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INTEGRALITY OF SUBRINGS OF MATRIX RINGS

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Let $A \subseteq B$ be commutative rings, and Γ a multiplicative monoid which generates the matrix ring $M_n(B)$ as a B-module. Suppose that for each $\gamma \in \Gamma$ its trace $\operatorname{tr}(\gamma)$ is integral over A. We will show that if A is an algebra over the rational numbers or if for every prime ideal P of A, the integral closure of A/P is completely integrally closed, then the algebra $A(\Gamma)$ generated by Γ over A is integral over A. This generalizes a theorem of Bass which says that if A is Noetherian (and the trace condition holds), then $A(\Gamma)$ is a finitely generated A-module.

Our generalizations of the theorem of Bass [**B**, Th. 3.3] yield a simplified proof of that theorem. Bass's proof used techniques of Procesi in [**P**, Ch. VI] and involved completion and faithfully flat descent. The arguments given here are based on elementary properties of integral closure and complete integral closure. They serve also to illuminate a couple of theorems of A. Braun concerning prime p.i. rings integral over the center.

One might expect that integrality of $\operatorname{tr}(\gamma)$ for $\gamma \in \Gamma$ would be sufficient to assure that $A(\Gamma)$ is integral over A. But this is not so, as we will show with a counterexample. As it frequently happens with traces, complications arise in prime characteristic.

1. Integrality and complete integral closure. Recall that if A is an integral domain and b lies in its quotient field, b is said to be *almost integral* over A if there is an $a \in A$, $a \neq 0$, such that $ab^i \in A$ for all integers $i \geq 1$. A is said to be *completely integrally closed* (c.i.c.) if every element almost integral over A lies in A. Recall that a Krull domain is completely integrally closed [**Bo**, §1, No. 3], as indeed is any intersection of rank 1 valuation rings. (However, examples are known of c.i.c. domains which are not intersections of rank 1 valuation rings — see [**Nk**] or [**G**, App. 4].) If A is a Noetherian domain, the Mori-Nagata Theorem [**N**, (33.10)] says that the integral closure of A is a Krull domain, hence is c.i.c.

LEMMA 1. *Let A be a completely integrally closed integral domain with quotient field F, and let B be the integral closure of A in any extension field of F. Then B is completely integrally closed.*

Proof. This is [**K**, Satz 11].

LEMMA 2. *Let R be a ring and A a subring of the center of R, such that A contains no zero divisors of R. Suppose the integral closure of A is completely integrally closed. If there is an $a \in A$, $a \neq 0$, with aR integral over A, then R is integral over A.*

Proof. If not, take $t \in R$ with t not integral over A. We may assume $R = A[t]$, which is commutative. Let $S = \{b \cdot f(t) | b \in A, b \neq 0$ and $f \in A[x], f$ monic$\}$, a multiplicatively closed subset of R not containing 0. Let P be an ideal of R maximal such that $P \cap S = \varnothing$. Then $P \cap A = (0)$ and, replacing R by R/P, we may assume that R is an integral domain. Let B be the integral closure of A in the quotient field of R. By hypothesis $aR \subseteq B$; hence, t is almost integral over B. By Lemma 1, $t \in B$, contradicting the choice of t.

Here is a variant of Lemma 2. It is proved in the same way, but using $S = \{a^i f(t) | f \in A[x], f \text{ monic}\}$ and applying the Mori-Nagata Theorem.

LEMMA 2′. *Let A be a Noetherian subring of the center of a ring R; let $a \in A$ be a regular element of R. If aR is integral over A, then R is integral over A.*

These lemmas can be applied to prime p.i. rings, yielding short proofs of one theorem of A. Braun and part of another. For, if R is a prime p.i. ring with center C, then a theorem of Amitsur using central polynomials [**A**, Th. 6] says that there is a $\delta \in C$, $\delta \neq 0$, such that δR lies in a ring which is a free C-module of finite rank. It follows by the usual determinant argument that δR is integral over C.

PROPOSITION 3 (*Braun*, [**Br$_1$**, Th. 2.7]). *Let R be a prime p.i. ring which is finitely-generated as an algebra over some commutative Noetherian ring A. Let C be the center of R. Then R is a finitely-generated C-module if and only if the integral closure of C is a Krull domain.*

Proof. If R is a finitely-generated C-module, then by the Artin-Tate Lemma [**AT**] C is a finitely-generated A-algebra. Hence, C is Noetherian, so by the Mori-Nagata Theorem its integral closure is a Krull domain. Conversely, suppose the integral closure of C is a Krull domain (hence completely integrally closed). By Lemma 2 and the remarks above, R is integral over C. Then, by a theorem of Procesi [**P**, p. 128], R is a finitely generated C-module.

PROPOSITION 4 (*Braun* [**Br$_2$**, pp. 13–14], *Schelter* [**S**, Cor. 2 to Th. 2]). *If R is a prime p.i. ring with center C, and if the integral closure of C is completely integrally closed, then R is integral over C.*

Proof. Apply Lemma 2 and the remarks preceding Prop. 3.

2. Integrality when traces are integral. We now return to Bass's theorem. Throughout this section, let $A \subseteq B$ be commutative rings, and Γ a multiplicative monoid in the $n \times n$ matrix ring $M_n(B)$ which generates $M_n(B)$ as a B-module. Let $A(\Gamma)$ be the A-module (and algebra) generated by Γ. We wish to consider when the following statement is true:

(*) If $\operatorname{tr}(\gamma)$ is integral over A, for each $\gamma \in \Gamma$, then $A(\Gamma)$ is integral over A.

PROPOSITION 5. *If A is an algebra over a field F, and if* $\operatorname{char} F = 0$ *or* $\operatorname{char} F = p > n$, *then* (*) *is true.*

Proof. Consider first the generic $n \times n$ matrix α, whose entries are the commuting indeterminates $x_{11}, x_{12}, \ldots, x_{nn}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of α in an algebraic closure of $F(x_{11}, \ldots, x_{nn})$, and let the characteristic polynomial of α be

$$\chi_\alpha(x) = x^n + c_1 x^{n-1} + \cdots + c_n.$$

For each i, let $t_i = \operatorname{tr}(\alpha^i) = \lambda_1^i + \cdots + \lambda_n^i$; these traces are related to the c_j's by Newton's identities (see, e.g., [**C**, pp. 436–437], or [**H**, p. 249]):

$$t_i + \sum_{j=1}^{i-1} c_j t_{i-j} + i c_i = 0, \qquad 1 \le i \le n. \tag{1}$$

Now, take any $\gamma \in A(\Gamma)$. Then $\operatorname{tr}(\gamma)$ is integral over A, since γ is an A-linear combination of elements of Γ. Specializing from α to γ we obtain formulas corresponding to (1) relating the traces $\operatorname{tr}(\gamma^i)$ and the coefficients of the characteristic polynomial $\chi_\gamma(x)$. The assumption on $\operatorname{char} F$ assures that we can divide by $2, 3, \ldots, n$. Therefore, we may solve recursively for the c_i in (1), obtaining expressions for the coefficients of $\chi_\gamma(x)$ as polynomials in $\{\operatorname{tr}(\gamma^i) \mid 1 \le i \le n\}$. Thus, the coefficients of $\chi_\gamma(x)$ are integral over A; hence γ is integral over A, as desired.

REMARKS. The argument for Prop. 5 is valid for any ring A in which the images of $2, 3, \ldots, n$ are all units. Note also that the assumption that $B(\Gamma) = M_n(B)$ was not used.

PROPOSITION 6. *Suppose that for every prime ideal P of A, the integral closure of A/P is completely integrally closed. Then $(*)$ is true.*

Proof. If not, take any $t \in A(\Gamma)$, t not integral over A. Let $S = \{f(t) | f \in A[x], f \text{ monic}\} \subseteq M_n(B)$. S is closed under multiplication and $0 \notin S$. Let Q be an ideal of $M_n(B)$, maximal with the property that $Q \cap S = \varnothing$. Then Q is a prime ideal and, reducing mod Q, we may assume that A and B are integral domains. Furthermore, since there is no harm in enlarging B or replacing A by an integral extension, we may assume that B is a field and A is integrally closed in B. Then, by Lemma 1, A is completely integrally closed.

Let $c_1, \ldots, c_{n^2} \in \Gamma$ be a basis for $M_n(B)$ as a vector space over B. Take any $\gamma \in A(\Gamma)$, and write $\gamma = \Sigma b_i c_i$. Then, for each j,

$$\operatorname{tr}(\gamma c_j) = \sum_i b_i \operatorname{tr}(c_i c_j). \tag{2}$$

By hypothesis, all the traces appearing in (2) lie in A. Viewing (2) as n^2 equations in the variables $b_1, \ldots, b_{n^2}$, it follows by Cramer's rule that $\delta b_i \in A$, $i = 1, \ldots, n^2$, where $\delta = \det(\operatorname{tr}(c_i c_j)) \in A$. By the nondegeneracy of the trace, $\delta \neq 0$. Let $T = \Sigma_{i=1}^{n^2} A(\delta c_i)$; since $\delta b_i \in A$, we have

$$\delta^2 A(\Gamma) \subseteq T. \tag{3}$$

To see that T is actually a ring we make a similar computation. Let

$$c_i c_j = \sum_k \beta_{ijk} c_k. \tag{4}$$

Multiplying (4) by any c_l and taking traces, we have

$$\operatorname{tr}(c_i c_j c_l) = \sum_k \beta_{ijk} \operatorname{tr}(c_k c_l). \tag{5}$$

Again, the traces in (5) lie in A, so (fixing i or j) by Cramer's rule $\delta\beta_{ijk} \in A$. Thus, rewriting (4) as

$$(\delta c_i)(\delta c_j) = \sum_k (\delta\beta_{ijk})(\delta c_k)$$

we see that T is closed under multiplication. Since T is also a finitely generated A-module, it is integral over A. Therefore, Lemma 2 and (3) above show that $A(\Gamma)$ is integral over A. This contradiction completes the proof.

COROLLARY 7 (*Bass*). *If, in addition to the hypotheses at the beginning of the section, A is Noetherian and Γ is a finitely generated monoid, then $A(\Gamma)$ is a finitely generated A-module.*

Proof. As noted earlier, the Mori-Nagata theorem assures that the integral closure of a Noetherian domain is c.i.c. Therefore, by Prop. 6, $A(\Gamma)$ is integral over A. Since, in addition, A is Noetherian and $A(\Gamma)$ is a finitely generated p.i. A-algebra, it follows by a theorem of Procesi [**P**, p. 128] that $A(\Gamma)$ is a finitely generated A-module.

EXAMPLE 8. Let F be any field of prime characteristic p, and let x and y be commuting indeterminates over F. Let $C = F[x, y]$; $J = yC$, A the subring $F + J$ of C, and B the quotient field of C. In the matrix ring $M_p(B)$, let I be the identity matrix, and $\{E_{ij}\}$ the usual matrix units. Let Γ be the monoid generated by $\{xI + yE_{ij} | 1 \leq i, j \leq p\}$. Then $B(\Gamma) = M_p(B)$, and for each $\gamma \in \Gamma$, $\mathrm{tr}(\gamma) \in A$. But none of the generators of Γ is integral over A. So, $(*)$ does not hold.

Proof. The p^2 generators of Γ are linearly independent over B; hence, $B(\Gamma) = M_p(B)$. Note that $\Gamma \subseteq M_p(C)$, and in $M_p(C/J)$ the image of Γ is generated by scalar matrices; so the image must consist entirely of scalar matrices, which have trace 0. Thus, for $\gamma \in \Gamma$, $\mathrm{tr}(\gamma) \in J \subseteq A$. However, $xI + yE_{ij}$ cannot be integral over A, since its image in $M_p(C/J)$ is clearly not integral over $A/J \cong F$.

REMARKS. Example 8 shows the need for the hypotheses in Prop. 5 and Prop. 6. In the example A is integrally closed, but its complete integral closure is C. By slightly modifying the example, one can obtains a counterexample to $(*)$ for any $n \geq p$ and any ring A with a prime ideal P such that A/P has characteristic p, and the integral closure of A/P is not c.i.c. E.g., to obtain a counterexample in characteristic 0, replace F in Ex. 8 by the ring $\mathbf{Z}$ of integers, and J by the ideal of $\mathbf{Z}[x, y]$ generated by p and y.

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