PACIFIC JOURNAL OF MATHEMATICS
Vol. 117, No. 1, 1985

GENERALIZED QUOTIENT MAPS THAT ARE INDUCTIVELY INDEX-σ-DISCRETE

R. W. HANSELL

E. Michael recently showed that a continuous quotient s-map between metrizable spaces can be contracted onto a G_δ-set so that the resulting map is index-σ-discrete; i.e., one that preserves σ-discretely decomposable families. Because of the potential utility of this result in descriptive set theory, we give a refinement that is less dependent upon the behavior of open sets under the map. Several types of generalized quotient maps are defined, not necessarily continuous, and we show that these are precisely the maps that are "inductively" index-σ-discrete under certain conditions similar to the above. The inter-relationships among these maps are also described. We further show that when the given map has a nice property (such as Borel measurability), then the restriction can be defined on a similarly nice subset of the domain. An application is made to maps that preserve analytic metric spaces; and additional applications to the existence of Borel measurable inverses will be given elsewhere.

1. Introduction and statements of principal results. The reader is referred to the paper of E. Michael [8] for background, general properties and detailed definitions of base-σ-discrete and index-σ-discrete maps. However, for convenience, we will briefly summarize the definitions here. Throughout, X and Y denote arbitrary topological spaces, and maps f: $X \to Y$ are *not* assumed to be continuous unless stated so. The power set of X is denoted $\mathscr{P}(X)$, and $\mathbf{N}$ denotes the set of natural numbers.

Let $\mathscr{B} \subset \mathscr{P}(X)$. We say that $\mathscr{B}$ is a *base* for a collection of sets $\mathscr{A}$, if each member of $\mathscr{A}$ is the union of a subcollection of $\mathscr{B}$. Recall that a base (not necessarily open) for the open sets of X is called a *network* for X. The collection $\mathscr{B}$ is *discrete* in X, if each point of X has a neighborhood meeting at most one member of $\mathscr{B}$; it is *σ-discrete*, if $\mathscr{B} = \bigcup_{n\in\mathbf{N}}\mathscr{B}_n$ and each $\mathscr{B}_n$ is discrete. A map f: $X \to Y$ is *base-σ-discrete* if the image of each discrete collection in X has a σ-discrete base in Y (see [**8**, §2] and [**2**, §3]). If $\mathscr{B} = \langle B_\lambda\rangle_{\lambda\in\Lambda}$, then $\mathscr{B}$ is *index-discrete* in X if each $x \in X$ has a neighborhood meeting B_λ for at most one λ (equivalently, $\mathscr{B}$ is discrete and $B_\lambda \cap B_{\lambda'} = \varnothing$ whenever $\lambda \neq \lambda'$; $\langle B_\lambda\rangle_{\lambda\in\Lambda}$ is *σ-discretely decomposable* if there are index-discrete families $\langle B_\lambda^n\rangle_\lambda$, for each $n \in \mathbf{N}$, such that

$B_\lambda = \bigcup_{n \in \mathbb{N}} B_\lambda^n$ for each $\lambda \in \Lambda$. Michael has shown that $\mathscr{B} = \langle B_\lambda \rangle_\lambda$ is σ-discretely decomposable if, and only if, $\langle B_\lambda \rangle_\lambda$ is index-point-countable[1] and $\mathscr{B}$ has σ-discrete base [**8**, Lemma 3.2.]. A map $f: X \to Y$ is *index-σ-discrete* if $\langle f(B_\lambda) \rangle_\lambda$ is σ-discretely decomposable in Y whenever $\langle B_\lambda \rangle_\lambda$ is index-discrete in X (see [**8**, §3]). We will also say that a map $f: X \to Y$ *has a σ-discrete base* if the collection of all images of open sets in X has a σ-discrete base in Y. (Note that any open map into a metrizable space has a σ-discrete base, but such need not be base-σ-discrete, even if the domain is metrizable [**2**, 3.12].)

Remark. Michael has shown that if $f: X \to Y$ is base-σ-discrete and has $\aleph_1$-compact[2] fibers, then f is index-σ-discrete; and that the converse holds whenever the fibers of f are closed [**8**, Proposition 4.3]. The assumption that a map has closed fibers is a relatively mild one from the standpoint of General Topology, but for Descriptive Set Theory it is very restrictive (maps of the first Baire class usually do not have closed fibers). However, if X is any space in which relatively discrete subsets are σ-discrete sets in X (for example, if X is a σ-space[3] or open sets in X are F_σ-sets), then the above equivalence holds (using the same proof) without assuming the fibers of the map are closed.[4]

A map $f: X \to Y$ is said to have a property *inductively* if there is a subset X' of X such that $f(X') = f(X)$ and $f|X'$ has the given property. The principal result in [**8**] is the following:

Theorem ([**8**, *Theorem* 6.1]). *If $f: X \to Y$ is a quotient s-map,[5] with X and Y metrizable, then f is inductively index-σ-discrete. Moreover, there is a G_δ-subset X' of X such that $f(X') = Y$ and $f|X'$ is index-σ-discrete.* (*Quotient maps are continuous surjections by assumption.*)

This result has considerable potential in the study of nonseparable descriptive set theory, particularly to the preservation of analytic sets under mappings and the existence of measurable selections. However, for

---

[1] $\langle B_\lambda \rangle_\lambda$ is *index-point-countable* if $\{\lambda: x \in B_\lambda\}$ is at most countable for each x.

[2] A space is called *$\aleph_1$-compact* if every closed discrete subset is at most countable.

[3] A *σ-space* is any space with a σ-discrete network (no separation axiom is assumed). These are precisely the base-σ-discrete continuous images of metrizable spaces (cf. [**8**, Theorem 8.1]).

[4] Similarly, we can prove that, if X is a metalindelöf (see §3) σ-space, then f is index-σ-discrete if, and only if, f has a σ-discrete base and $\aleph_1$-compact fibers.

[5] An *s-map* is a map with separable fibers

these applications, the assumptions of the theorem are much too restrictive in its present form. Indeed, since base-σ-discrete maps have little to do with continuity, let alone quotient maps, one would expect a result of this type to hold under more general assumptions.

Here we introduce generalizations of several types of quotient maps (not necessarily continuous), each of which turns out to be both a necessary and sufficient condition for certain maps to be inductively index-σ-discrete. We call these maps, "σ-quotient", "σ-bi-quotient", etc., to suggest that they generalize quotient maps onto metrizable spaces in a manner similar to the way σ-spaces generalize metrizable ones (e.g., f: $X \to Y$ is a σ-quotient map if $\{E \subset f(X)\colon f^{-1}(E)$ is open in $X\}$ has a σ-discrete base in Y). These maps are formally defined and discussed in §3. In §4 we prove the following theorem which gives a sharp form of the first part of Michael's theorem above.

THEOREM 1. *Let f: $X \to Y$ be a map having Lindelöf fibers, with X a metalindelöf σ-space. Then the following are equivalent.*

(a) *f is σ-bi-quotient.*

(b) *f is σ-almost-open.*

(c) *f is inductively index-σ-discrete.*

Moreover, if the quotient space X/f is Hausdorff, then the above are also equivalent to

(d) *f is σ-quotient.*

In §5 we define the notion of a "locally finitely additive" lattice of sets $\mathscr{L} \subset \mathscr{P}(X)$, and we prove that the descriptive classes $\Sigma_\alpha \mathscr{L}$, $\Pi_\alpha \mathscr{L}$, and Souslin $\mathscr{L}$ (defined in §5) will have this property whenever $\mathscr{L}$ does, provided $\mathscr{L}$ contains the closed sets of X. This is used to prove (in §5) the following generalization of the second part of Michael's theorem above.

THEOREM 2. *Let f: $X \to Y$ be inductively base-σ-discrete, with X a pseudometrizable space. Let $\mathscr{L}$ be a locally finitely additive lattice of subsets of X. Then the following hold.*

(a) *Suppose Y is a metalindelöf σ-space and $\mathscr{L}$ contains the open sets of X. If $f^{-1}(V) \in \mathscr{L}$ for every open V in Y, then there exists an $X^* \in \mathscr{L}_\delta$ such that $f(X^*) = f(X)$ and $f|X^*$ is index-σ-discrete.*

(b) *Suppopse $\mathscr{L}$ contains the closed sets of X. If $f^{-1}(F) \in \mathscr{L}$ for every closed F in Y, then there exists an $X^* \in \mathscr{L}_{\sigma\delta}$ such that $f(X^*) = f(X)$ and $f|X^*$ is index-σ-discrete.*

Note, for example, that Theorem 2 implies that if $f\colon X \to Y$ is of class α, say for metrizable spaces X and Y, and f is inductively base-σ-discrete, then there exists an X^* of multiplicative class $\alpha + 1$ in X with the above properties (see Corollary 5.3).

In §6 we given an application of the above theorems to the study of analytic metric spaces. Previously, it was shown that Souslin measurable base-σ-discrete maps and continuous almost-open s-maps preserve (non-separable) analytic spaces within the class of metric spaces (see [**2**] and [**5**]). For s-maps, each of these is a special case of the following result, in view of Theorems 1 and 2.

THEOREM 3. *Let $f\colon X \to Y$ be a Souslin measurable s-map from an analytic metric space X onto a metric space Y. If f is σ-bi-quotient, then Y is also analytic.*

Additional applications of Theorems 1 and 2 are given in [**4**].

Some examples are given to show that the assumptions on the fibers of the maps in most of the results cannot be weakened (see Examples 3.18, 3.19 and 6.4). However, we have been unable to determine exact conditions under which all σ-quotient maps are hereditarily σ-quotient, and all hereditarily σ-quotient maps are σ-bi-quotient (see 3.16 and 3.17 for partial solutions which cover the case for continuous maps).

2. Two lemmas on base-σ-discrete maps. The following lemma gives a useful sufficient condition for a map to be index-σ-discrete (it is clearly also necessary when X is a σ-space—cf. [**8**, Corollary 4.6]).

2.1. LEMMA. *For a given map $f\colon X \to Y$, if X has a network $\mathscr{B}$ such that $\{f(B)\colon B \in \mathscr{B}\}$ is σ-discretely decomposable in Y, then f is index-σ-discrete.*

Proof. Assume the hypothesis, and let $\mathscr{E} \subset \mathscr{P}(X)$ be a discrete family in X. Let $\mathscr{B}^*$ denote the family of all $B \in \mathscr{B}$ for which there is an $E_B \in \mathscr{E}$ satisfying $\{E_B\} = \{E \in \mathscr{E}\colon E \cap B \neq \varnothing\}$. Then $\{f(B \cap E_B)\colon B \in \mathscr{B}^*\}$ is σ-discretely decomposable, and it suffices to show that

$$f(E) = \bigcup\{f(B \cap E_B)\colon B \in \mathscr{B}^* \text{ and } E_B = E\}$$

for each $E \in \mathscr{E}$. But this follows easily from the fact that $\mathscr{B}$ is a network for X and $\mathscr{E}$ is discrete.

REMARK. If in the hypothesis of Lemma 2.1 we assume only that $\{f(B)\colon B \in \mathscr{B}\}$ has a σ-discrete base, we cannot conclude that f is base-σ-discrete, since there exist open maps between metrizable spaces that are not base-σ-discrete [**2**, example 3.12].

2.2. LEMMA. *Let X be a σ-space or a space in which open sets are F_σ-sets. If $f\colon X \to Y$ is base-σ-discrete, then so is $f|E$ for any $E \subset X$. In particular, f is inductively index-σ-discrete.*

Proof. The assumptions on X imply that any relatively discrete family of subsets of X has a σ-discrete base in X. For suppose $\mathscr{E} \subset \mathscr{P}(X)$ is discrete relative to the subspace $\bigcup \mathscr{E}$. For each $E \in \mathscr{E}$ we can choose an open set $U_E \supset E$ in X such that $U_E \cap E' = \varnothing$ for all E' in $\mathscr{E} - \{E\}$. If $\bigcup\{U_E\colon E \in \mathscr{E}\} = \bigcup_{n=1}^\infty F_n$, where each F_n is closed in X, then clearly each of the families $\{E \cap F_n\colon E \in \mathscr{E}\}$ is discrete in X. On the other hand, if $\bigcup_{n=1}^\infty \mathscr{B}_n$ is a σ-discrete network for X, for each $E \in \mathscr{E}$ and $n = 1, 2, \ldots$, define

$$E_n = \bigcup\{E \cap B\colon B \in \mathscr{B}_n \text{ and } B \subset U_E\}.$$

Then one easily observes that $\{E_n\colon E \in \mathscr{E}\}$ is discrete in X for each n, and the union of these families forms a base for $\mathscr{E}$. Since a map is base-σ-discrete if, and only if it preserves families having a σ-discrete base [**2**, Proposition 3.2], it follows that $f|E$ is base-σ-discrete for any $E \subset X$. Choosing E so that $f|E$ is one-to-one and $f(E) = f(X)$, shows that f is inductively index-σ-discrete. □

3. σ-quotient and related maps. If $f\colon X \to Y$ is a map, by a *quotient base* for f we mean any base for the collection

$$\{E \subset f(X) : f^{-1}(E) \text{ is open in } X\}.$$

If $\mathscr{B} \subset \mathscr{P}(Y)$ is a quotient base for f we will sometimes say that f is *$\mathscr{B}$-quotient*. Note that a surjective map is quotient in the usual sense if it has a quotient base of open sets (and is continuous). Also note that if f is a continuous quotient map, then $\mathscr{B}$ is a quotient base for f if, and only if, $\mathscr{B}$ is a network for the range of f. This establishes part of the following lemma, the routine proof of which we omit. (nat(x) will denote the equivalence class in a quotient space X/f containing the point $x \in X$.)

3.1. LEMMA. *For a given map $f\colon X \to Y$, let $f = \tilde{f} \circ$* nat *be the standard factorization of f through X/f, the quotient space of f. Then the following are equivalent for any $\mathscr{B} \subset \mathscr{P}(Y)$.*

(a) *$\mathscr{B}$ is a quotient base for f.*

(b) $\mathscr{B}$ *is a base for* $\tilde{f}$ (*hence also a quotient base*).

(c) $\tilde{f}^{-1}(\mathscr{B})$ *is a network for* X/f.

(d) $\tilde{f}^{-1}(\mathscr{B})$ *is a quotient base for* nat. □

3.2. DEFINITION. A map f: $X \to Y$ is said to be *σ-quotient* if it has a quotient base which is σ-discrete in Y.

REMARK. We do not require that a σ-quotient map be either continuous or surjective as is usually the case with quotient maps. If Y is a σ-space, or if open sets in Y are F_σ-sets, then any relatively discrete family of subsets of Y has a σ-discrete base in Y (see the proof of Lemma 2.2 above), and so any f: $X \to Y$ is σ-quotient if, and only if, f: $X \to f(X)$ is σ-quotient. Note that similar remarks apply to maps that are base-σ-discrete, index-σ-discrete, etc.

The terminology "σ-quotient" seems appropriate in view of the following.

3.3. PROPOSITION. *Each of the following implies that the map* f: $X \to Y$ *is σ-quotient. Conversely*, (d) *holds whenever f is σ-quotient, and* (e) *will hold whenever f is continuous and σ-quotient.*

(a) X *has a countable network.*

(b) f *inductively has a σ-discrete base.*

(c) f *is a quotient map* (*not necessarily continuous*) *and* Y *is a σ-space.*

(d) $\tilde{f}$: $X/f \to Y$ *has a σ-discrete base.*

(e) X/f *is a σ-space and* $\tilde{f}$ *is base-σ-discrete.*

Proof. (a) The image of any countable network for X is clearly a σ-discrete quotient base for f.

(b) Let $X' \subset X$ be such that $f(X') = f(X)$ and $\{f(U \cap X')$: U open in $X\}$ has a σ-discrete base $\mathscr{B}$ in Y. Thus, for any $x \in f^{-1}(y) \cap X'$, if $x \in U$ and U is open in X, then $y \in B \subset f(U \cap X')$ for some $B \in \mathscr{B}$. This evidently implies that f is σ-quotient with quotient base $\mathscr{B}$.

(c) If f is quotient, then $\{E \subset Y$: $f^{-1}(E)$ is open$\}$ is a family of open sets in Y, and thus has a σ-discrete base whenever Y is a σ-space.

(d) That f is σ-quotient precisely when (d) holds follows immediately from (a) ↔ (b) of Lemma 3.1.

(e) If $\mathscr{B}$ is a σ-discrete network for X/f and $\tilde{f}$ is base-σ-discrete, then $\tilde{f}(\mathscr{B})$ is a σ-discrete base for $\tilde{f}$, and hence (d) holds. Conversely, suppose f is continuous and $\mathscr{B}$ is a σ-discrete quotient base for f. Since $\tilde{f}$ is continuous, $\tilde{f}^{-1}(\mathscr{B})$ is a σ-discrete family in X/f, and will be a network for X/f by (a) $\leftrightarrow$ (c) of Lemma 3.1; hence X/f is a σ-space. Since we may assume that $\mathscr{B} \subset \mathscr{P}(f(X))$, $\tilde{f}(\tilde{f}^{-1}(\mathscr{B})) = \mathscr{B}$ and it follows from Lemma 2.1 that $\tilde{f}$ is index-σ-discrete. □

A quotient base $\mathscr{B}$ for $f\colon X \to Y$ will be called a *bi-quotient base* if, whenever $y \in f(X)$ and $\mathscr{U}$ is an open cover of $f^{-1}(y)$ in X, then there exist finite subfamilies $\mathscr{U}_y \subset \mathscr{U}$ and $\mathscr{B}_y \subset \mathscr{B}$ such that $y \in \bigcap \mathscr{B}_y \subset \bigcup f(\mathscr{U}_y)$. In this case we will say that f is *$\mathscr{B}$-bi-quotient*. A continuous surjection is a bi-quotient map in the usual sense [9] if it has an open bi-quotient base.

3.4. LEMMA. *Let $f = \tilde{f} \circ \mathrm{nat}$ be the usual factorization of $f\colon X \to Y$ through X/f. Then $\mathscr{B} \subset \mathscr{P}(Y)$ is a bi-quotient base for f if, and only if, $\tilde{f}^{-1}(\mathscr{B})$ is a bi-quotient base for* nat. *Consequently, if* nat *is a bi-quotient map, then any quotient base for f is a bi-quotient base.*

Proof. Let $z \in X/f$ and let $y = \tilde{f}(z)$. Suppose $\mathscr{U}$ is an open cover of $f^{-1}(y) = \mathrm{nat}^{-1}(z)$. If $\mathscr{B}$ is a bi-quotient base for f, then there are finite subfamilies $\mathscr{B}_y \subset \mathscr{B}$ and $\mathscr{U}_y \subset \mathscr{U}$ with $y \in \bigcap \mathscr{B}_y \subset \bigcup f(\mathscr{U}_y)$. Since for any $U \subset X$, $\tilde{f}^{-1}(f(U)) = \mathrm{nat}(U)$, it follows that $z \in \bigcap \tilde{f}^{-1}(\mathscr{B}_y) \subset \bigcup \mathrm{nat}(\mathscr{U}_y)$. This shows that $\tilde{f}^{-1}(\mathscr{B})$ is a bi-quotient base for nat. The converse is proved similarly.

Now suppose that nat is bi-quotient and $\mathscr{C}$ is any quotient base for f. Let $\mathscr{B} = \{V \subset f(X)\colon f^{-1}(V) \text{ is open in } X\}$. Then $\tilde{f}^{-1}(\mathscr{B})$ is the topology of X/f, and so is a bi-quotient base for nat. Thus $\mathscr{B}$ is a bi-quotient base for f by the above. Since $\mathscr{C}$ is a base for $\mathscr{B}$, it easily follows that $\mathscr{C}$ is a bi-quotient base for f. □

REMARK. The fundamental properties of bi-quotient maps given in [9] (in particular, Theorems 1.2 and 1.3) do not depend on the maps being continuous as is assumed in the introduction of [9]. Thus we will not assume here that bi-quotient maps are necessarily continuous. This enables us to use the following convenient way of showing that a $\mathscr{B}$-quotient map is $\mathscr{B}$-bi-quotient.

3.5. LEMMA. *Suppose $f\colon X \to Y$ is $\mathscr{B}$-quotient for some $\mathscr{B} \subset \mathscr{P}(f(X))$, and let τ denote the topology on $f(X)$ having $\mathscr{B}$ as a subbase. Then f is $\mathscr{B}$-bi-quotient if, and only if, $f\colon X \to (f(X), \tau)$ is bi-quotient.*

Proof. The proof is an immediate consequence of the definitions. □

REMARK. If $\mathscr{B}$ is a quotient base for f and $\mathscr{B}^*$ denotes the family of all finite intersections of members of $\mathscr{B}$, note that $\mathscr{B}$ is a bi-quotient base for

f if, and only if, $\mathscr{B}^*$ is. Thus when showing that $\mathscr{B}$ is a bi-quotient base we may assume $\mathscr{B}$ is closed to finite intersections. In particular, in 3.5 we may assume that the topology τ has $\mathscr{B}$ as a base.

3.6. Definition. A map f: $X \to Y$ is σ-bi-quotient if it has a bi-quotient base which is σ-discrete in Y.

Using Lemma 3..5 we can easily show that $\mathscr{B}$-bi-quotient maps retain most of the familiar properties of bi-quotient maps; and, in particular, that σ-bi-quotient maps could reasonably be called "bi-σ-quotient" maps (see 3.8 below).

3.7. Proposition. *Let f_α: $X_\alpha \to Y_\alpha$ be $\mathscr{B}_\alpha$-bi-quotient for each $\alpha \in A$. Let $\mathscr{B}$ be the family of all subsets of $\prod_\alpha Y_\alpha$ of the form*

$$\pi_{\alpha_1}^{-1}(B_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(B_n),$$

where $\alpha_i \in A$ and $B_i \in \mathscr{B}_{\alpha_i}$ for $i = 1,\ldots,n$, and $n \in \mathbf{N}$. Then the product map $\prod_\alpha f_\alpha$ is $\mathscr{B}$-bi-quotient for the product topologies.

Proof. We may suppose that $Y_\alpha = f(X_\alpha)$ for each α. Let τ_α be the topology on Y_α generated by $\mathscr{B}_\alpha$, for each $\alpha \in A$, and note that the product topology for $\prod_\alpha (Y_\alpha, \tau_\alpha)$ is precisely the topology τ generated by the family $\mathscr{B}$. Since each f_α: $X_\alpha \to (Y_\alpha, \tau_\alpha)$ is bi-quotient by Lemma 3.5, the product map $\prod_\alpha f$ from $\prod_\alpha X_\alpha$ to the space $(\prod_\alpha Y_\alpha, \tau)$ is bi-quotient by [**9**, Theorem 1.2]. Another application of Lemma 3.5 gives the desired result. □

3.8. Proposition. *A countable product of σ-bi-quotient maps is σ-bi-quotient.*

Proof. Suppose in the statement of Proposition 3.7 we let $A = \mathbf{N}$, and assume that $\mathscr{B}_n$ is a σ-discrete bi-quotient base for f_n, for each $n \in \mathbf{N}$. Since the projection maps are continuous, it follows that the family $\mathscr{B}$ of 3.7 is σ-discrete, and thus the product map is σ-bi-quotient. □

3.9. Proposition. *Let $\mathscr{B}$ be a quotient base for f: $X \to Y$, with Y Hausdorff. Then $\mathscr{B}$ is a bi-quotient base for f if, and only if,*

$$\mathscr{C} = \{B \times V\colon B \in \mathscr{B} \text{ and } V \text{ is open in } Z\}$$

is a quotient base for $f \times 1_Z$ for every space Z.

Proof. If f is $\mathscr{B}$-bi-quotient, then $\mathscr{C}$ is a bi-quotient base for $f \times 1_Z$ by 3.7.

Suppose $f \times 1_Z$ is $\mathscr{C}$-quotient for any space Z. Let Y^* denote the space $(f(X), \tau)$ where τ is the topology generated by $\mathscr{B}$. To show that f is $\mathscr{B}$-bi-quotient it suffices to show that, for any space Z,

$$f \times 1_Z\colon X \times Z \to Y^* \times Z$$

is a quotient map. For then it follows that $f\colon X \to Y^*$ is bi-quotient by [**9**, Theorem 1.3], and thus f is $\mathscr{B}$-bi-quotient by 3.5. But if $(f \times 1_Z)^{-1}(W)$ is open for some $W \subset f(X) \times Z$, then W is a union of sets from $\mathscr{C}$ (since $\mathscr{C}$ is a quotient base), and hence W is open in $Y^* \times Z$. □

3.10. Proposition. *Let $f\colon X \to Y$ be a bi-quotient map, $g\colon Y \to Z$ any map, and $\mathscr{B} \subset \mathscr{P}(g(Y))$.*

(a) *If g is $\mathscr{B}$-bi-quotient, then so is $g \circ f$.*

(b) *If f is continuous and $g \circ f$ is $\mathscr{B}$-bi-quotient, then g is $\mathscr{B}$-bi-quotient.*

(c) *If g is $\mathscr{B}$-bi-quotient and $M \subset Z$, then $\{B \cap M\colon B \in \mathscr{B}\}$ is a bi-quotient base for $g|g^{-1}(M)$.*

(d) *If there is a $Y' \subset Y$ such that $g(Y') = g(Y)$ and $g|Y'$ is $\mathscr{B}$-bi-quotient, then g is $\mathscr{B}$-bi-quotient.*

Proof. (a) Suppose $(g \circ f)^{-1}(z)$ is covered by the open family $\mathscr{U}$ in X, for a given z in Z. Then, for each y in $g^{-1}(z)$, $f^{-1}(y)$ is covered by $\mathscr{U}$, and so there exists an open V_y in Y and a finite $\mathscr{U}_y \subset \mathscr{U}$ such that $y \in V_y \subset \bigcup f(\mathscr{U}_y)$ (since f is bi-quotient). Since g is $\mathscr{B}$-bi-quotient and $\mathscr{V} = \{V_y\colon y \in g^{-1}(z)\}$ is an open cover of $g^{-1}(z)$, there exist finite subfamilies $\mathscr{B}_z \subset \mathscr{B}$ and $\mathscr{V}_z \subset \mathscr{V}$ such that $z \in \bigcap \mathscr{B}_z \subset \bigcup g(\mathscr{V}_z)$. Letting $\mathscr{U}(z)$ denote the union of all $\mathscr{U}_y$ such that $V_y \in \mathscr{V}_z$ we see that $\mathscr{U}(z)$ is finite and $z \in \bigcap \mathscr{B}_z \subset \bigcup (g \circ f)(\mathscr{U}(z))$. This shows that $g \circ f$ is $\mathscr{B}$-bi-quotient.

(b) Suppose $\mathscr{V}$ is an open cover of $g^{-1}(z)$ in Y. Since f is continuous, $f^{-1}(\mathscr{V})$ is an open cover of $(g \circ f)^{-1}(z)$. Applying the fact that $g \circ f$ is $\mathscr{B}$-bi-quotient immediately shows that g is $\mathscr{B}$-bi-quotient.

(c) This is routinely verified using the definitions.

(d) This follows immediately from the definitions. □

Our next result is similar to Lemmas 3.4 and 3.5 in that it allows us to show that a given map is generalized bi-quotient by showing that some other map with additional properties has a suitable bi-quotient base.

3.11. Proposition. *Let $f\colon X \to Y$ be a map, and let p denote the restriction of the projection map $X \times Y \to Y$ to the graph of f.*

(a) *If p is $\mathscr{B}$-bi-quotient, then so is f.*

(b) *If f is $\mathscr{B}$-bi-quotient, then $\{B \cap V\colon B \in \mathscr{B};\ V \text{ open in } Y\}$ is a bi-quotient base for p whenever Y is Hausdorff.*

Proof. (a) Suppose $y \in f(X)$, and $\mathscr{U}$ is an open cover of $f^{-1}(y)$ in X. For each $U \in \mathscr{U}$, let W_U be the intersection of $U \times Y$ with the graph of f, and note that $p(W_U) = f(U)$. Since $\{W_U: U \in \mathscr{U}\}$ is an open cover of $p^{-1}(y)$ relative to the graph of f, there exists finite subfamilies $\mathscr{U}_y \subset \mathscr{U}$ and $\mathscr{B}_y \subset \mathscr{B}$ satisfying

$$y \in \bigcap \mathscr{B}_y \subset \bigcup p(\{W_U: U \in \mathscr{U}_y\}) = \bigcup f(\mathscr{U}_y).$$

Thus f is $\mathscr{B}$-bi-quotient.

(b) Suppose Y is Hausdorff and f is $\mathscr{B}$-bi-quotient. By 3.9, the sets $B \times V$, where $B \in \mathscr{B}$ and V is open in Y, form a bi-quotient base for $f \times 1_Y$. Let Δ denote the diagonal of $Y \times Y$, and let $h: \Delta \to Y$ be the map taking (y, y) to y. Then $p = h \circ (f \times 1_Y)$. Since $h(B \times V) = B \cap V$ for any subsets B and V of Y, one now routinely verifies that $\{B \cap V: B \in \mathscr{B}, V \text{ open in } Y\}$ is a bi-quotient base for p. □

3.12. Corollary. *Let f and p be as in* 3.11. *Then f is σ-bi-quotient whenever p is, and p is σ-bi-quotient whenever f is and Y is a Hausdorff σ-space.*

Proof. That f is σ-bi-quotient whenever p is follows directly from 3.11 part (a). Conversely, suppose $\mathscr{B}$ is a σ-discrete bi-quotient base for f, and let $\mathscr{C}$ be a σ-discrete network for Y. Then $\{B \cap C: B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$ is a σ-discrete base for $\{B \cap V: B \in \mathscr{B} \text{ and } V \text{ open in } Y\}$. Thus, the former collection will be a σ-discrete bi-quotient base for p, when Y is Hausdorff, by 3.11 part (b). □

3.13. Definitions. A map $f: X \to Y$ is *hereditarily $\mathscr{B}$-quotient* if, whenever $E \subset f(X)$, then $f_E: f^{-1}(E) \to E$ is $\mathscr{B} \cap E$-quotient, where $f_E = f|f^{-1}(E)$ and $\mathscr{B} \cap E = \{B \cap E: B \in \mathscr{B}\}$; f is said to be *$\mathscr{B}$-pseudo-open* if, whenever $f^{-1}(y) \subset U$ with U open in X and $y \in f(X)$, then $y \in B \subset f(U)$ for some $B \in \mathscr{B}$. (Here, and in the following, unless otherwise stated, $\mathscr{B}$ is an arbitrary collection of subsets of $f(X)$.)

As with quotient maps (cf. [**0**, Theorem 1]), we have the following.

3.14. Proposition. *A map $f: X \to Y$ is hereditarily $\mathscr{B}$-quotient if, and only if, it is $\mathscr{B}$-pseudo-open.*

Proof. Assume f is hereditarily $\mathscr{B}$-quotient, and let $f^{-1}(y) \subset U$ with U open in X and $y \in f(X)$. Suppose for each $B \in \mathscr{B}$ with $y \in B$ there is a $y_B \in B - f(U)$. Let $E = \{y_B: B \in \mathscr{B}\} \cup \{y\}$. Since $f^{-1}(y) = U \cap f^{-1}(E)$, $f^{-1}(y)$ is open relative to $f^{-1}(E)$. Since f_E is $\mathscr{B} \cap E$-quotient by assumptions, it follows that $B \cap E \subset \{y\}$ for some $B \in \mathscr{B}$. This contradicts the fact that $y_B \notin f(U)$.

Now assume f is $\mathscr{B}$-pseudo-open, and let $E \subset f(X)$. Suppose $y \in M \subset E$, and $f^{-1}(M)$ is open relative to $f^{-1}(E)$, say $f^{-1}(M) = U \cap f^{-1}(E)$ with U open in X. Then $y \in B \subset f(U)$, for some B in $\mathscr{B}$, and so $y \in B \cap E \subset f(U) \cap E = M$. □

REMARK. It is clear that every $\mathscr{B}$-pseudo-open map is $\mathscr{B}$-quotient, and every $\mathscr{B}$-bi-quotient map will be $\mathscr{B}$-pseudo-open whenever $\mathscr{B}$ is closed to finite intersections. Our next results give conditions under which the converses are true. For continuous quotient maps, the principal results of this type can be found in [**9**, **10**, **14**, **17**]. Since we are mainly interested in the case when $\mathscr{B}$ is a σ-discrete family, and when maps are not necessarily continuous, we consider conditions on $\mathscr{B}$ and the map with this in mind. In view of the latter, it is somewhat disappointing that 3.16 below necessitates conditions on the fibers and the graph of the map closely related to continuity (see examples 3.18 and 3.19 below). The results here, however, leave open the question of what general conditions imply that σ-quotient maps are σ-pseudo-open, and that σ-pseudo-open maps are σ-bi-quotient (under possibly different quotient bases). A map is *σ-pseudo-open* if it is $\mathscr{B}$-pseudo-open for some σ-discrete family $\mathscr{B}$. We first state a useful preliminary result, the routine proof of which we omit.

3.15. PROPOSITION. *For each of the properties $\mathscr{B}$-quotient, $\mathscr{B}$-pseudo-open and $\mathscr{B}$-bi-quotient, $f\colon X \to Y$ has the property if, and only if, it does so inductively.* (*Compare* 3.10(d).)

3.16. PROPOSITION. *Let $f\colon X \to Y$ be such that $f^{-1}(y) \cap \partial f^{-1}(y)$ is a closed Lindelöf subset of X for each $y \in Y$. Let $\mathscr{B}$ be a point-countable network for $f(X)$.*

(a) *If f is $\mathscr{B}$-pseudo-open, then f is $\mathscr{B}$-bi-quotient.*

(b) *If f has a closed graph in $X \times Y$ and f is $\mathscr{B}$-quotient, then f is $\mathscr{B}$-bi-quotient.*

Proof. For $y \in f(X)$, if $f^{-1}(y) \cap \partial f^{-1}(y) = \varnothing$, then $f^{-1}(y)$ is open, and so (in either case) we must have $\{y\} \in \mathscr{B}$; hence f is $\mathscr{B}$-bi-quotient for y. We may thus assume that $f(X') = f(X)$, where X' is the union of the sets $f^{-1}(y) \cap \partial f^{-1}(y)$. Since, by 3.15, f is $\mathscr{B}$-bi-quotient whenever $f|X'$ is, we may as well assume that the fibers of f are closed Lindelöf subsets of X. We may also assume that $\mathscr{B}$ is closed to finite intersections (see the remark ater 3.5).

Now fix $y \in f(X)$, let $\{B'_n\}_{n\in\mathbf{N}}$ be an unumeration of $\{B \in \mathscr{B}: y \in B\}$, and put $B_n = B'_1 \cap B'_2 \cap \cdots \cap B'_n$ for each n. Since f has Lindelöf fibers, it is enough to prove that if $\{U_n\}_{n\in\mathbf{N}}$ is an increasing sequence of open sets in X that cover $f^{-1}(y)$, then $B_n \subset f(U_n)$ for some n. Assume the contrary, and for each $n \in \mathbf{N}$ let $y_n \in B_n - f(U_n)$, so that $y_n \neq y$ (since we may assume that each U_n meets $f^{-1}(y)$). Since $\{B_n\}_{n\in\mathbf{N}}$ is a local network at y, the sequence $\{y_n\}$ must converge to y. Put $M = \{y_n: n \in \mathbf{N}\}$.

If f is $\mathscr{B}$-pseudo-open, there is an $x \in f^{-1}(y) \cap \overline{f^{-1}(M)}$, since otherwise we must have $B_n \subset f(X - \overline{f^{-1}(M)}) \subset Y - M$, for some n, and this contradicts our choice of y_n. As $x \in U_n$ for some n, it follows that $U_n - \bigcup_{i=1}^n f^{-1}(y_i)$ is a neighborhood of x not meeting $f^{-1}(M)$. This contradiction proves the result in case (a).

Under the assumption of (b), there is an x in $\overline{f^{-1}(M)} - f^{-1}(M)$. For $f^{-1}(M)$ cannot be closed in X, since $Y - M$ contains y but no B_n so $f^{-1}(Y - M)$ cannot be open as f is $\mathscr{B}$-quotient. Since the fibers of f are closed, each neighborhood of x meets $f^{-1}(y_n)$ for infinitely many n, and it follows that (x, y) is in the closure of $\bigcup_{n=1}^\infty f^{-1}(y_n) \times \{y_n\}$ relative to $X \times Y$. Since the graph of f is closed in $X \times Y$, we must have $x \in f^{-1}(y)$, and the proof for the case (a) above can now be used. □

Most of the above argument is a familiar one and goes back to a result of A. H. Stone [**16**, Lemma 1, p. 694].

3.17. Corollary. *Let $f: X \to Y$ be σ-quotient and such that X/f is Haudorff. If $\partial f^{-1}(y)$ is Lindelöf for each $y \in f(X)$, then f is σ-bi-quotient.*

Proof. Let $\mathscr{B} \subset \mathscr{P}(f(X))$ be a σ-discrete quotient base for f. We may assume that $\mathscr{B}$ is closed to finite intersections. Using Lemma 3.4, and replacing f with nat, Y with Y/f, and $\mathscr{B}$ with $\tilde{f}^{-1}(\mathscr{B})$, we may further assume that $f: X \to Y$ is a continuous quotient map, with Y Hausdorff. $\partial f^{-1}(y)$ is Lindelöf for each $y \in Y$, and $\mathscr{B}$ is a network for Y (by 3.1(c)). The conclusion now follows from 3.16. □

The following two examples show that (i) the assumption on the fibers of f in 3.16(a) cannot be omitted, and (ii) the assumption that f has a closed graph in 3.16(b) cannot be omitted.

3.18. Example. Let $Y = \{0\} \cup \{1/n: n \in \mathbf{N}\}$ and let $X = \mathbf{N} \times Y$, both having their usual topology. Define $f: X \to Y$ by $f(n, y) = 0$ if

$y = 0$, and $f(n, y) = 1/n$ otherwise. One easily verifies that f is $\mathscr{B}$-pseudo-open, where $\mathscr{B}$ is the topology of Y. However, since $\mathscr{U} = \{\{n\} \times Y: n \in \mathbf{N}\}$ is an open cover of X, but no finite subcollection of $f(\mathscr{U})$ covers a neighborhood of 0, f is not $\mathscr{B}$-bi-quotient. □

3.19. EXAMPLE. Let $X = A \times B$, with $B = \{0\} \cup \{1/n: n \in \mathbf{N}\}$ and $A = \{-1\} \cup B$, and let $Y = B \cup \{-n: n \in \mathbf{N}\}$, both having their usual topology. Let $X_0 = X - \{(0,0)\}$ and define $f: X_0 \to Y$ by letting, $f^{-1}(0) = \{(-1,0)\}$, $f^{-1}(1/n) = \{1/n\} \times B$ and $f^{-1}(-n) = \{(-1, 1/n), (0, 1/n)\}$ for each $n \in \mathbf{N}$. Clearly f has closed fibers. One easily verifies that f is a (noncontinuous) quotient map. However, since $U = \{-1\} \times B$ is open and contains $f^{-1}(0)$, but $f(U)$ contains no neighborhood of 0, f is not pseudo-open. Note that $((0, 1/n), 0)$ is in the closure of the graph of f for each n. □

The following result is to be compared with [**9**, Propostion 3.2(b)]. Since the proof is very similar to (but easier than) that of 3.16 above we omit it.

3.20. PROPOSITION. *Let $f: X \to Y$ be such that $f^{-1}(y) \cap \partial f^{-1}(y)$ is a closed compact subset of X for each $y \in Y$. If f is hereditarily $\mathscr{B}$-quotient, then f is $\mathscr{B}$-bi-quotient.* □

Recall that a map $f: X \to Y$ is said to be *almost-open* if for each $y \in Y$ there is an $x_y \in f^{-1}(y)$ such that, whenever U is a neighborhood of x_y, y belongs to the interior of $f(U)$ [**14**, Proposition 1.3]. A map $f: X \to Y$ is said to be *$\mathscr{B}$-almost-open* if $\mathscr{B} \subset \mathscr{P}(Y)$ and for every $y \in f(X)$ there is an $x_y \in f^{-1}(y)$ such that, whenever U is a neighborhood of x_y in X, $y \in B \subset f(U)$ for some $B \in \mathscr{B}$. We say that f is *σ-almost-open* if f is $\mathscr{B}$-almost-open for some σ-discrete family $\mathscr{B}$ in Y. Observe that the proof of part (b) of Proposition 3.3 above actually shows that any map that has inductively a σ-discrete base (hence any inductively base-σ-discrete map defined on a σ-space) is σ-almost-open. We will presently show that under slightly stronger assumptions the converse is also true.

Recall that a space is said to be *metalindelöf* if every open cover has a point-countable open refinement. It is an easy exercise to see that in such spaces every discrete family $\mathscr{E}$ has a point-countable open expansion; i.e., there exists an index-point-countable family $\{U_E: E \in \mathscr{E}\}$ such that U_E is open and contains E for each E in $\mathscr{E}$.

3.21. PROPOSITION. *Let $f: X \to Y$ be σ-almost-open and have Lindelöf fibers, with X a metalindelöf σ-space. Then f is inductively base-σ-discrete.*

Proof. Let $\mathscr{A}$ be a σ-discrete network for X. Since X is metalindelöf, there is an index-point-countable open family in X of the form $\mathscr{U} = \{U_A: A \in \mathscr{A}\}$ such that $A \subset U_A$ for each A. Since $\mathscr{A}$ is also a σ-discrete base for $\mathscr{U}$, $\mathscr{U}$ is σ-discretely decomposable in X by a result of Michael [**8**, Lemma 3.2]. Consequently, each fiber of f can meet at most countably many members of $\mathscr{U}$, and so $\{f(U_A): A \in \mathscr{A}\}$ will be index-point-countable.

Let $\mathscr{C}$ be a σ-discrete family in Y such that f is $\mathscr{C}$-almost-open at $x_y \in f^{-1}(y)$ for each $y \in f(X)$. Let $X' = \{x_y: y \in f(X)\}$. Since $f(X') = f(X)$, f will be inductively base-σ-discrete if we can show that $\{f(A \cap X'): A \in \mathscr{A}\}$ is σ-discretely decomposalbe, in view of Lemma 2.1. Since $\{f(A \cap X')\}$ is point-countable, it suffices to show that this family has a σ-discrete base by the above mentioned result. We proceed to show that such a base is

$$\mathscr{B} = \{C \cap f(A \cap X'): C \subset f(U_A)\}.$$

Since $\mathscr{C}$ is σ-discrete and $\{f(U_A): A \in \mathscr{A}\}$ is index-point-countable, it follows that $\mathscr{B}$ is σ-discrete. Now let $y \in f(A \cap X')$, say $y = f(x)$ and $x \in A \cap X'$. Since $x \in U_A$ and f is $\mathscr{C}$-almost-open at x, $y \in C \subset f(U_A)$ for some $C \in \mathscr{C}$. This shows that $\mathscr{B}$ is a base for $\{f(A \cap X')\}$. □

Any $\mathscr{B}$-almost-open map is clearly $\mathscr{B}$-bi-quotient, but there exist perfect (hence bi-quotient) maps from the space of real numbers to itself that are not almost-open [**14**, Example 6]. However, we will show in the next section that there is a converse for σ-bi-quotient maps under the same assumptions (on X and the fibers of f) as in 3.21.

4. **Proof of Theorem 1.** The implication (b) → (c) follows from 3.21 and the fact that the fibers of f are $\aleph_1$-compact (see the remark in §1). That (c) → (d) follows from 3.3(b), and (d) → (a) is covered by 3.17. It remains only to prove that (a) → (b).

4.1. Proposition. *Let f: $X \to Y$ be σ-bi-quotient and have Lindelöf fibers, with X a metalindelöf σ-space. Then f is σ-almost-open.*

Proof. Let $\mathscr{A}$ be a σ-discrete network for X. As in the proof of 3.21, $\mathscr{A}$ has an open expansion $\{U_A: A \in \mathscr{A}\} = \mathscr{U}$ such that $\{f(U_A): A \in \mathscr{A}\}$ is index-point-countable. Let $\mathscr{B}$ be a σ-discrete bi-quotient base for f closed to finite intersections. By a result of A. S. Miščenko [**12**], for each B in $\mathscr{B}$ there are only countably many finite collections $\mathscr{F}_n(B) \subset \mathscr{U}$ $(n = 1, 2, \ldots)$ such that $f(\mathscr{F}_n(B))$ is a *minimal* cover of B. Let

$$\mathscr{C} = \left\{B \cap f(A): A \in \mathscr{A}, B \in \mathscr{B}, \text{ and } U_A \in \bigcup_{n=1}^{\infty} \mathscr{F}_n(B)\right\}.$$

Then $\mathscr{C}$ is σ-discrete (since $\mathscr{B}$ is and $\bigcup_{n=1}^{\infty} \mathscr{F}_n(B)$ is countable for each B in $\mathscr{B}$). We proceed to show that f is $\mathscr{C}$-almost-open (cf. the proof in [**8**, Theorem 6.1]).

Let $y \in f(X)$. Then there is an $x_y \in f^{-1}(y)$ such that, whenever $x_y \in A \in \mathscr{A}$, then $y \in B$ for some B in $\mathscr{B}$ with U_A belonging to $\bigcup_{n=1}^{\infty} \mathscr{F}_n(B)$. If not, for each $x \in f^{-1}(y)$ we can find an $A(x)$ in $\mathscr{A}$ containing x such that, whenever $y \in B \in \mathscr{B}$, $U_{A(x)} \notin \bigcup_{n=1}^{\infty} \mathscr{F}_n(B)$. Since f is $\mathscr{B}$-bi-quotient and $\{U_{A(x)}: x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$, there exists some $B \in \mathscr{B}$ and a finite $\mathscr{F} \subset \{U_{A(x)}: x \in f^{-1}(y)\}$ such that $y \in B \subset \bigcup f(\mathscr{F})$. Since $f(\mathscr{F})$ will contain some $f(\mathscr{F}_n(B))$ this leads to a contradiction. It follows that if U is any open neighborhood of x_y, then there is an $A \in \mathscr{A}$ such that $x_y \in A \subset U$ (since $\mathscr{A}$ is a network), and so $y \in B$ for some $B \in \mathscr{B}$ with $U_A \in \bigcup_{n=1}^{\infty} \mathscr{F}_n(B)$; i.e., $B \cap f(A)$ belongs to $\mathscr{C}$ and $y \in B \cap f(A) \subset f(U)$. This shows that f is $\mathscr{C}$-almost-open at each x_y. □

5. Locally finitely additive families and the proof of Theorem 2. If $f: X \to Y$ is inductively base-σ-discrete, then there is a subset X' of X such that $f(X') = f(X)$ and $f|X'$ is base-σ-discrete. The objective of the present section is to show that one may take X' to be a "nice" subset of X whenever f has a similarly "nice" property.

A collection of sets $\mathscr{L}$ is said to be a *lattice* if it contains the empty set and is closed under intersections and unions of pairs of its members. Let $\mathscr{L}$ be a lattice of subsets of a set X. For each countable ordinal α we define the Borel classes $\Sigma_\alpha \mathscr{L}$ and $\Pi_\alpha \mathscr{L}$ recursively as follows:

$$\Sigma_0 \mathscr{L} = \mathscr{L}_\sigma, \qquad \Pi_0 \mathscr{L} = \mathscr{L}_\delta,$$
$$\Sigma_{\alpha+1} \mathscr{L} = [\Pi_\alpha \mathscr{L}]_\sigma, \qquad \Pi_{\alpha+1} \Lambda = [\Sigma_\alpha \mathscr{L}]_\delta,$$
$$\Sigma_\lambda \mathscr{L} = \left[\bigcup_{\alpha<\lambda} \Sigma_\alpha \mathscr{L}\right]_\sigma, \quad \Pi_\lambda \mathscr{L} = \left[\bigcup_{\alpha<\lambda} \Pi_\alpha \mathscr{L}\right]_\delta \quad (\lambda \text{ a limit ordinal})$$

Here, for any collection $\mathscr{M}$, $\mathscr{M}_\sigma$ and $\mathscr{M}_\delta$ denote the families of all countable unions and countable intersections of sets from $\mathscr{M}$, respectively. Recall that for a topological space X, with open sets $\mathscr{G}$ and closed sets $\mathscr{F}$ satisfying $\mathscr{G} \subset \mathscr{F}_\sigma$, the (Borel) sets of additive class α are the members of $\Sigma_\alpha \mathscr{G}$, when α is even, and $\Sigma_\alpha \mathscr{F}$, when α is odd. The sets of multiplicative class α are just the complements of the sets of additive class α. We let Souslin-$\mathscr{L}$ denote the class of all sets obtained from $\mathscr{L}$ by applying the ($\mathscr{A}$)-operation of Souslin [7]. Note that $\Sigma_\alpha \mathscr{L}$, $\Pi_\alpha \mathscr{L}$, and Souslin-$\mathscr{L}$ are lattices whenever $\mathscr{L}$ is.

Let $\mathscr{L}$ be a lattice of subsets of a space X. We say that $\mathscr{L}$ is locally finitely additive if, whenever $\mathscr{E} \subset \mathscr{L}$ and $\mathscr{E}$ is locally finite in X, then $\bigcup \mathscr{E}$

belongs to $\mathscr{L}$. The following lemma appers to be new, although it is known for the Borel classes of a paracompact space (see, for example, [**1**]).

5.1. LEMMA. *Let $\mathscr{L}$ be a locally finitely additive lattice of subsets of a space X. If $\mathscr{L}$ contains the closed sets of X, then each of the lattices $\Sigma_\alpha \mathscr{L}$, $\Pi_\alpha \mathscr{L}$, and Souslin $\mathscr{L}$ is locally finitely additive. If $\mathscr{L}$ contains the open sets of X and each locally finite collection in X has a locally finite open expansion (for example, if X is paracompact), then the above lattices will again be locally finitely additive.*

Proof. The crux of the argument is the following set-theoretic fact proven in [**3**, Lemma 1.1]: If $\{E_{a\beta}\colon a \in A,\ \beta < \lambda\}$ (λ an ordinal) is a family of subsets of X such that $\{E_{a\beta}\}_\beta$ is decreasing for each a, and $\{E_{a\beta}\colon a \in A\}$ is index-point-finite for some β, then

(1) $$\bigcup_{a\in A}\bigcap_{\beta<\lambda} E_{a\beta} = \bigcap_{\beta<\lambda}\bigcup_{a\in A} E_{a\beta}.$$

Now suppose $\mathscr{L}$ contains the closed sets of X. Let $\mathscr{E} = \{E_a\colon a \in A\}$ be locally finite in X (as an indexed family). If $\mathscr{E} \subset \mathscr{L}_\sigma$, then it is clear that $\bigcup \mathscr{E} \subset \mathscr{L}_\sigma$ since $\mathscr{L}$ is locally finitely additive. If $\mathscr{E} \subset \mathscr{L}_\delta$, let $E_a = \bigcap E_{an}$ $(n < \omega)$, where $\{E_{an}\}$ is a decreasing sequence in $\mathscr{L}$ for each a. Since $\mathscr{L}$ contains the closed sets, we may assume that each E_{an} is contained in the closure of E_a. Since $\{\overline{E}_a\colon a \in A\}$ is locally finite, it now follow from (1) and the local finite additivity of $\mathscr{L}$ that

$$\bigcup_{a\in A} E_a = \bigcap_{n<\omega}\ \bigcup_{a\in A} E_{an} \in \mathscr{L}_\delta.$$

Proceeding inductively and using standard arguments, one sees that the proof for showing $\Sigma_\alpha \mathscr{L}$ and $\Pi_\alpha \mathscr{L}$ are locally finitely additive differs in no essential way from the case $\alpha = 0$ just shown. (Note that, for λ a limit ordinal, any member of $\Pi_\lambda \mathscr{L}$ can be expressed as $\bigcap_{\alpha<\lambda} E_\alpha$ where the (not necessarily distinct) sets E_α form a decreasing transfinite sequence.) The proof for the class Souslin-$\mathscr{L}$ is also entirely analogous to that given for $\mathscr{L}_\delta$ above, and so is omitted.

When $\mathscr{L}$ contains the open sets we can use the existence of a locally finite open expansion in exactly the same way we use the fact that the family of closures of members of a locally finite family is locally finite in proving the above. We omit the details. □

5.2. *Proof of Theorem* 2. Fix a pseudo-metric for X and let $\mathscr{U}_n$ $(n = 1, 2, \ldots)$ be a locally finite open cover of X by sets having diameter

$< 1/n$. By assumption there exists $X' \subset X$ such that $f(X') = f(X)$ and $f|X'$ is base-σ-discrete. By Lemma 2.2 we may assume that $f|X'$ is index-σ-discrete. Thus $\{f(U \cap X'): U \in \mathscr{U}_n\}$ is σ-discretely decomposable in Y, and so there exist F_σ sets H_U in Y such that $f(U \cap X') \subset H_U$ and $\{H_U: U \in \mathscr{U}_n\}$ is σ-discretely decomposable for each n. If Y is a metalindelöf σ-space, then we may assume each H_U is open in Y in addition. We now put

$$X^* = \bigcap_{n=1}^{\infty} \bigcup_{U \in \mathscr{U}_n} U \cap f^{-1}(H_U),$$

and proceed to prove both (a) and (b) for X^*. Clearly $X' \subset X^*$, and so $f(X^*) = f(X)$.

In case (a) we may assume each H_U is open, thus $\{U \cap f^{-1}(H_U): U \in \mathscr{U}_n\}$ is a locally finite subfamily of $\mathscr{L}$, and it follows that $X^* \in \mathscr{L}_\delta$. In case (b), each of the sets $U \cap f^{-1}(H_U)$ belongs to $\mathscr{L}_\sigma$, and so we have $X^* \in \mathscr{L}_{\sigma\delta}$.

It remains only to show that $f|X^*$ is index-σ-discrete. Since, for each U in $\bigcup_n \mathscr{U}_n$, we have

$$f\big(U \cap f^{-1}(H_U) \cap X^*\big) = H_U \cap f(U \cap X^*), \quad \text{and} \quad \{H_U: U \in \mathscr{U}_n\}$$

is σ-discretely decomposable in Y for each n, it will follow from Lemma 2.1 that $f|X^*$ is index-σ-discrete if we can show that the sets $U \cap f^{-1}(H_U) \cap X^*$ form a network for X^*. Given $x \in X^*$, choose U_n in $\mathscr{U}_n$ such that $x \in U_n \cap f^{-1}(H_{U_n})$. Then $\{U_n\}_{n=1}^{\infty}$ is a local base for x in X, since the diameter of U_n is $< 1/n$, and this implies that $\{U_n \cap f^{-1}(H_{U_n}) \cap X^*\}_{n=1}^{\infty}$ will be a local network for x in X^*. □

If X is a space in which open sets are F_σ sets, we say that a map f: $X \to Y$ is (Borel measurable) *of class* α, α some countable ordinal, if $f^{-1}(V)$ is of additive class α in X whenever V is open in Y; and we say that f is *Souslin measurable* if $f^{-1}(V)$ is a Souslin-$\mathscr{F}$ set in X for all open V. The following corollary is an immediate consequence of 5.1 and 5.2; for $\alpha = 0$ it was shown by Michael (see §1).

5.3 COROLLARY. *Let f: $X \to Y$ be inductively base-σ-discrete, with X and Y pseudometrizable spaces. If f is of class $\alpha < \omega_1$, then there is a set X^* of multiplicative class $\alpha + 1$ in X such that $f(X^*) = f(X)$ and $f|X^*$ is index-σ-discrete.* □

Let X be a regular σ-space. Then X has a σ-discrete network $\mathscr{A}$ of closed sets, and we may assume $\mathscr{A}$ is closed to finite intersections. Let τ be the topology on X having $\mathscr{A}$ as a base of open sets. Then (X, τ) is pseudometrizable, and the identity map $1_X: (X, \tau) \to X$ is continuous and takes open sets to F_σ sets (cf. [6]). 1_X is also index-σ-discrete by Lemma 2.1 (since $1_X(\mathscr{A}) = \mathscr{A}$ is σ-discrete). Using this device we can usually apply Theorem 2 in the case when X is a regular σ-space as illustrated by the following.

5.4. COROLLARY. *Let $f: X \to Y$ be inductively base-σ-discrete, with X a regular σ-space.*

(a) *If f is of class α and Y is a metalindelöf σ-space, then there is an X^* of multiplicative class $\alpha + 2$ in X such that $f(X^*) = f(X)$ and $f|X^*$ is index-σ-discrete.*

(b) *If f is Souslin measurable, then there is a Souslin-$\mathscr{F}$ set X^* in X such that $f(X^*) = f(X)$ and $f|X^*$ is index-σ-discrete.*

Proof. (a) Let $\mathscr{A}$, τ, and $1_X: (X, \tau) \to X$ have the meaning attached to them above. Then $f \circ 1_X$ is inductively base-σ-discrete and will be of class α whenever f is. Consequently, there is an $X^* \subset X$ of multiplicative class $\alpha + 1$ in X relative to τ such that $f \circ 1_X(X^*) = f(X)$ and $f \circ 1_X|X^*$ is index-σ-discrete. Since 1_X^{-1} is of class 1, X^* is of multiplicative class $\alpha + 2$ in X. It is routine to check that $f(X^*) = X$ and $f|X^*$ is index-σ-discrete.

The proof of (b) is entirely analogous. □

6. Maps that preserve analytic metric spaces and the proof of Theorem 3. Recall that if a metric space X is a Souslin-$\mathscr{F}$ subset of some complete metric space, then it is a Souslin-$\mathscr{F}$ set in any metric embedding, and we say that X is an *analytic space* (see [15] for a recent survey of the properties of these spaces). For separable metric spaces, the analytic spaces are precisely the continuous images of complete separable metric spaces (equivalently, the Baire space of weight $\aleph_0$), and so are preserved under arbitrary continuous maps. Of course, this is no longer the case for non-separable spaces, since any metric space is a continuous image of a discrete (hence complete) metric space. In fact, 6.3 below gives an example of a non-analytic metric space which is a one-to-one continuous image of the Baire space $B(\mathscr{k})$ (i.e., the countable product of discrete spaces of cardinality $\mathscr{k}$) for any given infinite cardinal $\mathscr{k} > \aleph_0$.

In [2] we showed that for the non-separable case one needs to work with base-σ-discrete maps (called "co-σ-discrete" there). (Note that any map defined on a separable metric space is trivially base-σ-discrete.) In particular, the following was shown in [2].

6.1. THEOREM. *The following are equivalent for any metric space X.*

(a) *X is analytic.*

(b) *X is a base-σ-discrete continuous image of $B(\mathscr{k})$, for some infinite cardinal $\mathscr{k}$.*

(c) *X is an index-σ-discrete continuous image of a closed subspace of $B(\mathscr{k})$, for some infinite cardinal $\mathscr{k}$.*

(d) *X is a base-σ-discrete Souslin measurable image of some analytic metric space.* □

REMARK. 6.1 (c) is implicit in the proof of [**2**, Theorem 4.1], since the set C constructed there is a closed subspace of $B(\mathscr{k})$ and the base-σ-discrete map g defined on C has separable fibers; hence [**8**, Proposition 4.3] applies. The rest of 6.1 is covered by Theorems 4.1 and 7.3 of [**2**].

For metric spaces, any closed map or any open s-map is base-σ-discrete [**2**, 3.10 and 3.11], so such maps when continuous or merely Souslin measurable will preserve analytic spaces by 6.1(d). What other quotient maps have this property? In [**5**] I showed that almost-open s-maps do, and asked whether this was true of quotient s-maps. Michael's result cited in the introduction can be combined with 6.1(d) to given an affirmative answer for continuous quotient s-maps. This result is a special case of Theorem 3, which we now show is a consequence of Theorems 1 and 2.

6.2. *Proof of Theorem* 3. Let $f: X \to Y$ be a Souslin measurable s-map from an analytic metric space X onto a metric space Y, and suppose f is σ-bi-quotient. Then, by Theorem 1, f is inductively base-σ-discrete, and so there is a Souslin $\mathscr{F}$ set X^* of X such that $f(X^*) = Y$ and $f|X^*$ is index-σ-discrete by 5.4(b). Since X^* must then be analytic, it follows that Y is analytic by 6.1(d). □

6.3. EXAMPLE. A non-analytic metric space X which is a one-to-one continuous image of $B(\mathscr{k})$, $\mathscr{k}$ an infinite cardinal $> \aleph_0$.

Proof. Let M be any non-analytic metric space of cardinality $\mathscr{k}$ (for example, if E is a non-analytic subset of the reals of cardinality $\aleph_1$, we may take M to be the topological sum of $\mathscr{k}$ copies of E). the product space $X = \prod M_n$ $(n = 1, 2, \ldots)$, where each M_n is a copy of M, is not analytic, since M is homeomorphic to a closed subspace of X. Let $f_n: T_n \to M_n$ be any bijection where T_n is a discrete space of cardinality $\mathscr{k}$. Then the product map is a one-to-one continuous map from $B(\mathscr{k})$ onto X. □

We conclude with an example which shows that in Theorems 1 and 3 we cannot weaken the assumption, that $f^{-1}(y)$ is Lindelöf for each y, to $\partial f^{-1}(y)$ is Lindelöf for each y.

6.4. EXAMPLE. An almost-open continuous surjection f: $X \to Y$, with X a complete metric space and Y a non-analytic subset of the reals, such that $\partial f^{-1}(y)$ contains at most one point for each $y \in Y$.

Proof. We give only an outline of the construction as the details are given in [**5**, p. 11]. Let Y be any non-analytic subset of the reals of cardinality $\aleph_1$. As observed in [**11**], there is an almost-open continuous surjetion g $B(\aleph_1) \to Y$. For each y in Y, let $x_y \in g^{-1}(y)$ be a point of openness for g, and let X_y denote the space obtained from $B(\aleph_1)$ by letting each $x \neq x_y$ be isolated and x_y have its usual neighborhoods. Let X be the topological sum of the spaces $\{y\} \times X_y$ $(y \in Y)$, and define f: $X \to Y$ by $f(y, x) = g(x)$. Then X is a complete metric space, f is continuous and almost-open, and $\partial f^{-1}(y)$ contains at most the point (y, x_y). Note that f is a σ-almost-open map (hence σ-bi-quotient), since it is an almost-open map onto a metrizable space. But f cannot be inductively base-σ-discrete, since then 5.4(b) and Theorem 3 would imply that Y is analytic. □

REFERENCES

[0] A. Arhangel'skiĭ, *Some types of factor mappings and the relations between classes of topological spaces*, Soviet Math. Dokl., **4** (1963), 1726–1729.

[1] R. Engelking, *On Borel sets and B-measurable functions in metric spaces*, Prace Mat., **10** (1967), 145–149.

[2] R. W. Hansell, *On characterizing non-separable analytic and extended Borel sets as types of continuous images*, Proc. London Math. Soc., (3) **28** (1974), 683–699.

[3] ______, *Hereditarily-additive families in descriptive set theory and Borel measurable multimaps*, Trans. Amer. Math. Soc., **278** (1983), 725–749.

[4] ______, *Measurable selection in non-separable spaces*, (in preparation).

[5] ______, *Analytic sets and almost-open mappings*, preprint.

[6] H. J. K. Junnila, *On σ-spaces and pseudometrizable spaces*, Topology Proc., **4** (1979), 121–132.

[7] K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York; PWN, Warsaw, 1966.

[8] E. A. Michael, *On maps related to σ-locally finite and σ-discrete collections of sets*, Pacific J. Math., **98** (1982), 139–152.

[9] ______, *Bi-quotient maps and Cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble), **18** (2) (1968), 287–302.

[10] ______, *A quintuple quotient quest*, General Topology and Appl., **2** (1972), 91–138.

[11] E. A. Michael and A. H. Stone, *Quotients of the space of irrationals*, Pacific J. Math., **28** (1969), 629–633.

[12] A. S. Miščenko, *Spaces with pointwise denumerable basis*, (Russian), Dokl. Akad. Nauk SSSR, **145** (1962), 985–988 [Soviet Math. Dokl., **3** (1962), 855–858].

[13] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl., **1** (1971), 143–154.

[14] F. Siwiec and V. J. Mancuso, *Relations among certain mappings and conditions for their equivalence*, General Topology and Appl., **1** (1971), 33–41.

[15] A. H. Stone, *Analytic Sets in Non-separable Metric Spaces*, Part 5 of Analytic Sets, Academic Press, London, 1980.

[16] _____, *Metrisability of decomposition spaces*, Proc. Amer. Math. Soc., **7** (1956), 690–700.

[17] G. T. Whyburn, *Accessibility spaces*, Proc. Amer. Math. Soc., **24** (1970), 181–185.

Received February 9, 1983 and in revised form January 31, 1984.

UNIVERSITY OF CONNECTICUT
STORRS, CT 06268