U})} \in S(\Gamma/\Lambda, \Gamma/\Lambda)$$

by Theorem 5.8. Now, if we set

$$\alpha(\gamma, \delta) = \bar{\alpha}(\phi(\gamma), \phi(\delta)), \qquad \gamma, \delta \in \Gamma,$$

then $\alpha \in S(\Gamma, \Gamma)$, $\alpha^2 = \alpha$, and $\alpha|_\Delta = \chi_{\tilde{U}}$.

COROLLARY 5.10. *If G is nondiscrete, then $M(G \times G)$ is not dense in $BM(G, G)$, neither in the BM-norm topology, nor in the topology of uniform convergence of Fourier-Stieltjes transforms on $\Gamma \times \Gamma$.*

Proof. If $\mu \in M(G \times G)$, then since Δ is a closes subgroup of $\Gamma \times \Gamma$, $\hat{\mu}|_\Delta \in B(\Delta)$. But not every bounded uniformly continuous function on Δ lies in the uniform closure $B(\Delta)^-$ of $B(\Delta)$. Since

$$\|\hat{u}\|_{\Gamma\times\Gamma}k \leq \|u\|, \qquad u \in BM(G, G),$$

our assertion follows from Theorem 5.8.

REMARK 5.11. A characterization of the space $B(\Gamma)^-$ for any LCA group appears in [**22**].

As an application of Corollary 5.10 we can obtain a proof of the following theorem of Saeki [**25**, Thm. 3] in the spirit of the present work.

THEOREM 5.12. *Let X and Y be locally compact spaces. Then $M(X \times Y) = C_0(X \times Y)^*$ is dense in $BM(X, Y)$ if and only if either X or Y contains no nonvoid perfect set.*

Proof. Suppose both X and Y contain nonvoid perfect sets. Let $D_2 = (\mathbf{Z}_2)^\omega$ denote the Cantor group. Then there exist compact perfect sets $E \subset X$ and $F \subset Y$ and surjective continuous maps

$$\phi\colon E \to D_2, \qquad \psi\colon F \to D_2.$$

Define

$$\Phi: BM(E, F) \to BM(D_2, D_2)$$

by

$$\langle f, \Phi(u) \rangle = \langle f \circ (\phi \times \psi), u \rangle, \qquad f \in V(D_2, D_2).$$

Then Φ is easily seen to be a norm-reducing linear map which maps $M(E \times F)$ onto $M(D_2 \times D_2)$. If $M(X \times Y)$ is dense in $BM(X, Y)$, then $M(E \times F)$ is dense in $BM(E, F)$ (cf. Definition 1.6), which then implies $M(D_2 \times D_2)$ is dense in $BM(D_2, D_2)$. But Corollary 5.10 says this is not the case.

Conversely, suppose, say, X contains no nonvoid perfect sets. Since the support of a continuous regular Borel measure is obviously a perfect set, we see that every regular Borel measure on X is discrete. In particular, if $u \in BM(X, Y)$ and λ_X, λ_Y are a Grothendieck measure pair for u, then λ_X is discrete. So given $\varepsilon > 0$ and $C > 0$ as in (1), it follows easily from (1) that there is a finite set $E \subset X$ such that

$$\left\| u - u|_{E \times Y} \right\| < C\varepsilon,$$

and $u|_{E \times Y}$ is clearly a measure.

6. Some thin sets associated with bimeasures. Throughout this section we shall restrict our attention to *compact groups* G_1 and G_2 and their discrete duals and study the notion of a Sidon set relative to the bimeasure algebra. Extensions to noncompact groups G_1 and G_2, including the notion of a BM-Helson set, may be supplied by the informed reader.

DEFINITION 6.1. A subset E of $\Gamma_1 \times \Gamma_2$ is called a *BM-Sidon set* if for every bounded, complex-valued function f on E there exists $u \in BM(G_1, G_2)$ such that $\hat{u} = f$ on E, Note that, as in the case of measures, if E is a BM-Sidon set, there exists $C > 0$ such that, given f as above, the bimeasure u can be chosen so that $\|u\| \leq C\|f\|_E$. The smallest such C is called the *BM-Sidon constant* of E.

Just as in the case of measures, there is a series of conditions on a set E which are equivalent to the assertion that E is a BM-Sidon set. The proof of the following theorem rests on standard arguments such as may be found in [**24**, §5.7] and ideas developed earlier in this work. We shall omit the details. Recall that a function $f \in L^1(G_1 \times G_2)$ is called an E-function if $\hat{f}(\gamma, \delta) = 0$ for all $(\gamma, \delta) \notin E$ [**24**, 5.7.1].

THEOREM 6.2. *The following conditions on a subset E of $\Gamma_1 \times \Gamma_2$ are equivalent.*

(i) *E is a BM-Sidon set with BM-Sidon constant C.*

(ii) *For every trigonometric E-polynomial f on $G_1 \times G_2$,*

$$\sum_{(\gamma,\delta)\in E} |\hat{f}(\gamma,\delta)| \le C\|f\|_V. \tag{24}$$

(iii) *Condition* (24) *holds for every E-function $f \in V(G_1, G_2)$.*

(iv) *For every E-function $f \in \mathfrak{L}^\infty(G_1) \hat{\otimes} \mathfrak{L}^\infty(G_2)$,*

$$\sum_{(\gamma,\delta)\in E} |\hat{f}(\gamma,\delta)| \le C\|f\|_{\infty,\infty}.$$

Moreover, E is a BM-Sidon set if and only if there exists $C' > 0$ such that for every $f \in C_0(E)$ there exists $u \in BM_a(G_1, G_2)$ such that $\hat{u} = f$ on E and $\|u\| \le C'\|f\|_E$. If K is the infimum of all such numbers C', and E satisfies (i), *then $C \le K \le 2K_G^2 C$.*

We know from Lemma 1.12 or Theorem 5.8 that if $\Gamma_1 = \Gamma_2 = \Gamma$ then the diagonal Δ in $\Gamma \times \Gamma$ is a *BM*-Sidon set with *BM*-Sidon constant 1. We shall now extend this result, thereby exibiting more *BM*-Sidon sets that are not $M(G_1 \times G_2)$-Sidon sets.

THEOREM 6.3. *Let $\phi\colon \Gamma_1 \to \Gamma_2$ be any mapping such that*

$$\max_{\delta\in\Gamma_2} |\phi^{-1}(\delta)| = n < \infty. \tag{25}$$

Then the graph $E = \{(\gamma, \phi(\gamma))\colon \gamma \in \Gamma_1\}$ of ϕ is a BM-Sidon set with BM-Sidon constant at most $n^{1/2}$.

Proof. We shall verify condition (ii) of Theorem 6.2. Let f be a trigonometric E-polynomial and let $\varepsilon > 0$ be given. Choose trigonometric polynomials $g_1, \ldots, g_m$ on C_1 and $h_1, \ldots, h_m$ on G_2 such that $f = \sum_1^m g_j \otimes h_j$ and

$$\sum_{j=1}^m \|g_j\|_{G_1} \|h_j\|_{G_2} < (1+\varepsilon)\|f\|_V.$$

It follows from (25) that $\|\hat{h}_j \circ \phi\|_2 \leq n^{1/2}\|\hat{h}_j\|_2$. Thus

$$\begin{aligned}\sum_{(\gamma,\delta)\in E} |\hat{f}(\gamma,\delta)| &\leq \sum_{j=1}^{m} \sum_{(\gamma,\delta)\in E} |\hat{g}_j(\gamma)|\,|\hat{h}_j(\delta)| \\ &= \sum_{j=1}^{m} \sum_{\gamma\in\Gamma_1} |\hat{g}_j(\gamma)|\,|\hat{h}_j(\phi(\gamma))| \leq \sum_{j=1}^{m} \|\hat{g}_j\|_2 \|\hat{h}_j \circ \phi\|_2 \\ &\leq \sum_{j=1}^{m} n^{1/2}\|g_j\|_2 \|h_j\|_2 \leq n^{1/2} \sum_{j=1}^{m} \|g_j\|_{G_1} \|h_j\|_{G_2} \\ &< n^{1/2}(1+\varepsilon)\|f\|_V,\end{aligned}$$

and our Theorem follows.

EXAMPLES 6.4. (i) Let a and b be nonzero integers, and set $E = \{(m,n): m, n \in \mathbf{Z}, am + bn = 0\}$. Then by Theorem 6.3 E is a BM-Sidon set in $\mathbf{Z}^2$.

(ii) If Γ_1 and Γ_2 are infinite, then it is easy to see that for any $\gamma_0 \in \Gamma_1$ and $\delta_0 \in \Gamma_2$, $\{(\gamma, \delta_0): \gamma \in \Gamma_1\}$ and $\{(\gamma_0, \delta): \delta \in \Gamma_2\}$ are not BM-Sidon sets. For if $u \in BM(G_1, G_2)$, then it is clear that the functions $\phi(\gamma) = \hat{u}(\gamma, \delta_0)$ and $\psi(\delta) = \hat{u}(\gamma_0, \delta)$ are Fourier-Stieltjes transforms. Hence some condition like (25) is necessary if the conclusion of Theorem 6.3 is to be valid.

REMARK 6.5. We conclude our discussion of BM-Sidon sets with some remarks regarding the union of such sets. It is a straightforward exercise to modify the argument of Drury **[4]**, **[5**, §5.5] by substituting bimeasures for measures; we shall omit the details. If this is done and the fact that $M(G_1 \times G_2)$ is a Banach algebra is replaced by Theorem 2.6, one arrives at the quantitative result embodied in the following theorem. A standard argument then leads to Corollary 6.7 below. We have explored the possibility of adapting to the context of bimeasures Rider's proof **[23]** of the union theorem for Sidon sets, which proof uses the notion of "almost surely continuous" functions. If that adaptation could be accomplished, a better estimate (on the order of C^3) of the norm in Theorem 6.6 would follow. However, there are several places in the proof of Rider where such an adaptation appears difficult to achieve.

THEOREM 6.6. *Let E be a BM-Sidon set in $\Gamma_1 \times \Gamma_2$ with BM-Sidon constant C. There exists $u \in BM(G_1, G_2)$ such that $\hat{u} = 1$ on E, $|\hat{u}| \leq 1/2$ outside E, and $\|u\| \leq 128 K_G^6 C^4$.*

COROLLARY 6.7. *If E and F are BM-Sidon sets in $\Gamma_1 \times \Gamma_2$, then $E \cup F$ is a BM-Sidon set.*

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NORTHWESTERN UNIVERSITY
EVANSTON, IL 60201

AND

WAYNE STATE UNIVERSITY
DETROIT, MI 48202

Current address of C. Graham: Dept. of Mathematics
University of British Columbia
Vancouver BC V6T 1W5, Canada