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p-ADIC OSCILLATORY INTEGRALS AND WAVE FRONT SETS

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For K a p-adic field, we examine oscillatory integrals

$$I(\phi, p)(\lambda) = \int_{K^n} \phi(x)\Psi(\lambda p(x))\,dx$$

where ϕ is a Schwartz function on K^n, Ψ is an additive character, $\lambda \in K^x$, and $p\colon K^n \to K$ is locally analytic. If $Dp \neq 0$ on the support of ϕ, $\lambda \mapsto I(\phi, p)(\lambda)$ has bounded support. If $Dp(x_0) = 0$ at exactly one point x_0 in the support of ϕ but $D^2p(x_0)$ is non-degenerate, then

$$I(\phi, p)(\lambda) = |\lambda|^{-n/2}\gamma\Psi(p(x_0))|\det D^2p(x_0)|^{-1/2}\phi(x_0)$$

for sufficiently large $|\lambda|$, where γ is a complex eighth root of unity. An invariant definition of wave front set, $WF_\Lambda(u)$, for distributions u relative to an open subgroup Λ of $K^\times$ is proved to exist, analogous to the classical case, with "rapidly decreasing" replaced by "bounded support". Definitions of pull backs and push forwards of distributions, distribution products, and kernel maps are made, again similar to the classical case, and their wave front sets computed. Wave front sets $WF_\Lambda(\rho)$ of representations ρ of p-adic groups are also defined (cf. Howe, Automorphic forms, representation theory, and arithmetic, Tata Inst., 1979, for the Lie group analogue). For admissible representations ρ of, say, a semi-simple group G, with character χ_ρ, we show that $WF_\Lambda^0(\rho) = WF_\Lambda^0(\chi_\rho)$, where $WF_\Lambda^0(\cdot) \subseteq \mathrm{Lie}(G)$ is $WF_\Lambda(\cdot)$ above the identity element. Functorial properties of $WF_\Lambda(\rho)$ are developed and examples computed.

Introduction. The motivation for this work is to apply to p-adic groups the approach of Howe [H] who has applied classical wave front set theory to Lie group representations. Chapter I develops a stationary phase formula for p-adic oscillatory integrals. In Chapter II, p-adic wave front sets are defined, relative to multiplicative subgroups of the multiplicative group of a p-adic field. The wave front set of a representation of a p-adic group is then defined and developed in Chapter III.

We denote by K a locally compact field of characteristic 0 with valuation $|\cdot|_K$, O_K its ring of integers, and P_K the unique maximal prime ideal of O_K. Denote by $\bar{\omega}$ a fixed uniformizing element. If $|\bar{\omega}|_K = q^{-1}$, define $\mathrm{Ord}(x)$, $x \in K^\times$, by $|x| = q^{-\mathrm{Ord}(x)}$. On the n-dimensional space K^n, we use the norm $|(x_1,\ldots,x_n)|_{K^n} = \max\{|x_i|_K\}$. The unit sphere is denoted Σ^{n-1}. Identify K^n with its dual $(K^n)^*$ by the symmetric bilinear form $\{x, y\} = \Sigma_{i=1}^n x_i y_i$. Fix an additive character Ψ of K with conductor O_K.

Any additive character of K^n is then of the form $x \mapsto \Psi(\langle x, y\rangle)$. Taking dx to be a Haar measure on K^n, the Fourier transform is defined by

$$\hat{\phi}(y) = \int_{K^n} \phi(x)\Psi(\langle x, y\rangle)\, dx$$

for $\phi \in L^1(K^n)$, and can be extended to $L^2(K^n)$ where, if $\int_{O_K} dx = 1$, we have $\hat{\hat{\phi}}(x) = \tilde{\phi}(x) = \phi(-x)$.

For X an analytic n-dimensional K manifold, let $H(X)$ denote the set of locally analytic functions on X. By $C^\infty(X)$, we denote the space of locally constant complex valued functions on X, $S(X)$ the subspace of compactly supported functions in $C^\infty(X)$, and $S'(X)$ the space of all linear functionals on $S(X)$. If ω is a fixed strictly positive smooth density on X, $C^\infty(X)$ is then identified with a subset of $S'(X)$ by $f \mapsto f\omega$. By T^*X we denote the analytic co-tangent space on X.

1. ***p*-adic oscillatory integrals.** For $\phi \in S(X)$, $X \subseteq K^n$ open, and $p \in H(X \times K^r)$ let

$$I_\eta(p, \phi)(\lambda) := \int_X \phi(x)\Psi(\lambda p(x, \eta))\, dx, \qquad \lambda \in K^\times.$$

We compute asymptotic expansions for $\lambda \mapsto I_\eta(p, \phi)(\lambda)$ in the two cases where $\operatorname{grad}_x p$ is non-zero on $\operatorname{supp}(\phi)$, and where p has one non-degenerate critical point on $\operatorname{supp}(\phi)$.

PROPOSITION 1.1. *Let $X \subseteq K^n$ and $V \subseteq K^r$ be open. Suppose that $p \in H(X \times V)$, $\phi \in S(X)$, and $|\operatorname{grad}_x p(x, \eta)| \geq \delta > 0$ for $(x, \eta) \in \operatorname{supp}(\phi) \times V$. Suppose further that $|R(x, y, \eta)|$ is bounded for x, $x + y \in \operatorname{supp}(\phi)$, and $\eta \in V$, where R is defined by*

$$p(x + y, \eta) = p(x, \eta) + \langle \operatorname{grad}_x p(x, \eta), y\rangle + \langle R(x, y, \eta)y, y\rangle.$$

Then $\lambda \mapsto I_\eta(p, \phi)(\lambda)$ has bounded support on $K^\times$, with bound independent of $\eta \in V$.

Proof. Assume $n = 1$, the proof for $n > 1$ being analogous. Since $U = \operatorname{supp}(\phi)$ is open-compact, there is an integer N such that if $m > N$,

$$U = \bigcup_{i \in I} x_i + P_K^m$$

for some finite set $\{x_i\}_{i \in I} \subset \operatorname{supp}(\phi)$. Suppose

$$q^a = \inf_{x \in U, \eta \in V} |\operatorname{grad}_x p(x, \eta)|, \qquad q^b = \sup_{\substack{x \in U, \eta \in V \\ x + y \in U}} |R(x, y, \eta)|,$$

and let $M_U = q^{\max\{b - 2a, 2N - b\}}$.

If $|\lambda| > M_U$, then

$$-\mathrm{Ord}(\lambda) > b - 2a \quad \text{and} \quad -\mathrm{Ord}(\lambda) + a > (-\mathrm{Ord}(\lambda) + b)/2.$$

Choose m_0 such that $-\mathrm{Ord}(\lambda) + a > m_0 \geq (-\mathrm{Ord}(\lambda) + b)/2$. Note that $m_0 > N$, since $-\mathrm{Ord}(\lambda) > 2N - b$.

Supposing $\phi \equiv 1$ on U, we have

$$I_\eta(p, \phi)(\lambda) = \sum_{i \in I} \int_{P_K^{m_0}} \Psi(\lambda p(x_i + x, \eta))\, dx,$$

for a finite set $\{x_i\}_{i \in I} \subset U$. Now for each $i \in I$,

$$\begin{aligned}(1.1)\quad & \int_{P_K^{m_0}} \Psi(\lambda p(x_i + x, \eta))\, dx \\ &= \int_{P_K^{m_0}} \Psi\left(\lambda\left\{p(x_i, \eta) + \frac{dp}{dx}(x_i, \eta) + R(x_i, x, \eta)x^2\right\}\right) dx \\ &= \Psi(\lambda p(x_i, \eta)) \int_{P_K^{m_0}} \Psi\left(\lambda \frac{dp}{dx}(x_i, \eta)x\right) \cdot \Psi\left(\lambda R(x_i, x, \eta)^2\right) dx.\end{aligned}$$

For $x \in P_K^{m_0}$, $|\lambda R(x_i, x, \eta)x^2| \leq |\lambda| q^{-2m_0 + b} \leq 1$, hence $\Psi(\lambda R x^2) = 1$. If $|x| = q^{-m_0}$ then

$$\left|\lambda \frac{dp}{dx}(x_i, \eta)x\right| \geq |\lambda| q^{a - m_0} > 1,$$

thus $x \mapsto \Psi(\lambda\, dp(x_i, \eta)x/dx)$ is a non-trivial character on $P_K^{m_0}$. Returning to (1.1), if $|\lambda| > M_U$, for any $\eta \in V$,

$$\int_{P_K^{m_0}} \Psi(\lambda p(x_i + x, \eta))\, dx = \Psi(\lambda p(x_i, \eta)) \int_{P_K^{m_0}} \Psi\left(\lambda \frac{dp}{dx}(x, \eta)x\right) dx = 0,$$

hence $I_\eta(p, \phi)(x) = 0$. □

Now suppose $p \in H(X)$ (we drop the parameter in K^r for the moment), $\phi \in S(X)$, and $\operatorname{grad} p(x_0) = 0$ for some $x_0 \in \operatorname{supp}(\phi)$. If $D^2 p(x_0)$ is non-degenerate, we have a p-adic stationary phase formula, analogous to the classical case [**G–S**, p. 6].

PROPOSITION 1.2. *Let $X \subseteq K^n$ be open, $p \in H(X)$, and $\phi \in S(X)$. Suppose $\{x_i\}_{i \in I}$ is the set of critical points of p in* supp(ϕ), *and $D^2 p(x_i)$ is non-degenerate for each $i \in I$. Then if $|\lambda|$ is sufficiently large,*

$$(1.2)\qquad I(p, \phi)(\lambda) = |\lambda|^{-n/2} \sum_{i \in I} c(p, x_i)\phi(x_i)$$

for $\lambda \in (K^\times)^2$, where

$$c(p, x_i) = \gamma \Psi(p(x_i)) \left|\det(D^2 p(x_i))\right|^{-1/2},$$

γ a complex eighth root of unity.

Proof. First note that the Morse Lemma holds true for p-adic locally analytic functions (cf. [**L**, p. 174]).

Thus assuming p has only one critical point $x_0 \in U = \operatorname{supp}(\phi)$, if $D^2p(x_0)$ is non-degenerate, by the Morse Lemma (and reducing $\operatorname{supp}(\phi)$ is necessary), there is a coordinate system y on U such that

$$p(x) = p(x_0) + \langle Ay(x), y(x)\rangle, \qquad x \in U,$$

where $A = D^2p(x_0)$ is symmetric, non-degenerate. Thus

$$(1.3) \quad I(p,\phi)(\lambda) = \int_X \phi(x)\Psi[\lambda(p(x_0) + \langle Ay(x), y(x)\rangle)]\, dx$$

$$= \Psi(\lambda p(x_0))\int_X \theta(y)\Psi(\lambda\langle Ay, y\rangle)\, dy$$

where $\theta = \phi|\det(dx/dy)| \in S(X)$.

By [**W**, p. 161], if $\Psi_{\lambda A}(x) := \Psi(\langle A\sqrt{\lambda}x, \sqrt{\lambda}x\rangle)$, then

$$(\Psi_{\lambda A})\hat{\ }(y) = \gamma|\det A|^{-1/2}|\lambda|^{-n/2}\Psi\left(-\lambda^{-1}\langle y, {}^t(A^{-1})y\rangle\right),$$

where γ is a complex eighth root of 1.

Applying the Fourier multiplication formula to (1.3), we then have

$$I(p,\phi)(\lambda) = \Psi(\lambda p(x_0))\gamma|\det A|^{-1/2}|\lambda|^{-n/2}$$

$$\cdot\int_X \tilde{\hat{\theta}}(y)\Psi\left(-y^{-1}\langle y, {}^t(A^{-1})y\rangle\right) dy.$$

To evaluate the integral, note that $y \mapsto \Psi(-\lambda^{-1}\langle y, {}^t(A^{-1})y\rangle)$ is identically 1 on $\operatorname{supp}(\tilde{\hat{\theta}})$ for $|\lambda|$ sufficiently large. Thus

$$I(p,\phi)(\lambda) = \Psi(\lambda p(x_0))\gamma|\det A|^{-1/2}|\lambda|^{-n/2}\int_X \hat{\theta}(y)\, dy$$

$$= \Psi(\lambda p(x_0))\gamma|\det A|^{-1/2}|\lambda|^{-n/2}\phi(x_0),$$

proving the proposition. □

2. ***P*-adic wave front sets.** We first define the notion of wave front set for open subsets of K^n (c.f. [**Ho**, p. 119–133] for the classical case).

DEFINITION. Let $X \subseteq K^n$ be open, $u \in S'(X)$, and Λ be an open subgroup of $K^\times$ with $[K^\times : \Lambda] < \infty$. We say that u is Λ-smooth at $(x_0, \xi_0) \in X \times (K^n - \{0\})$ if there are open neighborhods U of x_0 and V of ξ_0 such that for any $\phi \in S(U)$ there is a $N_\phi > 0$ for which $|\lambda| > N_\phi$

implies that $(\phi u)\hat{}(\lambda\xi) = 0$ for any $\xi \in V$. The complement of the set of smooth directions of u is called the Λ-wave front set of u, denoted $WF_\Lambda(u)$.

The local nature of $WF_\Lambda(u)$ is shown by the p-adic version of [**Ho**, Prop. 2.5.4].

PROPOSITION 2.1. *Let $X \subseteq K^n$ and $U \subseteq X$ be open. Then for any $u \in S'(X)$,*

$$WF_\Lambda(u|U) = WF_\Lambda(u) \cap \pi_X^{-1}(U),$$

where π_X: $X \times (K^n - \{0\})$ is the projection onto X.

The following lemma notes a useful computational simplification.

LEMMA 2.2. *Let $X \subseteq K^n$ be open and $u \in S'(X)$. Suppose $U \subseteq X$ is open, ϕ_U is the characteristic function on U, and $\lambda \mapsto (\phi_U u)\hat{}(\lambda\xi)$ has bounded support on Λ, uniformly for $\xi \in V$, V some open-compact subset of $K^n - \{0\}$. Then $\lambda \mapsto (\phi u)\hat{}(\lambda\xi)$ has bounded support on Λ, uniformly for $\xi \in V$, for any $\phi \in S(U)$.*

Proof. By hypothesis there is an $N_U > 0$ such that when $\lambda \in \Lambda$, $|\lambda| > N_U$ implies $(\phi_U u)\hat{}(\lambda\xi) = 0$ for any $\xi \in V$. If $\phi \in S(U)$, $\operatorname{supp}(\hat{\phi})$ is compact, so since V is compact, there is an $N_V > 0$ such that $|\lambda| > N_V$ implies $\lambda^{-1}\eta + \xi \in V$ for $\eta \in \operatorname{supp}(\hat{\phi})$, $\xi \in V$. Thus $N_\phi = \max\{N_U, N_V\}$ and $|\lambda| > N_\phi$ implies $(\phi_U u)\hat{}(\eta + \lambda\xi) = (\phi_U u)\hat{}(\lambda(\lambda^{-1}\eta + \xi)) = 0$ for $\eta \in \operatorname{supp}(\hat{\phi})$, $\xi \in V$. Hence

$$(\phi u)\hat{}(\lambda\xi) = (\phi \cdot \phi_U u)\hat{}(\lambda\xi) = \int \hat{\phi}(-\eta)(\phi_U u)\hat{}(\eta + \lambda\xi)\, d\eta = 0$$

proving the lemma. □

EXAMPLE. Suppose the residual characteristic of K is not 2, let $\varepsilon_0 \in O_K^\times$ be an element of order $q - 1$, Λ the image of $K(\sqrt{\varepsilon_0})^\times$ in $K^\times$ under the map N_{ε_0}: $K(\sqrt{\varepsilon_0}) \to K$, $x + \sqrt{\varepsilon_0}y \mapsto x^2 - \varepsilon_0 y^2$, and H_Λ the characteristic function of Λ. If Φ_0 is the characteristic function on O_K and $u = ((1 - \Phi_0)H_\Lambda)\check{}$, then $WF_\Lambda(u) = \{(0, \xi) \in K \times (K - \{0\}): \xi \in \Lambda\}$.

The invariance of the definition of wave front set is a consequence of the following proposition.

PROPOSITION 2.3. *Let $p \in H(X \times K^r)$ and $X \subseteq K^n$ be open. Suppose* $\operatorname{grad}_x p(x_0, \eta_0) = \xi_0 \neq 0$ *for some $\eta_0 \in K^r$, and (x_0, ξ_0) is smooth for $u \in S'(X)$. Then there are neighborhood U_0 of x_0 and W_0 of η_0 such that for any $\phi \in S(U_0)$ there exists an $N_\phi > 0$ for which $|\lambda| \geq N_\phi$, $\lambda \in \Lambda$, implies*

$$\langle u, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$$

for any $\eta \in W_0$.

Proof. (After **[D]**, Prop. 1.3.2.) Suppose $u \in S'(K^n)$ is smooth on the neighborhood $U_0 \times V$ of $(x_0, \xi_0) \in T^*K^n$, and $p \in H(X \times K^r)$ satisfies $\operatorname{grad}_x p(x_0, \eta_0) = \xi_0$ for some $\eta_0 \in K^r$. Let $\phi \in S(U_0)$ and $\phi' \in S(K^n)$ be 1 on $\operatorname{supp}(\phi)$. Then

$$(2.1)\quad \langle u, \phi\Psi(\lambda p(\cdot, \eta))\rangle = |\lambda|^n \int_{K^n} (\phi u)\hat{}(\lambda\xi) \times \int_X \phi'(x)\Psi(\lambda(p(x, \eta) - \langle \xi, x\rangle))\, dx\, d\xi.$$

By hypothesis there is a $M_\phi > 0$ such that for $\lambda \in \Lambda$, $|\lambda| > M_\phi$ implies $(\phi u)\hat{}(\lambda\xi) = 0$ for $\xi \in V$. So suppose $\xi \notin V$. Since $\operatorname{grad}_x p(x_0, \eta_0) = \xi_0$, by shrinking U_0 if necessary, we have

$$|\operatorname{grad}_x p(x, \eta) - \xi| \geq \delta > 0$$

when $x \in U_0$, and η is close to η_0. Applying Proposition 1.1 shows that $\lambda \mapsto \int_X \phi'(x)\Psi(\lambda(p(x, \eta) - \langle \xi, x\rangle))\, dx$ has bounded support, and we are done. □

A well defined wave front set for a distribution on an analytic manifold X can thus be determined using any coordinate system on X. Note that Proposition 2.1 and Lemma 2.2 are valid for analytic manifolds.

DEFINITION. For $S \subseteq K^n$, an S-cone is a set $\Gamma \subseteq K^n \times (K^n - \{0\})$ satisfying $(x, s\xi) \in \Gamma$ whenever $(x, \xi) \in \Gamma$ and $s \in S$. S-cones on manifolds are defined analogously.

The proof of the next theorem is straightforward.

THEOREM 2.4. *Let $u \in S'(X)$, X be an n-dimensional analytic K manifold, and Λ be an open subgroup of finite index in $K^\times$. Then*

(i) *$WF_\Lambda(u)$ is a closed Λ-cone in $T^*X - \{0\}$,*

(ii) *If Λ' is a open subgroup of Λ with $[\Lambda : \Lambda'] < \infty$, then $WF_\Lambda(u) = \bigcup_{\tau \in \Lambda/\Lambda'} \tau \cdot WF_{\Lambda'}(u)$, where $\tau \cdot WF_\Lambda(u) := \{(x, \tau\xi) \in T^*X\colon\ (x, \xi) \in WF_{\Lambda'}(u)\}$.*

(iii) $\operatorname{Sing\,supp}(u) = \pi_X(WF_\Lambda(u))$, *where* π_X *is the projection* $T^*X \to X$.
(iv) $WF_\Lambda(u_1 + u_2) \subseteq WF_\Lambda(u_1) \cup WF_\Lambda(u_2)$, *for* $u_1, u_2 \in S'(X)$.

We now develop functorial properties of WF_Λ. First we derive some lemmas on the geometry of Λ-cones. For $S \subseteq K^\times$ and $T \subseteq K^n - \{0\}$ let $\Gamma_S(T) := \{s\xi : s \in S, \xi \in T\}$.

If T is open then $\Gamma_S(T)$ is open since $s \cdot U$ is a neighborhood of $s\xi$ in $\Gamma_S(T)$ when U is a neighborhood of $\xi \in T$. Further, if T is compact and S closed then $\Gamma_S(T)$ is closed in $K^n - \{0\}$.

DEFINITION. Let Λ be a subgroup of $K^\times$ with $[K^\times : \Lambda] < \infty$ and suppose the order of $\bar{\omega}\Lambda$ in $K^\times/\Lambda$ is l. Set

$$\Sigma_\Lambda^{n-1} := \bigcup_{i=0}^{l} \bar{\omega}^i \cdot \Sigma^{n-1}.$$

LEMMA 2.5. *If* $\Gamma \subseteq K^n$ *is a* Λ*-cone and* $\Gamma^1 = \Gamma \cap \Sigma_\Lambda^{n-1}$, *then* $\Gamma = \Gamma_\Lambda(\Gamma^1)$.

Proof. That $\Gamma_\Lambda(\Gamma^1) \subseteq \Gamma$ is clear. If $\xi \in \Gamma$ there is an integer m such that $\xi' = (\bar{\omega}^l)^m \xi \in \Gamma^1$. Since $\bar{\omega}^l \in \Lambda$, $\xi = (\bar{\omega}^l)^{-m}\xi' \in \Gamma$, and $\Gamma = \Gamma_\Lambda(\Gamma^1)$. □

LEMMA 2.6. *Let* Γ_1 *and* Γ_2 *be* Λ*-cones, closed in* $K^n - \{0\}$, *and suppose* $\Gamma_1 \cap \Gamma_2 = \varnothing$. *Then there exists a* Λ*-cone* V, *open in* K^n *and closed in* $K^n - \{0\}$, *such that* $\Gamma_1 \subseteq V$ *and* $\Gamma_2 \subseteq V' = K^n - V$.

Proof. Let $\Gamma_1^1 = \Gamma_1 \cap \Sigma_\Lambda^{n-1}$. Since Γ_1^1 is compact and $\Gamma_1^1 \subseteq K^{-n} - \Gamma_2$, an open set, there is a finite set $\{V_{\xi_i}\}_{i \in I}$ of open-closed balls V_{ξ_i} such that $\Gamma_1^1 \subseteq V^1 = \bigcup_{i \in I} V_{\xi_i}$, and $V^1 \cap \Gamma_2 = \varnothing$. Let $V = \Gamma_\Lambda(V^1)$. Then since V^1 is open, V is an open cone, and it is also closed, since Λ is. By Lemma 2.5, $\Gamma_1 = \Gamma_\Lambda(\Gamma_1^1) \subseteq \Gamma_\Lambda(V^1) = V$, and $V \cap \Gamma_2 = \Gamma_\Lambda(V^1) \cap \Gamma_2 = \varnothing$. Hence $\Gamma_2 \subseteq V' = K^n - V$ and V' is open. □

LEMMA 2.7. *Let* X *be an open subset of* K^n, $u \in S'(X)$, *and* $x_0 \in X$. *Suppose* V *is an open-closed* Λ*-cone containing*

$$\Gamma = \{\xi \in K^n - \{0\} : (x_0, \xi) \in WF_\Lambda(u)\},$$

and set $V' = (K^n - \{0\}) - V$. *Take* $p \in H(X \times K^r)$ *with* $\operatorname{grad}_x p(x_0, \eta) \in V'$, *for* η *in some open* $W_0 \subseteq K^r$. *Then there is an open neighborhood* $U \subseteq X$ *of* x_0 *such that* $\phi \in S(U)$ *implies there is an* $M_\phi > 0$ *for which, if* $|\eta| > M_\phi$, $\eta \in V'$, *then* $\langle u, \phi\Psi(p(\cdot, \eta))\rangle = 0$.

Proof. Assume that X, u, x_0, V, and V' are as in the hypothesis, and that $p(x, n) = \langle x, \eta \rangle$, the proof for arbitrary p following from Proposition 2.3.

Let $V_1 = V \cap \Sigma_\Lambda^{n-1}$. Now V_1 is open-closed so $V_1' = \Sigma_\Lambda^{n-1} - V_1$ is open-closed, hence compact. Since V_1' is contained in $K^n - \Gamma$, there exists a finite set $\{V_{\xi_i}\}_{i \in I}$ of open-closed balls such that (i) Each V_{ξ_i} is an open-closed neighborhood of some $\xi_i \in V_1'$ for which there exists a neighborhood U_{ξ_i} of x_0 such that if $\phi \in S(U_{\xi_i})$, there is an $M_{\phi_i} > 0$ for which $|\lambda| > M_{\phi_i}$ implies that $(\phi u)\hat{}(\lambda \xi) = 0$, for any $\xi \in V_{\xi_i}$, and (ii) $V_1' \subseteq \bigcup_{i \in I} V_{\xi_i}$.

Let $U = \bigcap_{i \in I} U_{\xi_i}$, $\phi \in S(U) \subseteq S(U_{\xi_i})$, $M_\phi = \operatorname{Max}_{i \in I}\{M_{\phi_i}\}$, and $i \in I$. If $\xi = \lambda \xi'$, $\xi' \in V_{\xi_i}$, and $|\lambda| \geq M_\phi$, then $(\phi u)\hat{}(\xi) = (\phi u)\hat{}(\lambda \xi') = 0$. Hence $(\phi u)\hat{}$ restricted to $\Gamma_\Lambda(V_{\xi_i})$ has bounded support. By (ii), $V' = \Gamma_\Lambda(V_1')$ is contained in $\bigcup_{i \in I} \Gamma_\Lambda(V_{\xi_i})$. Hence $(\phi u)\hat{}$ has bounded support in V'. □

A locally analytic map $f: X \to Y$ between analytic K manifolds X and Y induces a pull back $f^*: C^\infty(Y) \to C^\infty(X)$, $\phi \mapsto \phi \circ f$. If f is also proper (f^{-1} (compact) is compact), then the push forward $f_*: S'(X) \to S'(Y)$ of f is defined by $\langle f_* u, \phi \rangle = \langle u, f^* \phi \rangle$, $u \in S'(X)$, $\phi \in S(Y)$. We want to compute relations between $WF_\Lambda(u)$, $WF_\Lambda(f^*u)$ and $WF_\Lambda(f_* u)$.

We have so far defined f^* only on $C^\infty(Y)$, where $WF_\Lambda(f^*u) = \varnothing$, an uninteresting case. So we next extend the definition of f^*.

DEFINITION. For any closed Λ-cone $\Gamma \subseteq T^*X - \{0\}$, let $S'_\Gamma(X)$ denote the set $\{u \in S'(X): WF_\Lambda(u) \subseteq \Gamma\}$.

THEOREM 2.8. *Let X and Y be analytic m- and n-dimensional K-manifolds respectively, and suppose $f: X \to Y$ is locally analytic. Let $N_f = \{(y, \eta) \in T^*Y - \{0\}: y = f(x)$ for some $x \in X$ and ${}^tDf(x)\eta = 0\}$. Let Γ' be a closed Λ-cone in $T^*Y - \{0\}$ with $N_f \cap \Gamma' = \varnothing$. If $\Gamma = f^*\Gamma' := \{(x, {}^tDf(x)\eta) \in T^*X$: there is a $y \in Y$ with $y = f(x)$ and $(y, \eta) \in \Gamma'\}$, then $f^*: C^\infty(Y) \to C^\infty(X)$ has a unique continuous extension to a map $f^*: S'_{\Gamma'}(Y) \to S'_\Gamma(X)$. Moreover*

$$WF_\Lambda(f^*v) \subseteq f^*(WF_\Lambda(v))$$

when $v \in S'_{\Gamma'}(Y)$.

Proof. By Proposition 2.1 we may assume $X \subseteq K^m$ and $Y \subseteq K^n$ are open and $\operatorname{supp}(v)$ is compact, $v \in S'_{\Gamma'}(Y)$.

Now N_f and Γ' are closed cones, so if $N_f \cap \Gamma' = \varnothing$, by Lemma 2.6 there exists an open-closed Λ-cone $V \subseteq K^n - 0$ such that $N_f \subseteq V$ and $\Gamma' \subseteq V' = (K^n - 0) - V$.

If $v \in S'_{\Gamma'}(Y)$, by Lemma 2.7 for each $y \in Y$, there is an open-compact neighborhood U_y such that $\chi \in S(U_y)$ implies $\eta \mapsto (\chi v)\hat{}(\eta)$ has bounded support when restricted to V. Thus $\int_V (\chi v)\hat{}(\eta) I_f(\phi)(\eta)\, d\eta$ converges, where

$$I_f(\phi)(\eta) := \int_X \phi(x)\Psi(\langle f(x), -n\rangle)\, dx.$$

Assume $\operatorname{supp}(v) \subseteq U_{y_0}$ for some $y_0 \in Y$.

Now let $V'_1 = V' \cap \Sigma_\Lambda^{n-1}$. Since $N_f \cap V' = \varnothing$, there is a δ such that

$$|\langle f(x), -\eta\rangle| = |{}^t Df(x)\eta| \geq \delta > 0$$

for $(x, \eta) \in U \times V'_1$, $U = f^{-1}(U_{y_0})$. By Proposition 1.1, if $\phi \in S(U)$, there is a $M_\phi > 0$ such that $|\lambda| > M_\phi$, $\lambda \in \Lambda$, implies $I_f(\phi)(\lambda\eta) = 0$ for any $\eta \in V'_1$. So if $\xi = \lambda\eta$ is an element of $V' = \Gamma_\Lambda(V'_1)$, $\lambda \in \Lambda$, $\eta \in V'_1$, then $|\lambda| > M_\phi$ implies $|\lambda| > M_\phi$ and $I_f(\phi)(\xi) = I_f(\phi)(\lambda\eta) = 0$. Hence $\eta \mapsto I_f(\phi(\eta))$ has bounded support on V', and $\int_{V'} \hat{v}(\eta) I_f(\phi)(\eta)\, d\eta$ converges.

Thus $\int_{K^n} \hat{v}(\eta) I_f(\phi)(\eta)\, d\eta$ exists and defines an extension of f^* from $C^\infty(Y)$ to $S'_\Gamma(Y)$.

To show that we have really extended f^* to a map $S'_{\Gamma'}(Y) \to S'(X)$ and that this extension is unique, take $v \in S(Y)$ ($\subseteq S_{\Gamma'}(Y)$ for any Γ, using the standard identification). Then $\hat{v} \in S(Y)$,

$$\int_{K^n} \hat{v}(\eta) I_f(\phi)(\eta)\, d\eta = \int_X \left[\int_{K^n} \Psi(\langle f(x), -\eta\rangle)\hat{v}(\eta)\, d\eta\right]\phi(x)\, dx$$

converges for any $\phi \in S(X)$, and

$$\int_X \left[\int_{K^n} \Psi(\langle f(x), -\eta\rangle)\hat{v}(\eta)\, d\eta\right]\phi(x)\, dx$$

$$= \int_X v(f(x))\phi(x)\, dx = \int_X f^*v(x)\phi(x)\, dx = \langle f^*v, \phi\rangle.$$

Uniqueness comes from the fact that $C^\infty(Y)$ is dense in $S'_\Gamma(Y)$, proved by using an approximate identity as in the classical case.

Next we show that f^* maps $S'_{\Gamma'}(Y)$ into $S'_\Gamma(X)$ by proving that $WF_\Lambda(f^*v) \subseteq f^*WF_\Lambda(v)$ for $v \in S'_{\Gamma'}(Y)$. So suppose that (x_0, ξ_0), $\xi_0 = {}^t Df(x_0)\eta_0$, is not an element of $f^*WF_\Lambda(v)$, that is, $(f(x_0), \eta_0) = (y_0, \eta_0) \notin WF_\Lambda(v)$. Let U_{y_0} and V_{y_0} be open-closed neighborhoods of y_0 and η_0, respectively, such that for any $\chi \in S(U_{y_0})$, $\lambda \mapsto (\chi v)(\lambda\eta)$ has bounded support on Λ, uniformly for $\eta \in V_{y_0}$. We may assume that $\operatorname{supp}(v) \subseteq U_{y_0}$.

Now for any $\phi \in S(X)$,

$$(2.2)\quad (\phi f^* v)^\wedge(\lambda\xi) = \int_{K^n} \hat{v}(\eta)\left[\int_X \phi(x)\Psi(\langle f(x), -\eta\rangle + \langle x, \lambda\xi\rangle)\, dx\right] d\eta$$

$$= |\lambda|^n \int_{K^n} \hat{v}(\lambda\eta)\left\{\int_X \phi(x)\Psi[\lambda(\langle f(x), -\eta\rangle + \langle x, \xi\rangle)]\, dx\right\} d\eta$$

$$= |\lambda|^n \int_{K^n} \hat{v}(\lambda\eta) I_\phi(\lambda, \xi, \eta)\, d\eta,$$

where

$$I_\phi(\lambda, \xi, \eta) := \int_X \phi(x)\Psi(\lambda p(x, \xi, \eta))\, dx,$$

with

$$p(x, \xi, \eta) := \langle f(x), -\eta\rangle + \langle x, \xi\rangle.$$

Let T denote the subset of K^{m+n},

$$\{(\xi, \eta) \colon {}^t Df(x)\eta = \xi, \eta \in V, f(x) \in U\}.$$

For any Λ-conic open-closed neighborhood Ω of T there exists a $\delta > 0$ such that

$$|\operatorname{grad}_x p(x, \xi, \eta)| = |{}^t Df(x)\eta - \xi| \geq \delta > 0,$$

if $x \in f^{-1}(U)$ and (ξ, η) is not in Ω. Hence by Proposition 1.1, $\lambda \mapsto I_\phi(\lambda, \xi, \eta)$ has bounded support on Λ, uniformly for $(\xi, \eta) \notin \Omega$. However if $\eta \in V_{y_0}$, $\lambda \mapsto \hat{v}(\lambda, \eta)$ has bounded support for $\lambda \in \Lambda$. Hence (2.2) does too, $(x_0, \xi_0) \notin WF_\Lambda(f^* v)$, and the theorem is proved. □

Next we examine $WF_\Lambda(f_* u)$.

THEOREM 2.9. *If $u \in S'(X)$ and $f\colon X \to Y$ is a locally analytic proper map on* supp(u), *then $f_* u$ is defined and*

$$WF_\Lambda(f_* u) \subseteq f_*(WF_\Lambda(u)) \cup N_f,$$

where $f_*(WF_\Lambda(u)) := \{(y, \eta) \in T^*Y\colon\ y = f(x)$ *for some* $x \in X$ *and* $(x, {}^t Df(x)\eta) \in WF_\Lambda(u)\}$.

Proof. See [**G**, Theorem 2]. □

In the classical case distribution products can be shown to exist, assuming certain criteria on wave front sets [**Ho**, Theorem 2.5.10]. The same is true in the p-adic case.

THEOREM 2.10. *Let X be an n-dimensional analytic K-manifold, and Γ_1, $\Gamma_2 \subseteq T^*X - 0$ be closed Λ-cones with $\Gamma_1 \cap -\Gamma_2 = \varnothing$. Then there is a unique continuous extension of the product on $S(X) \times S(X)$ to a product on $S'_{\Gamma_1}(X) \times S'_{\Gamma_2}(X)$. If $u_i \in S'_{\Gamma_i}(X)$, $i = 1, 2$, then*

$$\operatorname{supp}(u_1 \cdot u_2) \subseteq \operatorname{supp}(u_1) \cap \operatorname{supp}(u_2) \tag{2.3}$$

and

$$WF_\Lambda(u_1 \cdot u_2) \subseteq (WF_\Lambda(u_1) + WF_\Lambda(u_2)) \cup WF_\Lambda(u_1) \cup WF_\Lambda(u_2). \tag{2.4}$$

Proof. The proof is analogous to the classical case. We assume $X \subseteq K^n$ open. If the closed cones Γ_1 and $-\Gamma_2$ are disjoint, by Lemma 2.6 there is an open-closed Λ-cone V such that $\Gamma_1 \subseteq V$ and $-\Gamma_2 \subseteq V' = (K^n - 0) - V$.

Take $u_1 \in S'_{\Gamma_1}(X)$, $u_2 \in S'_{\Gamma_2}(X)$, and $\phi \in S(X)$. Since $WF_\Lambda(u_1) \cap V' = \varnothing$, by Lemma 2.7 and assuming $\operatorname{supp}(\phi)$ sufficiently small, $\xi \mapsto (\phi^{1/2}u_1)\hat{}\,(\xi)$ has bounded support in V'. Similarly, $\xi \mapsto (\phi^{1/2}u_2)\hat{}\,(-\xi)$ has bounded support in V. Thus the integral

$$\int_X (\phi^{1/2}u_1)\hat{}\,(\xi)(\phi^{1/2}u_2)\hat{}\,(-\xi)\, d\xi$$

converges, defining $\langle u_1 \cdot u_2, \phi \rangle$.

Now suppose $x_0 \notin \operatorname{supp}(u_1)$. Let $U \subseteq X$ be an open-closed ball with $x_0 \in U$ and $U \cap \operatorname{supp}(u_1) = \varnothing$. Set $W = X - U$ and suppose χ_W is the characteristic function on W. Pick $\{v_n\}$ and $\{v'_n\}$ to be sequences of C^∞ functions convergent to u_1 and u_2, respectively, in $S'(X)$. Note that $\chi_W v_n \to u_1$. Let $w_n = (\chi_W v_n) v'_n$, so $w_n \to u_1 \cdot u_2$. Then if $\phi \in S(U)$,

$$\langle u_1 \cdot u_2, \phi \rangle = \lim_{n \to \infty} \langle w_n \phi \rangle = \lim_{n \to \infty} \langle v_n v'_b, \chi_W \phi \rangle = 0,$$

so $x_0 \notin \operatorname{supp}(u_1 \cdot u_2)$. Similarly, $\operatorname{supp}(u_1 \cdot u_2) \subset \operatorname{supp}(u_2)$, proving (2.3).

For the proof of (2.4) see **[G**, pp. 241–242], whose functorial approach we have now proved to be valid for the p-adic case. □

As noted at the end of the proof of Theorem 2.10, Theorems 2.8–2.9 on the wave front sets of pull-backs and push-forwards give us p-adic versions of the basic functorial machinery in **[G]**. The following two results on kernel maps, which we just state, can then be proved as in **[G]**.

THEOREM 2.11. *Let X and Y be analytic K manifolds.*

(i) *Suppose $\kappa \in S'(X \times Y)$ is properly supported ($\pi_*\kappa$ is defined for both projections $\pi = \pi_X$: $X \times Y \to X$ and $\pi = \pi_Y$: $X \times Y \to Y$), and*

$v \in S(Y)$. *Then the product* $L_\kappa v = \kappa \cdot (1 \otimes v)$ *is defined, inducing a map* $L_\kappa\colon S(Y) \to S'(X)$, *where* $\langle L_\kappa v, \phi\rangle := \langle \kappa \cdot (1 \otimes v), \phi \otimes 1\rangle$, *and* $WF_\Lambda(L_\kappa v) \subseteq \pi_{X_*}(WF_\Lambda(\kappa))$.

(ii) *If* $\Gamma \subseteq T^*Y - 0$ *is a closed* Λ*-cone and* $-\Gamma$ *is disjoint from* $\{(y,\eta) \in T^*Y\colon ((x,y),(0,\eta)) \in WF_\Lambda(\kappa)$ *for some* $x \in X\}$, *then* L_κ *can be uniquely extended from* $S(Y)$ *to a continuous map* $L_\kappa\colon S'_\Gamma(Y) \to S'(X)$. *If* $v \in S'_\Gamma(Y)$, $WF_\Lambda(L_\kappa v) \subseteq R^{*\prime}_\kappa(WF_\Lambda(v)) \cup \pi_{X_*}(WF_\Lambda(\kappa))$, *where*

$$R'^*_\kappa(WF_\Lambda(v)) := \{(x,\xi) \in T^*X\colon ((x,y),(\xi,-\eta)) \in WF_\Lambda(\kappa) \text{ for some } (y,\eta) \in WF_\Lambda(v)\}.$$

Further, if $\pi_{X_*}(WF_\Lambda(\kappa)) = \varnothing$, *then by Theorem* 2.4(iii), $\operatorname{Sing\,supp}(L_\kappa v)$ *is contained in*

$$\{x \in X\colon (x,y) \in \operatorname{sing\,supp}(\kappa) \text{ for some } y \in \operatorname{sing\,supp}(v)\}.$$

THEOREM 2.12. *Let* X, Y, *and* Z *be analytic* K*-manifolds,* $\kappa_1 \in S'(X \times Y)$, $\kappa_2 \in S'(Y \times Z)$ *be properly supported, and suppose* $WF_\Lambda(\kappa_1) \times 0$ *is disjoint from* $-(0 \times WF_\Lambda(\kappa_2))$. *Then*

(i) *The composition*

$$\kappa_1 \circ \kappa_2 := \pi_{X \times Z_*}((\kappa_1 \otimes 1)\cdot(1 \otimes \kappa_2))$$

exists.

(ii) $WF_\Lambda(\kappa_1 \circ \kappa_2)$ *is contained in the set*

$$WF'_\Lambda(\kappa_1) \circ WF_\Lambda(\kappa_2) \cup \big(R^*_{\kappa_1}(O_Y) \times O_Z\big) \cup \big(O_X \times R_{\kappa_{2_*}}(O_Y)\big),$$

where

$$WF'_\Lambda(\kappa_1) := \{((x,y),(\xi,\eta)) \in T^*(X \times Y)\colon ((x,y),(\xi,-\eta)) \in WF_\Lambda(\kappa_1)\},$$

$$R^*_{\kappa_1}(O_Y) := \{(x,\xi) \in T^*X\colon ((x,y),(\xi,0)) \in WF_\Lambda(\kappa_1) \text{ for some } y \in Y\},$$

and

$$R_{\kappa_{2_*}}(O_Y) := \{(z,\zeta) \in T^*Z\colon ((y,z),(0,\zeta)) \in WF_\Lambda(\kappa_2) \text{ for some } y \in Y\}.$$

Here O_X, O_Y, *and* O_Z *are the zero sections in* T^*X, T^*Y, *and* T^*Z *respectively.*

(iii) $\operatorname{supp}(\kappa_1 \circ \kappa_2) \subseteq \operatorname{supp}(\kappa_1) \circ \operatorname{supp}(\kappa_2)$.

3. P-adic wave front sets of group representations. In this section we turn to representations ρ of p-adic groups, defining the notion of the wave front set of ρ similar to that of **[H]** for Lie groups. Our novelty consists not of formulation of results, but in placement of emphasis, and verification of the p-adic case.

As usual, K is a locally compact field with discrete valuation and Λ is an open subgroup of $K^\times$ with $[K^\times : \Lambda] < \infty$. Let G be a unimodular closed subgroup of $\mathrm{GL}_n(K)$, closed in K^{n^2}, and dx be a Haar measure on G. We consider only unitary representations (ρ, H_ρ) of G on the Hilbert space H_ρ. (Our results are valid for a wider range of representations, but for concreteness we take ρ unitary.)

Our first definition generalizes that of WF_Λ given in Chapter 2 to operator valued distributions.

DEFINITION. We say that ρ is Λ-smooth at $(g_0, \xi_0) \in T^*G$ if for any locally analytic function p on $G \times K^r$, there are neighborhoods U_0 of g_0 and $W_0 \subseteq K^r - \{0\}$ of ξ_0 such that $\phi \in S(U_0)$ implies that

$$\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = \int_G \phi(x)\Psi(\lambda p(x, \eta))\rho(x)\, dx = 0,$$

for any $\eta \in W_0$, when $|\lambda| > M_\phi$, $\lambda \in \Lambda$, for some $M_\phi > 0$. The set of $(x, \xi) \in T^*G$ where ρ is not smooth at (x, ξ) is denoted $WF_\Lambda(\rho)$ and called the Λ-wave front set of the representation ρ.

If $L_g(x) := g^{-1}x$ and $R_g(x) := xg$ for $g, x \in G$, then $L_g^*(WF_\Lambda(\rho)) = R_g^*(WF_\Lambda(\rho)) = WF_\Lambda(\rho)$. For if $\phi \in S(G)$, and $p \in H(G \times K^r)$, we have

$$\begin{aligned}\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle &= \int_G \phi(g^{-1}x)\Psi(\lambda p(g^{-1}x, \eta))\rho(g^{-1}x)\, dx \\ &= \rho(g^{-1})\langle \rho, L_g^*\phi\Psi(\lambda p(\cdot, \eta))\rangle,\end{aligned}$$

and similarly for R_g^*. Thus $WF_\Lambda(\rho)$ can be identified with an $\mathrm{Ad}G$ invariant subset of the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G:

$$WF_\Lambda^0(\rho) = \{\xi \in \mathfrak{g}^* : (e, \xi) \in WF_\Lambda(\rho)\},$$

where we identify $T^*G \simeq G \times \mathfrak{g}^*$. If $\xi \in \mathfrak{g}^* - WF_\Lambda(\rho)$, we say that ρ is Λ-smooth at ξ.

We give three equivalent conditions for ρ-smoothness, which are the p-adic versions of [**H**, Theorem 1.4 i, vi, vii].

LEMMA 3.1. *Let (ρ, H_ρ) be a unitary representation of the unimodular p-adic group G and suppose that $V \subseteq \mathfrak{g}^*$ is open. Then the following statements are equivalent*:

(i) $V \cap WF_\Lambda^0(\rho) = \varnothing$.

(ii) *If* $\phi \in S(G)$ *and* $p \in H(G \times K^r)$ *satisfies* $\operatorname{grad}_x p(\eta)(\operatorname{supp}(\phi)) \subseteq V$ *for all* η *in some open subset* W_0 *of* K^r, *then the operator* $\langle \rho, \phi\psi(\lambda p(\cdot, \eta))\rangle$ *is* 0 *for large enough* $|\lambda|$, *uniformly for* $\eta \in W_0$.

(iii) *There is a neighborhood* U *of the identity element* e *such that whenever* $\phi \in S(U)$, $p \in H(G \times K^r)$, *and* $\operatorname{grad}_x p(\eta)(\operatorname{supp}(\phi)) \subseteq V$ *for* $\eta \in W_0$, W_0 *an open subset of* K^r, *then* $\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ *when* $|\lambda|$ *is sufficiently large, uniformly for* $\eta \in W_0$.

Next we examine the collective behavior of the scalar distributions defined by the matrix coefficients of ρ. Denote by $\Upsilon(H_\rho)$, or just Υ_ρ, the set of trace class operators on H_ρ. For any $g \in G$ and $T \in \Upsilon_\rho$, $\rho(g)T$ is of trace class, and $g \mapsto \operatorname{Tr}_\rho(T)(g) = \operatorname{tr}(\rho(g)T)$ defines a bounded continuous function on G. Let $TWF_\Lambda(\rho)$ denote the closed Λ-cone in T^*G:

$$\operatorname{Cl}\left[\bigcup_{T \in \Upsilon_\rho} WF_\Lambda(\operatorname{Tr}_\rho(T))\right] \quad (\operatorname{Cl} = \text{closure}).$$

We show that $TWF_\Lambda(\rho) = WF_\Lambda(\rho)$. The first step is to note that $TWF_\Lambda(\rho)$ is invariant under L_g^* and R_g^*. This is proved for the p-adic case analogously to the classical proof [**H**, Prop. 1.1]. We may then identify $TWF_\Lambda(\rho)$ with an $\operatorname{Ad}G$ invariant subset of $\mathfrak{g}^*$:

$$TWF_\Lambda^0(\rho) = \{\xi \in \mathfrak{g}^* \colon (e, \xi) \in TWF_\Lambda(\rho)\}.$$

Again it is useful to have various equivalent criteria for Λ "$TWF_\Lambda^0(\rho)$" smoothness.

LEMMA 3.2. *Let* (ρ, H_ρ) *be a unitary representation of the unimodular* p*-adic group* G, *and* V *be an open subset of* $\mathfrak{g}^*$. *Then the following are equivalent*:

(i) $V \cap TWF_\Lambda^0(\rho) = \varnothing$.

(ii) *If* $T \in \Upsilon_\rho$ *and* $p \in H(G \times K^r)$ *satisfies* $\operatorname{grad}_x p(e, \eta) \in V$ *for* $\eta \in W_0$, W_0 *an open subset of* K^r, *then there is an open neighborhood* U *of* e *such that for any* $\phi \in S(U)$ *there exists an* $M_\phi > 0$ *for which* $|\lambda| > M_\phi$, $\lambda \in \Lambda$, *implies* $\langle \operatorname{Tr}_\rho(T), \Phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ *for any* $\eta \in W_0$.

(iii) *If* $T \in \Upsilon_\rho$, $\phi \in S(G)$, *and* $p \in H(G \times K^r)$ *with* $\operatorname{grad}_x p(\eta) \cdot (\operatorname{supp}(\phi)) \subseteq V$ *for* $\eta \in W_0$, W_0 *an open subset of* K^r, *then* $\lambda \mapsto \langle \operatorname{Tr}_\rho(T), \phi\Psi(\lambda p(\cdot, \eta))\rangle$ *has bounded support on* Λ, *uniformly for* $\eta \in W_0$.

(iv) *Same as* (iii) *except that we require only* $\phi \in S(U)$ *for some fixed neighborhood* U *of* e.

(v) *Same as* (iii) *except that the support of* $\lambda \mapsto \langle \operatorname{Tr}_\rho(T), \phi\Psi(\lambda p(\cdot, \eta))\rangle$ *for fixed* ϕ *and* p *is independent of* T.

Again the proof for the p-adic case is analogous to the classical [**H**, Theorem 2.4i–v].

Lemmas 3.1 and 3.2 now imply that $WF_\Lambda(\rho) = TWF_\Lambda(\rho)$.

THEOREM 3.3. *$WF_\Lambda^0(\rho) = TWF_\Lambda^0(\rho)$ for any unitary representation ρ of a unimodular p-adic group G.*

Proof. (After [**H**].) Let $V \subseteq \mathfrak{g}^*$ be open. We must show that $V \cap TWF_\Lambda^0(\rho) = \varnothing$ iff $V \cap WF_\Lambda^0(\rho) = \varnothing$.

So suppose V and $TWF_\Lambda^0(\rho)$ are disjoint. To show that $V \cap WF_\Lambda^0(\rho) = \varnothing$, suppose $\phi \in S(G)$, $p \in H(G \times K^r)$, and $\operatorname{grad}_x p(\eta)\,(\operatorname{supp}(\phi)) \subseteq V$ for all $\eta \in W_0$, for some open $W_0 \subseteq K^r$. By Lemma 3.1(ii) it suffices to show that for some $M_\phi > 0$, $|\lambda| > M_\phi$, $\lambda \in \Lambda$, implies that $\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ for any $\eta \in W_0$.

Now by Lemma 3.2(v), there is a $M_\phi > 0$ such that $|\lambda| > M_\phi$, $\lambda \in \Lambda$, implies that for all $T \in \Upsilon_\rho$ and $\eta \in W_0$,

$$\langle \operatorname{Tr}_\rho(T), \phi\Psi(\lambda p(\cdot, \eta))\rangle = \operatorname{tr}\left[\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle T\right] = 0.$$

Let $\{e_i\}_{i \in I}$ be any orthonormal basis of H_ρ. For $i, j \in I$ define $T_{ij} \in \Upsilon_\rho$ by setting $T_{ij}v := \langle v, e_j\rangle e_i$ for $v \in H_\rho$. Then if $|\lambda| > M_\phi$,

$$\begin{aligned}\operatorname{tr}\left[\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle T_{ij}\right] &= \sum_{k \in I} \langle \langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle T_{ij} e_k, e_k\rangle \\ &= \langle \langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle e_i, e_j\rangle = 0.\end{aligned}$$

Hence $\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ when $|\lambda| > M_\phi$, and $V \cap WF_\Lambda^0(\rho) = \varnothing$.

Conversely, suppose $V \cap WF_\Lambda^0(\rho) = \varnothing$. By Lemma 3.1(ii), assuming $\operatorname{grad}_x p(\eta)(\operatorname{supp}(\phi)) \subseteq V$ for $\phi \in S(G)$, $\eta \in W_0$, W_0 open in K^r, there is a $M_\phi > 0$ for which $|\lambda| > M_\phi$, $\lambda \in \Lambda$, implies $\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ when $\eta \in W_0$. But then $\langle Tr_\rho(T), \phi\Psi(\lambda p(\cdot, \eta))\rangle = \operatorname{tr}(0T) = 0$ for $|\lambda| > M_\phi$, for any $T \in \Upsilon_\rho$. Hence by Lemma 3.2(v), $V \cap TWF_\Lambda^0(\rho) = \varnothing$, and theorem is proved. □

Some representations have a natural distribution and wave front set associated with them. For example, take G to be semi-simple and (ρ, H_ρ) to be a unitary representation of G. We say that ρ is admissible if (i) every element of H_ρ is fixed by some open subgroup of G, and (ii) the subspace of H_ρ fixed by any open subgroup of G is finite dimensional [**Go**, p. 1.2]. It follows from (ii) that for any $\phi \in S(G)$, $\langle \rho, \phi\rangle$ is a finite rank operator. Hence

$$\langle \chi_\rho, \phi\rangle = \operatorname{tr}(\langle \rho, \phi\rangle)$$

defines a distribution χ_ρ on G called the character of ρ, which is conjugation invariant. Denote by $WF_\Lambda^0(\chi_\rho)$ the $\mathrm{Ad}G$ invariant Λ-conical subset of $\mathfrak{g}^*$:

$$\{\xi \in \mathfrak{g}^*: (e, \xi) \in WF_\Lambda(\chi_\rho)\}.$$

THEOREM 3.4. *If G is a unimodular semi-simple p-adic group and ρ is a unitary admissible representation of G, then $WF_\Lambda^0(\chi_\rho) = WF_\Lambda^0(\rho)$.*

Proof. That $WF_\Lambda^0(\rho) \subseteq WF_\Lambda^0(\chi_\rho)$ is proved analogously to the Lie group case [**H**, Theorem 1.8].

To show the converse suppose that ρ is smooth at ξ_0, $p \in H(G \times K^r)$ satisfies $\mathrm{grad}_x\, p(e, \eta_0) = \xi_0$, and $U_0 \times W_0$ is a neighborhood of (e, η_0) such that if $\phi \in U_0$, $\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle = 0$ for any $\eta \in W_0$, when $|\lambda| > M_\phi$, $\lambda \in \Lambda$, for some $M_\phi > 0$. Then $|\lambda| > M_\Phi$ implies that for all $\eta \in W_0$,

$$\begin{aligned}\langle \chi_\rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle &= \mathrm{tr}\left[\int_G \phi(x)\Psi(\lambda p(x, \eta))\rho(x)\, dx\right]\\ &= \mathrm{tr}\left[\langle \rho, \phi\Psi(\lambda p(\cdot, \eta))\rangle\right] = \mathrm{tr}(0) = 0.\end{aligned}$$

Hence χ_ρ is smooth at ξ_0. □

Not surprisingly $WF_\Lambda(\rho)$ obeys the same elementary properties as the scalar wave front set. Proposition 2.1, Lemma 2.2, and Theorem 2.4 (i–ii) hold true with "X" replaced by "G", and "$WF_\Lambda(u)$" replaced by "$WF_\Lambda(\rho)$."

The result on sums of representations is actually stronger than in the case of sums of distributions on manifolds (Theorem 2.4iv).

THEOREM 3.5. *Let (ρ_1, H_ρ) and (ρ_2, H_{ρ_2}) be unitary representation of the unimodular p-adic group G, and $(\rho_1 \oplus \rho_2, H_{\rho_1} \oplus H_{\rho_2})$ their direct sum. Then*

$$WF_\Lambda^0(\rho_1 \oplus \rho_2) = WF_\Lambda^0(\rho_1) \cup WF_\Lambda^0(\rho_2).$$

Proof. It suffices to show that $WF_\Lambda^0(\rho_i) \subseteq WF_\Lambda^0(\rho_1 \oplus \rho_2)$, $i = 1, 2$, containment in the converse being implied by Theorem 2.4(iv).

Now Υ_{ρ_1} can be included in $\Upsilon_{\rho_1 \oplus \rho_2}$ by identifying

$$T \leftrightarrow \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \Upsilon_{\rho_1 \oplus \rho_2}, \qquad T \in \Upsilon_{\rho_1}.$$

Then

$$WF_\Lambda\big(\mathrm{Tr}_{\rho_1\oplus\rho_2}(T)\big) = WF_\Lambda\big(\mathrm{Tr}_{\rho_1}(T)\big),$$

for any $T \in \Upsilon_{\rho_1} \subseteq \Upsilon_{\rho_1\oplus\rho_2}$. Hence $TWF_\Lambda(\rho_1) \subseteq TWF_\Lambda(\rho_1 \oplus \rho_2)$, so

$$WF_\Lambda^0(\rho_1) \subseteq WF_\Lambda^0(\rho_1 \oplus \rho_2)$$

by Theorem 3.3. Similarly $WF_\Lambda^0(\rho_2) \subseteq WF_\Lambda^0(\rho_1 \oplus \rho_2)$, proving the theorem. □

The wave front set of an outer tensor product behaves as the scalar case [**G**, Theorem 3].

THEOREM 3.6. *Let (ρ_1, H_{ρ_1}) and (ρ_2, H_{ρ_2}) be unitary representations of the unimodular p-adic group G, and suppose $(\rho_1 \otimes \rho_2, H_{\rho_1\otimes\rho_2})$ is their outer Kronecker product. Then*

$$WF_\Lambda^0(\rho_1 \otimes \rho_2) \subseteq WF_\Lambda^0(\rho_1) \times WF_\Lambda^0(\rho_2) \cup \big(0 \times WF_\Lambda^0(\rho_2)\big) \cup \big(WF_\Lambda^0(\rho_1) \times 0\big).$$

Next suppose $G = G_1 = G_2$. The inner Kronecker product $\rho_1 \times \rho_2$ on G is given by $\rho_1 \times \rho_2(g) := \rho_1(g) \otimes \rho_2(g)$. It is tempting to postulate that

$$WF_\Lambda(\rho_1 \times \rho_2) \subseteq (WF_\Lambda(\rho_1) + WF_\Lambda(\rho_2)) \cup WF_\Lambda(\rho_1) \cup WF_\Lambda(\rho_2).$$

Computing coefficients of $\langle \rho_1 \otimes \rho_2, \phi\Psi(\lambda p(\cdot, \eta))\rangle$ we have formally

$$\begin{aligned}(3.1)\quad &\langle\langle \rho_1 \otimes \rho_2, \phi\Psi(\lambda p(\cdot, \eta))\rangle u \otimes w, u' \otimes w'\rangle \\ &= \left\langle \int_G \rho_1(x)u \otimes \rho_2(x)w\phi(x)\Psi(\lambda p(x, \eta))\,dx, u' \otimes w'\right\rangle \\ &= \int_G \langle \rho_1(x)u, u'\rangle\langle \rho_2(x)w, w'\rangle\phi(x)\Psi(\lambda p(x, \eta))\,dx \\ &= \int_G \mathrm{Tr}_{\rho_1}(S)(x) \cdot \mathrm{Tr}_{\rho_2}(T)(x)\phi(x)\Psi(\lambda p(x, \eta))\,dx,\end{aligned}$$

where $S(v) := \langle v, u'\rangle u$ and $T(v) := \langle v, w'\rangle w$, for $u \in H_{\rho_1}$, $w \in H_{\rho_2}$, $u' \in H^*_{\rho_1}$, and $w' \in H^*_{\rho_2}$.

If $WF_\Lambda(\mathrm{Tr}_{\rho_1}(S))$ and $-WF_\Lambda(\mathrm{Tr}_{\rho_2}(T))$ are disjoint, (3.1) converges, as in Theorem 2.10. However, if $\rho = \rho_1 = \rho_2$, $S = T$, $-1 \in \Lambda$, and $WF_\Lambda(\mathrm{Tr}_\rho(S)) = \varnothing$, then

$$WF_\Lambda\big(\mathrm{Tr}_{\rho_1}(S)\big) \cap WF_\Lambda\big(\mathrm{Tr}_{\rho_2}(T)\big) = WF_\Lambda\big(\mathrm{Tr}_\rho(S)\big) \neq \varnothing,$$

and we do not know if (3.1) converges. So, in general, it is not clear how to compute $WF_\Lambda(\rho_1 \otimes \rho_2)$.

If (ρ, H_ρ) is a unitary representation of a group G, and H is a subgroup of G, then $(\rho|H, H_\rho)$ is a unitary representation of H. A natural question is to ask what is the relation between $WF_\Lambda(\rho)$ and $WF_\Lambda(\rho|H)$ when G is a unimodular p-adic group.

THEOREM 3.7. *Let H be a closed subgroup of the unimodular p-adic group G, and $\mathfrak{h}$ denote the Lie algebra of H. If $\iota\colon H \to G$ is the inclusion map and $\rho|H$ is the restriction to H of the unitary representation (ρ, H_ρ) of G, then*

$$ {}^tD\iota(e)\big(WF^0_\Lambda(\rho)\big) \subseteq WF^0_\Lambda(\rho|H). $$

Further, if N_ι is disjoint from $WF^0_\Lambda(\rho)$, then

$$ {}^tD\iota(e)\big(WF^0_\Lambda(\rho)\big) = WF^0_\Lambda(\rho|H). $$

Proof. The proof for the Lie group case [**H**, Props. 1.5–6] also handles the p-adic case. □

We conclude with non-trivial examples of representation wave front sets.

EXAMPLE 1. Let $G = K^n$ and $(\rho, L^2(G))$ be the regular representation of G. ρ is equivalent by the Fourier transform to the representation

$$ \rho(x)f(y) = \overline{\Psi(\langle x, y\rangle)}f(y), \qquad x \in K^n, f \in L^2(G). $$

Take ϕ to be the characteristic function of $(O_K)^n$, and set $p(x, \eta) = \langle x, \eta\rangle$. Then

$$ \langle \rho, \phi\Psi(\lambda\langle\,\cdot\,, \xi\rangle)\rangle f(y) = \int_{K^n} \phi(x)\Psi(\lambda\langle x, \xi\rangle)(\rho(x)f)(y)\,dx $$

$$ = \int_{K^n} \phi(x)\Psi(\lambda\langle x, \xi\rangle)\overline{\Psi(\langle x, y\rangle)}f(y)\,dx = \hat{\phi}(\lambda\xi - y)f(y), $$

and $WF^0_\Lambda(\rho) = K^n - 0$.

EXAMPLE 2. Suppose G is nilpotent. The computation is the same as [**H**, p. 131], since the Lie and p-adic theory are the same [**K**]. To each irreducible unitary representation (ρ, H_ρ) there is associated an orbit $\Omega_\rho = \{\mathrm{Ad}^*(x)\nu_0\colon x \in G\}$ for some $\nu_0 \in \mathfrak{g}^*$, such that

$$ \langle \chi_\rho, \phi\rangle = \operatorname{tr}\left(\int_G \phi(x)\rho(x)\,dx\right) \tag{3.2} $$

$$ = \int_{\Omega_\rho} \hat{\phi}(\nu)\,d\Omega(\nu), \qquad \phi \in S(G), $$

where

$$\hat{\phi}(\nu) = \int_{\mathfrak{g}^*} \phi(\exp(X))\Psi(\langle X, \nu\rangle)\, dX,$$

and $d\Omega$ is an invariant measure on Ω_ρ. Note that exp is defined on all of since $\exp(X)$ is a finite sum and $X \in \mathfrak{g}^*$.

We use (3.2) to compute $WF_\Lambda^0(\rho)$. Let $\phi \in S(G)$ and $p \in H(G \times K')$. Then

$$(3.3)\quad \langle \chi_\rho, \phi\Psi(\lambda p(\cdot, \xi))\rangle = \int_{\Omega_\rho} [\phi\Psi(\lambda p(\cdot, \xi))]\hat{\ }(\nu)\, d\Omega(\nu)$$

$$= \int_{\Omega_\rho} \left\{ \int_{\mathfrak{g}^*} \phi(\exp(X))\Psi[\lambda p(\exp(X), \eta)]\Psi(\langle X, \nu\rangle)\, dK \right\} d\Omega(\nu)$$

$$= \int_{\Omega_\rho} \left\{ \int_{\mathfrak{g}^*} \phi(\exp(X))\Psi[\lambda p(\exp(X), \eta) + \langle X, \nu\rangle]\, dX \right\} d\Omega(\nu).$$

If we take $p(\exp(X), \xi) = \langle X, \xi\rangle$, (3.3) becomes

$$\int_{\Omega_\rho} \hat{\phi}(\nu + \lambda\xi)\, d\Omega(\nu).$$

Hence $\langle \chi_\rho, \phi\Psi(\lambda p(\cdot, \xi))\rangle$ has bounded support on Λ iff $\Omega_\rho + \lambda\xi$ is disjoint from $\operatorname{supp}(\hat{\phi})$ for $|\lambda|$ sufficiently large, $\lambda \in \Lambda$. Thus by Theorem 3.3, $\xi \notin WF_\Lambda^0(\rho)$ iff there is a neighborhood $U \subseteq \mathfrak{g}^*$ of 0 such that $\Omega_\rho + \lambda\xi \cap U = \varnothing$ for $|\lambda|$ sufficiently large.

EXAMPLE 3. Take G to be a unimodular semi-simple p-adic group. Let (ρ, H_ρ) be an irreducible admissible representation of G. Then χ_ρ is a locally summable function of G, and is locally constant on an open, dense subset G' of G. χ_ρ is not C^∞ at $g = e$ ($\notin G'$). However, there is a neighborhood V of 0 in $\mathfrak{g}$, and constants c_Ω such that

$$(3.4)\qquad \chi_\rho(\exp(X)) = \sum_\Omega c_\Omega \cdot \hat{\mu}_\Omega(X), \qquad X \in V,$$

the (finite) sum being over the nilpotent orbits in Ω in $\mathfrak{g}^*$ [**HC**, p. 180]. Here the locally summable functions $\hat{\mu}_\Omega$ are defined as follows [**R**]: Fix $X_0 \in \mathfrak{g}$ and let $\Omega_{X_0} = \{\operatorname{Ad}(x)X_0 \colon x \in G\}$. If G_{X_0} is the stabilizer of X_0, then G_{X_0} is unimodular and G/G_{X_0} has an invariant measure $d\Omega$, and we define

$$(3.5)\qquad \langle \mu_\Omega, \phi\rangle = \int_{G/G_{X_0}} \phi(\operatorname{Ad}(x)X_0)\, d\Omega(x), \qquad \phi \in S(\mathfrak{g}).$$

Note that we have identified $\mathfrak{g}$ with $\mathfrak{g}^*$ using a fixed non-degenerate, symmetric, G-invariant, bi-linear form $\langle , \rangle$.

Using (3.4) for χ_ρ on $W = \exp(V)$, if $\phi \in S(W)$, we have that for any $p \in H(G \times K^r)$,

$$\begin{aligned}\langle \chi_\rho, \phi\Psi(\lambda p(\cdot, \xi))\rangle &= \int_G \phi(x)\Psi(\lambda p(x, \xi))\chi_\rho(x)\, dx \\ &= \int_V \phi'(X)\Psi[\lambda p(\exp(X), \xi)]\chi_\rho(\exp(X))\, dx \\ &= \int_V \phi'(X)\Psi[\lambda p(\exp(X), \xi)]\Big\{\sum_\Omega c_\Omega \cdot \hat{\mu}_\Omega(X)\Big\}\, dX \\ &= \sum_{\Omega \text{ nilpotent}} c_\Omega \cdot \langle \hat{\mu}_\Omega, \phi'\Psi(\lambda p(\cdot, \xi))\rangle,\end{aligned}$$

where $\phi' \in S(V)$ and $p'(\cdot, \xi) = \exp^* p(\cdot, \xi)$. Thus by Prop. 2.4(iv), $WF^0_\Lambda(\chi_\rho)$ is contained in

$$\bigcup_{\Omega \text{ nilpotent}} WF^0_\Lambda(\hat{\mu}_\Omega),$$

where

$$WF^0_\Lambda(\hat{\mu}_\Omega) = \{\xi \in \mathfrak{g} : (e, \xi) \in WF_\Lambda(\hat{\mu}_\Omega)\}.$$

So let $\Omega \subseteq \mathfrak{g}$ be a nilpotent orbit generated by $X_0 \in \mathfrak{g}$. For $\phi' \in S(V)$, by (3.5),

$$\begin{aligned}(3.6)\quad \langle \hat{\mu}_\Omega, \phi'\Psi(\lambda p'(\cdot, \xi))\rangle &= \langle \mu_\Omega, [\phi'\Psi(\lambda p'(\cdot, \xi))]\hat{}\ \rangle \\ &= \int_{G/G_{X_0}} [\phi'\Psi(\lambda p'(\cdot, \xi))]\hat{}\ (\mathrm{Ad}(x)X_0)\, d\Omega(x) \\ &= \int_{G/G_{X_0}} \left[\int_{\mathfrak{g}} \phi'(Y)\Psi(\lambda p'(Y, \xi))\Psi(\langle \mathrm{Ad}(x)X_0, Y\rangle)\, dY\right] d\Omega(x).\end{aligned}$$

Letting $p'(Y, \xi) = \langle Y, \xi\rangle$, (3.6) becomes

$$\begin{aligned}&\int_{G/G_{X_0}} \left[\int \phi'(Y)\Psi(\langle \lambda\xi + \mathrm{Ad}(x)X_0, Y\rangle)\, dY\right] d\Omega(x) \\ &\qquad = \int_{G/G_{X_0}} \hat{\phi}'(\lambda\xi + \mathrm{Ad}(x)X_0)\, d\Omega(x).\end{aligned}$$

But this is 0 for large $|\lambda|$ iff $\lambda\xi + \Omega$ is disjoint from $\mathrm{supp}(\hat{\phi})$ for $|\lambda|$ sufficiently large. Thus by Theorem 3.3, $\xi \notin WF^0_\Lambda(\rho)$ iff there is an open neighborhood $U \subseteq \mathfrak{g}$ of 0 such that $N + \lambda\xi \cap U = \varnothing$ for $|\lambda|$ sufficiently large, where N is the set of nilpotent elements of $\mathfrak{g}$.

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